Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 5, pp. 1087–1095
Online ISSN: 1735-8515

# EXISTENCE OF A GROUND STATE SOLUTION FOR A CLASS OF $p$-LAPLACE EQUATIONS

Y. H. DENG

(Communicated by Mohammad Asadzadeh)

Abstract. According to a class of constrained minimization problems, the Schwartz symmetrization process and the compactness lemma of Strauss, we prove that there is a nontrivial ground state solution for a class of $p$-Laplace equations without the Ambrosetti-Rabinowitz condition.
**Keywords:** Ground state solution, $p$-Laplace equation, minimization problem, the Schwartz symmetrization process.
**MSC(2010):** Primary: 35J20; Secondary: 35J60.



## 1. Introduction

In [1, 2, 5, 6, 9], the authors studied the existence of a ground state solution for the following problem

$$\begin{cases} -\triangle u + W(x)u = g(x,u) + f \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.1}$$

subject to the condition that $W > 0$. In the case $W < 0$, various difficulties arise in the study of (1.1). On this subject, the existence of solutions has been studied by Ghimenti, Micheletti, Benrhouma and Ounaies in [3, 4, 8, 11] under some special conditions.

It is well known that problems involving the $p$-Laplacian operator appear in many areas of applied mathematics and physics. For example, they may be found in the study of non-Newtonian fluids, non-linear

Article electronically published on October 27, 2014.
Received: 8 February 2013, Accepted: 28 July 2010.

elasticity and reaction-diffusions. In [7] and [12], the authors discussed the existence of a ground state solution and the asymptotic behavior of ground states for the following equation

$$-\triangle_p u + P(|x|)u^{p-1} = Q(|x|)u^{q-1}, \tag{1.2}$$

under the condition that $P(|x|) > 0$. In [10], Liu studied the existence of ground states for a class of more general $p$-Laplacian equations.

To the best of author's knowledge, not much is known about the existence of a ground state solution to (1.2) and their general versions in $\mathbb{R}^N$ under the condition $P(|x|) < 0$.

In this paper, we study the existence of a ground state solution for the following problem

$$\begin{cases} -\triangle_p u - |u|^{p-2}u + |u|^{q-2}u = f(u) \\ u > 0 \\ u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \end{cases} \tag{1.3}$$

where $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $N \geq 3$, $1 \leq q < p < N$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following standard condition

$$f(s) \leq C(s^{p^*-1} + s^{p-1}), \tag{1.4}$$

for all $s > 0$ and some constants $C > 0$.

Let $F(s) = \int_0^s f(t)dt$ and

$$G(s) = \frac{1}{p}|s|^p + F(s) - \frac{1}{q}|s|^q. \tag{1.5}$$

To guarantee the existence of a solution for problem (1.3), we suppose that there exists $\xi > 0$ such that $G(\xi) > 0$ which is a necessary condition for existence of a solution of problem (1.3) (see [5]).

It is worth pointing out that if there exist constants $\lambda > 0$ and $m \in (p, p^*)$ such that $f(s) \geq \lambda s^{m-1}$ holds for every $s > 0$, then $\lambda s^{m-1} \leq f(s) \leq C(s^{p^*-1} + s^{p-1})$ and $G(s) = \frac{1}{p}|s|^p + F(s) - \frac{1}{q}|s|^q > 0$ can be satisfied by large enough $s > 0$. Therefore, the hypotheses $f(s) \leq C(s^{p^*-1} + s^{p-1})$ for all $s > 0$ and $G(\xi) > 0$ for some $\xi > 0$ are reasonable. The main result of this paper is

**Theorem 1.1.** *Suppose that there exists a constant $C > 0$ such that $f(s) \leq C(s^{p^*-1} + s^{p-1})$ for all $s > 0$. If there exists $\xi > 0$ such that $G(\xi) > 0$, then (1.3) possesses a nontrivial ground state solution.*

Similar to [1], our result is obtained without the Ambrosetti-Rabinowitz condition and the condition that $\frac{f(s)}{s}$ is increasing in $(0, \infty)$.

## 2. **Notations and preliminaries**

Since we seek positive solutions, without loss of generality, we may assume that $f(s) = 0$ for $s \leq 0$. In order to discuss the existence of a ground state solution for (1.3), we consider the following minimization problem

$$A = \inf\{\frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p : u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u) = 1\}, \tag{2.1}$$

where $G(s)$ is defined in (1.5) and $F(s) = \int_0^s f(t)dt$ with $f$ satisfying condition (1.4).

Similar to [4] and [11], we let $E = W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. It is obvious that $E$ is a Banach space under the following norm

$$||u|| = ||\nabla u||_p + ||u||_q,$$

where $|| \cdot ||_r$ denotes the standard normal in $L^r(\mathbb{R}^N)$.

We recall that the Schwartz symmetrized function $f^*$ of $f \in L^1(\mathbb{R}^N)$ is a radial, nonincreasing function of $r = |x|$ such that

$$\int_{\mathbb{R}^N} H(f)dx = \int_{\mathbb{R}^N} H(f^*)dx \tag{2.2}$$

for every continuous function $H$ with $H(f)$ is integrable (for more details, please see [5]). Since (1.3) is an autonomous problem, by (2.2) we conclude that under the Schwartz symmetrization process we can minimize problem (2.1) on the space $E_{rad}$, the subspace of $E$ formed by radially symmetric functions. Furthermore, according to the same method as in [5], we can easily prove that the set $\{u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) = 1\}$ is not empty.

## 3. **Some lemmas**

To prove Theorem 1.1, we need to establish some useful lemmas.

**Lemma 3.1.** *There exists a constant $d > 0$ such that for any $u \in E$ we have*

$$\frac{1}{q}||u||_q^q \geq (C + \frac{2}{p})||u||_p^p - d||u||_{p^*}^{p^*},$$

*where $p^* = \frac{pN}{N-p} > p > q$.*

*Proof.* Consider the following function

$$h(s) = \frac{(C + \frac{2}{p})|s|^p - \frac{1}{q}|s|^q}{|s|^{p^*}}, \ s \neq 0.$$

We observe that if $0 < |s| < (\frac{1}{q(C+\frac{2}{p})})^{\frac{1}{p-q}}$, then $h(s) < 0$. On the other hand, since $p^* = \frac{pN}{N-p} > p > q$, we have $\lim_{|s|\to+\infty} h(s) = 0$. Therefore we conclude that there exists $d > 0$ such that

$$(C + \frac{2}{p})|s|^p - \frac{1}{q}|s|^q \leq d|s|^{p^*}. \tag{3.1}$$

Putting $s = |u|$ in (3.1) and then integrating, the lemma is proved. □

**Lemma 3.2.** *Any minimizing sequence $\{u_n\}$ for (2.1) is bounded in $E_{rad}$.*

*Proof.* If $\{u_n\}$ is a minimizing sequence for (2.1), then we have

$$\lim_{n\to\infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = A \text{ and } \int_{\mathbb{R}^N} G(u_n) = 1. \tag{3.2}$$

By (1.4), we obtain

$$F(s) = \int_0^s f(t)dt \leq C(s^{p^*} + s^p). \tag{3.3}$$

According to (1.5), (3.2) and (3.3), we get

$$1 \leq \frac{1}{p}||u_n||_p^p + C||u_n||_p^p + C||u_n||_{p^*}^{p^*} - \frac{1}{q}||u_n||_q^q. \tag{3.4}$$

By Lemma 3.1 and (3.4), we get

$$1 + \frac{1}{p}||u_n||_p^p \leq (C + d)||u_n||_{p^*}^{p^*}. \tag{3.5}$$

Since $\lim_{n\to\infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = A$, then $\int_{\mathbb{R}^N} |\nabla u_n|^p$ is bounded. By the Gagliardo-Nirenberg inequality we conclude that $||u_n||_{p^*}^{p^*}$ is also bounded. Thus, it follows from (3.5) that $||u_n||_p^p$ is bounded. By (3.4), $||u_n||_q^q$ is bounded, and consequently, we conclude that $\{u_n\}$ is bounded in $E_{rad}$. □

**Lemma 3.3.** *The number $A$ given by (2.1) is positive, that is, $A > 0$.*

*Proof.* From the definition of $A$, it is clear that $A \geq 0$. Assume by contradiction that $A = 0$. Similar to [1], we let $\{u_n\}$ be a minimizing sequence in $E_{rad}$ to $A = 0$, then we have

$$\lim_{n\to\infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = 0 \text{ and } \int_{\mathbb{R}^N} G(u_n) = 1.$$

Therefore, by the Gagliardo-Nirenberg inequality we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} = 0.$$

On the other hand, by (3.5) we have $||u_n||_{p^*}^{p^*} \geq \frac{1}{C+d}$. Therefore, we get a contradiction which means that $A > 0$. □

**Lemma 3.4.** *([5]) If $u \in L^p(\mathbb{R}^N)$, and $1 \leq p < +\infty$ is a radial nonincreasing function, then*

$$|u(x)| \leq |x|^{-\frac{N}{p}} (\frac{N}{|S^{N-1}|})^{\frac{1}{p}} ||u||_p, \quad x \neq 0,$$

*where $|S^{N-1}|$ is the volume of the unit sphere in $\mathbb{R}^N$.*

**Lemma 3.5.** *The number $A$ given by (2.1) is attained by some functions in the following set*

$$W = \{u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) = 1\}.$$

*Proof.* Let $\{u_n\} \subset E_{rad}$ be a minimizing sequence for (2.1). By Lemma 3.2, we conclude that there is a subsequence of $\{u_n\}$, we also denoted $\{u_n\}$ such that $\{u_n\}$ converges weakly in $E$ almost everywhere in $\mathbb{R}^N$ to a function $u \in E$. Since every $u_n$ is radial, nonnegative and nonincreasing with $r = |x|$, then $u$ is radial, nonnegative and nonincreasing with $r = |x|$. Note that $u_n \in L^q(\mathbb{R}^N)$, and by Lemma 3.4 we have

$$|u_n(x)| \leq |x|^{-\frac{N}{q}} (\frac{N}{|S^{N-1}|})^{\frac{1}{q}} ||u_n||_q. \tag{3.6}$$

Since $||u_n||_q^q$ is bounded, by (3.6) we conclude that there exists a constant $b > 0$ such that $|u_n(x)| \leq b|x|^{-\frac{N}{q}}$. Therefore, we have

$$|u_n(x)|^p \leq b^p |x|^{-\frac{pN}{q}} \text{ and } |u_n(x)|^{p^*} \leq b^{p^*} |x|^{-\frac{p^* N}{q}}. \tag{3.7}$$

Since $p > q$ and $p^* > q$, we have $|x|^{-\frac{pN}{q}} \in L^1(\mathbb{R}^N)$ and $|x|^{-\frac{p^*N}{q}} \in L^1(\mathbb{R}^N)$. Thus, by (3.7) we get

$$F(u_n) \leq C(|u_n|^{p^*} + |u_n|^p) \leq C(b^p |x|^{-\frac{pN}{q}} + b^{p^*} |x|^{-\frac{p^* N}{q}}) \in L^1(\mathbb{R}^N). \tag{3.8}$$

Since $\{u_n\}$ converges almost everywhere in $\mathbb{R}^N$ to $u$ and $F$ is continuous, then we have $F(u_n) \to F(u)$ almost everywhere. Therefore, by (3.8) and Lebesgue's dominated convergence theorem we obtain

$$F(u_n) \to F(u) \text{ in } L^1(\mathbb{R}^N). \tag{3.9}$$

On the other hand, since $||u_n||_q^q$ and $||u_n||_{p^*}^{p^*}$ are bounded,

$$\sup_n \int_{\mathbb{R}^N} (|u_n|^q + |u_n|^{p^*}) < +\infty. \tag{3.10}$$

By (3.6), we have $u_n(x) \to 0$ as $|x| \to +\infty$, uniformly with respect to $n$. It follows from $p^* > p > q \geq 1$ that

$$\lim_{|s|\to 0} \frac{|s|^p}{|s|^q + |s|^{p^*}} = \lim_{|s|\to 0} \frac{|s|^{p-q}}{1 + |s|^{p^*-q}} = 0, \tag{3.11}$$

and

$$\lim_{|s|\to +\infty} \frac{|s|^p}{|s|^q + |s|^{p^*}} = 0. \tag{3.12}$$

Since $|u_n|^p$ converges to $|u|^p$ almost everywhere in $\mathbb{R}^N$, by (3.10), (3.11), (3.12) and the compactness lemma of Strauss we conclude that

$$\lim_{n\to +\infty} \int_{\mathbb{R}^N} |u_n|^p = \int_{\mathbb{R}^N} |u|^p. \tag{3.13}$$

By (1.5), (3.9), (3.13) and Fatou's lemma, we have

$$1 \leq \frac{1}{p} \int_{\mathbb{R}^N} |u|^p + \int_{\mathbb{R}^N} F(u) - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q. \tag{3.14}$$

The inequality (3.14) means that $\int_{\mathbb{R}^N} G(u) \geq 1$. If $u$ is not in $W$, one should have

$$\int_{\mathbb{R}^N} G(u) > 1. \tag{3.15}$$

Similar to [1], we define a function $h : [0, 1] \to \mathbb{R}$ as $h(t) = \int_{\mathbb{R}^N} G(tu)$. It is obvious that $h$ is continuous. Since $G(tu) = \frac{1}{p}|tu|^p + F(tu) - \frac{1}{q}|tu|^q$, $F(tu) \leq C(|tu|^{p^*} + |tu|^p)$ and $p^* > p > q \geq 1$, we conclude that $h(t) < 1$ for $t$ close to 0. By (3.15), we have $h(1) > 1$. Therefore, there exists $t_0 \in (0, 1)$ such that $h(t_0) = 1$, which means that $t_0 u \in W$. On the other hand, since the minimizing sequence $\{u_n\}$ for (2.1) converges weakly to $u$, then

$$\frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \leq \liminf_{n\to +\infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = A. \tag{3.16}$$

Since $t_0 \in (0,1)$ and $t_0 u \in W$, by (3.16) we have

$$A \leq \frac{1}{p}\int_{\mathbb{R}^N} |\nabla t_0 u|^p = \frac{t_0^p}{p}\int_{\mathbb{R}^N} |\nabla u|^p < A.$$

This is a contradiction. Therefore, $u \in W$ and $\frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p = A$. □

Let $T(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p$ and $V(u) = \int_{\mathbb{R}^N} G(u)$. It is well known that $T$ and $V$ are $C^1$ functionals on E.

**Lemma 3.6.** *Suppose that* $J(w) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w|^p - \int_{\mathbb{R}^N} H(w)$ *is a* $C^1$ *function on a suitable Banach space. If* $u$ *is a critical point of* $J$*, then*

$$(N-p)\int_{\mathbb{R}^N} |\nabla u|^p = pN\int_{\mathbb{R}^N} H(u). \tag{3.17}$$

*Proof.* Let $\sigma > 0$ and

$$u_\sigma = u(\frac{x}{\sigma}) = u(\frac{x_1}{\sigma}, \frac{x_2}{\sigma}, \cdots, \frac{x_N}{\sigma}) = u(y_1, y_2, \cdots, y_N).$$

Direct calculation shows that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_\sigma|^p dx &= \frac{1}{\sigma^p}\int_{\mathbb{R}^N} \{(\frac{\partial u}{\partial y_1})^2 + (\frac{\partial u}{\partial y_2})^2 + \cdots + (\frac{\partial u}{\partial y_N})^2\}^{\frac{p}{2}} dx \\ &= \frac{1}{\sigma^p}\int_{\mathbb{R}^N} \sigma^N \{(\frac{\partial u}{\partial y_1})^2 + (\frac{\partial u}{\partial y_2})^2 + \cdots + (\frac{\partial u}{\partial y_N})^2\}^{\frac{p}{2}} dy \\ &= \sigma^{N-p}\int_{\mathbb{R}^N} |\nabla u|^p.\end{aligned}$$

Similarly, we have $\int_{\mathbb{R}^N} H(u_\sigma) = \sigma^N \int_{\mathbb{R}^N} H(u)$. Thus, we obtain

$$J(u_\sigma) = \frac{\sigma^{N-p}}{p}\int_{\mathbb{R}^N} |\nabla u|^p - \sigma^N \int_{\mathbb{R}^N} H(u).$$

Since $u$ is a critical point of $J$, then $\frac{d}{d\sigma}|_{\sigma=1} J(u_\sigma) = 0$, which means that (3.17) holds. □

**Lemma 3.7.** *If* $u$ *is a solution of (1.3), then* $S(u) = \frac{1}{N}T(u) > 0$*, where* $S(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p - \int_{\mathbb{R}^N} G(u)$, $T(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p$.

*Proof.* By Lemma 3.6, we have

$$S(u) = \frac{1}{p}(1 - \frac{N-p}{N})T(u) = \frac{1}{N}T(u) > 0.$$

□

## 4. **The proof of Theorem 1.1**

*Proof.* Suppose that $u_n$, $u$, $V$ and $T$ are functions defined in Section 3. Since $V$ and $T$ are $C^1$ functionals on $E$, there exists a Lagrange multiplier $\theta$ such that $T'(u) = \theta V'(u)$. If $\theta = 0$ or $V'(u) = 0$, then $A = 0$ which contradicts Lemma 3.3. Therefore, $\theta \neq 0$ and $V'(u) \neq 0$. Choose a function $w \in C_0^\infty(\mathbb{R}^N)$ such that $\langle V'(u), w\rangle > 0$. It is obvious that $V(u+\varepsilon w) = V(u) + \varepsilon\langle V'(u), w\rangle + o(\varepsilon)$ and

$$T(u+\varepsilon w) = T(u) + \varepsilon\theta\langle V'(u), w\rangle + o(\varepsilon) \text{ for } \varepsilon \to 0.$$

If $\theta < 0$, then one can find $\varepsilon > 0$ small enough so that $v = u + \varepsilon w$ satisfies $V(v) > V(u) = 1$ and $T(v) < T(u) = A$. Therefore, there exists $\sigma \in (0,1)$ such that $v_\sigma = v(\frac{x}{\sigma})$ satisfies $V(v_\sigma) = 1$ and $T(v_\sigma) < A$, which is impossible. Hence $\theta > 0$. Thus $u$ satisfies, at least in the distribution sense, the equation

$$-\triangle_p u = \theta(|u|^{p-2}u - |u|^{q-2}u + f(u)) \text{ in } \mathbb{R}^N.$$

Set $u_\sigma = u(\frac{x}{\sigma})$. Direct calculation shows that $\nabla u_\sigma = \frac{1}{\sigma}\nabla u$ and $|\nabla u_\sigma|^{p-2} = \frac{1}{\sigma^{p-2}}\nabla u$. Therefore, we have

$$\triangle_p u_\sigma = |\nabla u_\sigma|^{p-2}\triangle u_\sigma + (p-2)|\nabla u_\sigma|^{p-3}\nabla u_\sigma \cdot \nabla|\nabla u_\sigma| = \frac{1}{\sigma^p}\triangle_p u.$$

Thus, we conclude that $u(\frac{x}{\sqrt[p]{\theta}}) = u_{\sqrt[p]{\theta}}$ is a solution of problem (1.3). Using Lemma 3.6 and Lemma 3.7, similar to the method in the proof of Theorem 3 in [5], we have

$$0 < S(u_{\sqrt[p]{\theta}}) \leq S(v),$$

where $v$ is any solution of problem (1.3). Therefore, $u_{\sqrt[p]{\theta}}$ is a ground state solution of problem (1.3). □

**Acknowledgments**

This research is supported by Hunan Provincial Natural Science Foundation of China (No. 14JJ2120) and by Scientific Research Fund of Hunan Provincial Education Department (No.14A020), and partly supported by the construct program of the key discipline in Hunan province (No. [2011]76).

## References

[1] C. O. Alves, M. A. S. Souto and M. Montemegro, Existence of a ground state solution for a nonlinear scalar field equation with critical growth, *Calc. Var. Partial Differential Equations* **43** (2012), no. 3-4, 537–554.

[2] A. Ambrosetti, V. Felli and A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, *J. Eur. Math. Soc. (JEMS)* **7** (2005), no. 1, 117–144.

[3] M. Benrhouma and H. Ounaies, Existence and uniqueness of positive solution for nonhomogeneous sublinear elliptic equation, *J. Math. Anal. Appl.* **358** (2009), no. 2, 307–319.

[4] M. Benrhouma and H. Ounaies, Existence of solutions for a perturbation sublinear elliptic equation in $\mathbb{R}^N$, *Nonlinear Differential Equations Appl.* **17** (2010), no. 5, 647–662.

[5] H. Berestycki and P. Lions, Nonlinear scalar field equations, I. Existence of a ground state, *Arch. Rat. Mech. Anal.* **82** (1983), no. 4, 313–345.

[6] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), no. 4, 437–477.

[7] F. Gazzola, B. Peletier, P. Pucci and J. Serrin, Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, *II. Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003), no. 6, 947–974.

[8] M. Ghimenti and A. M. Micheletti, Solutions for a nonhomogeneous nonlinear Schrödinger equation with double power nonlinearity, *Differential Integral Equations* **20** (2007), no. 10, 1131–1152.

[9] L. Jeanjean and K. Tanaka, A remark on least energy solutions in $\mathbb{R}^N$, *Proc. Amer. Math. Soc.* **131** (2003), no. 8, 2399–2408

[10] S. B. Liu, On ground states of superlinear $p$-Laplacian equations in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **361** (2010), no. 1, 48–58

[11] H. Ounaies, Study of an elliptic equation with a singular potential, *Indian J. Pure Appl. Math.* **34** (2003), no. 1, 111–131.

[12] J. B. Su, Z. Q. Wang and M. Willem, Weighted sobolev embedding with unbounded and decaying radial potentials, *J. Differential Equations* **238** (2007), no. 1, 201–219.

(Yi Hua Deng) Department of Mathematics and Computational science, Hengyang Normal University, P.O. Box 421002, Hengyang, China

*E-mail address*: dengchen4032@126.com