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ON L_1-WEAK ERGODICITY OF NONHOMOGENEOUS CONTINUOUS-TIME MARKOV PROCESSES

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ABSTRACT. In the present paper we investigate the L_1-weak ergodicity of nonhomogeneous continuous-time Markov processes with general state spaces. We provide a necessary and sufficient condition for such processes to satisfy the L_1-weak ergodicity. Moreover, we apply the obtained results to establish L_1-weak ergodicity of quadratic stochastic processes.
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1. **Introduction**

The main aim of the present paper is to establish necessary and sufficient conditions for nonhomogeneous continuous-time Markov process to satisfy the L_1-weak ergodicity. Here we are going to employ measure-theoretic methods to get the desired assertions. Note that a similar condition was studied for homogeneous Markov processes in [18]. We recall that the ergodicity of Markov process means the tendency for a chain to forget the distant past. There is a huge number of investigations devoted to the ergodicity of such processes with countable state spaces (see for example, [1–6, 8, 9, 19, 20]). For example, in [7] it was studied weak ergodicity of nonhomogeneous Markov processes. In [13] weak and strong ergodicity of nonhomogeneous Markov processes were studied in terms of the Dobrushun's ergodic coefficient [1]. In [10, 21, 22] using methods

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of differential equations, some sufficient conditions for weak and strong ergodicity of nonhomogeneous continuous-time Markov processes were given.

The paper is organized as follows. In Section 2 we provide necessary notions. In Section 3 we prove our main result, i.e., necessary and sufficient conditions for nonhomogeneous continuous-time Markov process to satisfy the L_1-weak ergodicity. Finally, in Section 4 we provide some applications of the main result to L_1-weak ergodicity of quadratic stochastic processes which improves the result of [17]. Note that such processes relate to quadratic operators [11, 12] as Markov processes relate to linear operators. For the recent review on quadratic operator we refer to [5].

2. L_1-Weak ergodicity

Let $(X, \mathcal{F}, \mu)$ be a probability space. In what follows, we consider the standard $L^1(X, \mathcal{F}, \mu)$ and $L^\infty(X, \mathcal{F}, \mu)$ spaces. Note that $L^1(X, \mathcal{F}, \mu)$ can be identified with the space of finite signed measures on X which are absolutely continuous with respect to μ. By $\mathfrak{M}$ we denote the set of all probability measures on X which are absolutely continuous w.r.t. μ. We recall that a set of transition probabilities $P^{[s,t]}(x, A)$, $x \in X$, $A \in \mathcal{F}$ $(s, t \in \mathbb{R}_+)$ forms *a non-homogeneous continuous-time Markov process (NHCTMP)* if the following conditions are satisfied:

1. for each s, t $(s \le t)$ the function of two variables $P^{[s,t]}(x, A)$ is a Markov kernel, and it is μ-measurable, i.e., $\mu(A) = 0$ implies $P^{[s,t]}(x, A) = 0$ a.e. on X.
2. Kolmogorov-Chapman equation: for every $s \le h \le t$

$$P^{[s,t]}(x, A) = \int P^{[s,h]}(x, dy) P^{[h,t]}(y, A) \tag{2.1}$$

In the sequel, we will deal with μ-measurable NHCTMP. In this case, for each s and t one can define a positive linear contraction operator on L^1 (respectively L^∞) denoted by $P_*^{[s,t]}$ (respectively $P^{[s,t]}$). Namely,

$$(2.2) \qquad (P_*^{[s,t]}\nu)(A) = \int P^{[s,t]}(x,A)d\nu(x), \quad \nu \in L^1$$

$$(2.3) \qquad (P^{[s,t]}f)(x) = \int P^{[s,t]}(x,dy)f(y), \quad f \in L^\infty.$$

It is clear that $\|P_*^{[s,t]}\nu\|_1 = \|\nu\|_1$ for every positive measure $\nu \in L^1$.

From (2.2) it follows that (2.1) can be rewritten as follows

$$P_*^{[s,t]} = P_*^{[h,t]} P_*^{[s,h]}$$

where $s \le h \le t$.

Recall that if for a NHCTMP $P^{[s,t]}(x,A)$ one has $P_*^{[s,t]} = \big(P_*^{[0,1]}\big)^{t-s}$, then such a process becomes *homogeneous* and, therefore, it is denoted by $P^t(x,A)$.

Definition 2.1. *A NHCTMP $P^{[s,t]}(x,A)$ is said to satisfy:*

(i) *the weak ergodicity if for any $s \in \mathbb{R}_+$ one has*

$$\lim_{n\to\infty} \sup_{x,y\in X} \|P^{[s,t]}(x,\cdot) - P^{[s,t]}(y,\cdot)\|_1 = 0;$$

(i) *the L_1-weak ergodicity if for any probability measures $\lambda, \nu \in \mathfrak{M}$ and $s \in \mathbb{R}_+$ one has*

$$\lim_{n\to\infty} \|P_*^{[s,t]}\lambda - P_*^{[s,t]}\nu\|_1 = 0;$$

(ii) *the L_1-strong ergodicity if there exists a probability measure μ_1 such that for every $s \in \mathbb{R}_+$ and $\lambda \in \mathfrak{M}$ one has*

$$\lim_{n\to\infty} \|P_*^{[s,t]}\lambda - \mu_1\|_1 = 0.$$

It is clear that the weak ergodicity implies the L_1-weak ergodicity. In the paper we will deal with L_1-weak ergodicity. Note that historically, one of the most significant conditions for the weak erodicity is the Doeblin's Condition (for homogeneous Markov processes), which is formulated as follows: there exist a probability measure ν, an integer $n_0 \in \mathbb{N}$ and constants $0 < \varepsilon < 1$, $\delta > 0$ such that for every $A \in \mathcal{F}$ if $\nu(A) > \varepsilon$ then

$$\inf_{x\in X} P^{n_0}(x,A) \ge \delta.$$

Such a condition does not imply either the aperiodicity or the weak ergodicity of the process. In [15] the aperiodicity is studied by minorization type conditions, i.e., there exist a non-trivial positive measure λ and $n_0 \in \mathbb{N}$ such that

$$P^{n_0}(x, A) \geq \lambda(A), \quad \forall x \in X, \ \forall A \in \mathcal{F}.$$

But this condition is not sufficient for the strong ergodicity. In [18] it was introduced a variation of the above condition, i.e., Condition (A_0): there exists a non-trivial positive measure $\mu_0 \in L^1$, $\|\mu_k\|_1 \neq 0$, and for every $\lambda \in \mathfrak{M}$ one can find a sequence $\{X_n\} \subset \mathcal{F}$ with $\mu(X \setminus X_n) \to 0$, as $n \to \infty$, and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has

$$P_*^n \lambda \geq \mu_0 1_{X_n},$$

where 1_A stands for the indicator function of a set A. It has been proved that such a condition is necessary and sufficient for the L_1-strong ergodicity of the homogeneous Markov process. In the present paper we shall introduce a simple variation of the condition (A_0) for NHCTMP, and prove that it is necessary and sufficient for the L_1-weak ergodicity. Note that another variation of the Deoblin's Condition has been studied in [2], which also provides a necessary and sufficient condition for the weak ergodicity.

3. Main results

In this section we introduce a simple variation of condition (A_0).

Definition 3.1. *We say that a NHCTMP $P^{[s,t]}(x, A)$ given on $(X, \mathcal{F}, \mu)$ satisfies condition (B) if for each $s \in \mathbb{R}_+$ there exist a positive measure $\mu_s \in L^1$, $\|\mu_s\|_1 \neq 0$, and for every $\delta > 0$ and $\lambda, \nu \in \mathfrak{M}$ one can find sets $X_s, Y_s \in \mathcal{F}$ with $\mu(X \setminus X_s) < \delta$, $\mu(X \setminus Y_s) < \delta$ and a number $t_s \geq 1$ such that*

$$P_*^{[s,s+t_s]}\lambda \geq \mu_s 1_{X_s}, \quad P_*^{[s,s+t_s]}\nu \geq \mu_s 1_{Y_s}. \tag{3.1}$$

Remark 3.2. *In* (3.1), (3.19) *without loss of generality we may assume that $\|\mu_s\|_1 < 1/2$, otherwise we will replace μ_s by $\mu'_s = \mu_s/2$.*

Before we formulate the main result, we need the following auxiliary lemma.

Lemma 3.3. *For every $\lambda, \nu \in \mathfrak{M}$ there exist $\lambda_1, \nu_1 \in \mathfrak{M}$ with $supp(\lambda_1) \cap supp(\nu_1) = \emptyset$ such that*

$$\lambda - \nu = \frac{\|\lambda - \nu\|_1}{2}(\lambda_1 - \nu_1). \tag{3.2}$$

The proof is obvious.

Now we are ready to formulate our main result.

Theorem 3.4. *Let $P^{[s,t]}(x, A)$ be a NHCTMP given on $(X, \mathcal{F}, \mu)$. Then the following assertions are equivalent:*

(i) *$P^{[s,t]}(x, A)$ satisfies condition (B) with*

$$\liminf_{s\to\infty} \|\mu_s\|_1 > 0; \tag{3.3}$$

(ii) *for each $s \in \mathbb{Z}_+$ and any $\lambda, \nu \in \mathfrak{M}$ there is a number $\gamma_s \in [0, 1)$ and $t_0 \geq 1$ such that*

$$\|P_*^{[s,s+t_0]}\lambda - P_*^{[s,s+t_0]}\nu\|_1 \leq \gamma_s\|\lambda - \nu\|_1; \tag{3.4}$$

where

$$\limsup_{s\to\infty} \gamma_s < 1; \tag{3.5}$$

(iii) *$P^{[s,t]}(x, A)$ satisfies the L_1-weak ergodicity.*

Proof. (i)⇒(ii). Take any $\lambda, \nu \in \mathfrak{M}$ and fix $s \in \mathbb{R}_+$. Then due to Lemma 3.4 one can find measures $\lambda_1, \nu_1 \in \mathfrak{M}$ such that (3.2) holds. For λ_1, ν_1 due to condition (B) one can find a measure μ_s. Then according to absolute continuity of Lebesgue integral, there is $\delta_1 > 0$ such that for any $Z \in \mathcal{F}$ with $\mu(Z) < 2\delta_1$ one has

$$\int \mu_s 1_Z d\mu < \frac{\|\mu_s\|_1}{2}. \tag{3.6}$$

Now due to condition (B) there are $X_1, Y_1 \subset \mathcal{F}$ and $t_1 \geq 1$ such that one has $\max\{\mu(X \setminus X_1), \mu(X \setminus Y_1)\} < \delta$ and

$$P_*^{[s,s+t_1]}\lambda_1 \geq \mu_s 1_{X_1}, \quad P_*^{[s,s+t_1]}\nu_1 \geq \mu_s 1_{Y_1}. \tag{3.7}$$

Denoting $Z_1 = X_1 \cap Y_1$, one gets $\mu(X \setminus Z_1) < 2\delta$, and from (3.7) we find

$$P_*^{[s,s+t_1]}\lambda_1 \geq \mu_s 1_{Z_1}, \quad P_*^{[s,s+t_1]}\nu_1 \geq \mu_s 1_{Z_1}. \tag{3.8}$$

It follows from (3.8) that

$$\begin{aligned} \|P_*^{[s,s+t_1]}\lambda_1 - \mu_s 1_{Z_1}\|_1 &= \int \big(P_*^{[s,s+t_1]}\lambda_1 - \mu_s 1_{Z_1}\big) d\mu \\ &= \int P_*^{[s,s+t_1]}\lambda_1 d\mu - \int \mu_0 1_{Z_{n,1}} d\mu \\ &= 1 - \int \mu_0 1_{Z_1} d\mu \\ &= \int P_*^{[s,s+t_1]}\nu_1 d\mu - \int \mu_0 1_{Z_1} d\mu \\ &= \|P_*^{[s,s+t_1]}\nu_1 - \mu_0 1_{Z_1}\|_1. \end{aligned} \tag{3.9}$$

Therefore, let us define

$$\gamma_s = \|P_*^{[s,s+t_1]}\lambda_1 - \mu_s 1_{Z_1}\|_1.$$

One can see that

$$1 - \int \mu_s 1_{Z_1} d\mu \geq 1 - \int \mu_s d\mu \geq \frac{1}{2}. \tag{3.10}$$

Due to $\mu(X \setminus Z_1) < 2\delta_1$ from (3.6) we have

$$\frac{1}{2}\int \mu_s d\mu \geq \int \mu_s 1_{X\setminus Z_1} d\mu = \int \mu_s d\mu - \int \mu_s 1_{Z_1} d\mu$$

which yields

$$\int \mu_s 1_{Z_1} d\mu \geq \frac{\|\mu_s\|_1}{2}.$$

Therefore,

$$1 - \int \mu_s 1_{Z_1} d\mu \leq 1 - \frac{\|\mu_s\|_1}{2}. \tag{3.11}$$

Hence, from (3.10),(3.11) we infer

$$\frac{1}{2} \leq \gamma_s \leq 1 - \frac{\|\mu_s\|_1}{2}.$$

This with (3.3) yields (3.5). Thus, we obtain

$$
\begin{aligned}
\|P_*^{[s,s+t_1]}\lambda_1 - P_*^{[s,s+t_1]}\nu_1\|_1 &= \|(P_*^{[s,s+t_1]}\lambda_1 - \mu_s 1_{Z_1}) \\
&\quad -(P_*^{[s,s+t_1]}\nu_1 - \mu_s 1_{Z_1})\|_1 \\
&= \gamma_s\|\lambda_2 - \nu_2\|_1, \qquad (3.12)
\end{aligned}
$$

where

$$
\lambda_2 = \frac{1}{\gamma_s}\big(P_*^{[s,s+t_1]}\lambda_1 - \mu_s 1_{Z_1}\big)
$$

$$
\nu_2 = \frac{1}{\gamma_s}\big(P_*^{[s,s+t_1]}\nu_1 - \mu_s 1_{Z_1}\big).
$$

It is clear that $\lambda_2, \nu_2 \in \mathfrak{M}$. Now from (3.2) and (3.12) one gets

$$
\begin{aligned}
\|P_*^{[s,s+t_1]}\lambda - P_*^{[s,s+t_1]}\nu\|_1 &= \frac{\|\lambda-\nu\|_1}{2}\|P_*^{[s,s+t_1]}\lambda_1 - P_*^{[s,s+t_1]}\nu_1\|_1 \\
&\le \gamma_s\|\lambda-\nu\|_1
\end{aligned}
$$

which implies the required assertion.

(ii)⇒(iii). Take any $\lambda, \nu \in \mathfrak{M}$ and fix $s \in \mathbb{R}_+$. Then due to condition (ii) one finds $t_0 \ge 1$ and $\gamma_s \in [0,1)$ such that one has

$$
\|P_*^{[s,s+t_0]}\lambda - P_*^{[s,s+t_0]}\nu\|_1 \le \gamma_s\|\lambda-\nu\|_1. \qquad (3.13)
$$

Let us prove by induction that there are numbers $\{t_i\}_{i=0}^{\ell}$ ($t_i \ge 1$) such that for any $\ell \in \mathbb{N}$ one has

$$
\|P_*^{[s,K_\ell]}\lambda - P_*^{[s,K_\ell]}\nu\|_1 \le \left(\prod_{i=0}^{\ell}\gamma_{s+i}\right)\|\lambda-\nu\|_1, \qquad (3.14)
$$

where $K_\ell := s + \sum\limits_{i=0}^{\ell} t_i$, $\gamma_{s+i} \in [0,1)$, $i = 0,1,\dots,\ell$.

We have proved (3.14) at $\ell = 0$. Now assume that (3.14) holds at $i = \ell$.

Let us prove (3.14) at $i = \ell + 1$. Denote

$$
\lambda_\ell = P_*^{[s,K_\ell]}\lambda, \quad \nu_\ell = P_*^{[s,K_\ell]}\nu.
$$

Now from (ii) for λ_ℓ, ν_ℓ and K_ℓ one finds $t_{\ell+1} \geq 1$ and $\gamma_{s+\ell+1}$ such that

$$\|P_*^{[K_\ell, K_\ell + t_{\ell+1}]}\lambda_\ell - P_*^{[K_\ell, K_\ell+t_{\ell+1}]}\nu_\ell\|_1 \leq \gamma_{s+\ell+1}\|\lambda_\ell - \nu_\ell\|_1. \tag{3.15}$$

Denoting $K_{\ell+1} = K_\ell + t_{\ell+1}$ with (3.15),(3.14) we get

$$\begin{aligned} \|P_*^{[s,K_{\ell+1}]}\lambda - P_*^{[s,K_{\ell+1}]}\nu\|_1 &= \big\|P_*^{[K_\ell,K_{\ell+1}]}\big(\lambda_\ell - \nu_\ell\big)\big\|_1 \\ &\leq \gamma_{s+\ell+1}|P_*^{[s,K_\ell]}\lambda - P_*^{[s,K_\ell]}\nu\|_1 \\ &\leq \left(\prod_{i=0}^{\ell+1}\gamma_{s+i}\right)\|\lambda - \nu\|_1. \end{aligned}$$

Hence, (3.14) is valid for all $\ell \in \mathbb{N}$.

Take any $t > s$, then one can find $m \in \mathbb{N}$ such that

$$t = K_m + r, \quad 0 \leq r < n_{m+1}$$

Now due to (3.14) we obtain

$$\begin{aligned} \|P_*^{[s,t]}\lambda - P_*^{[s,t]}\nu\|_1 &= \big\|P_*^{[K_m,t]}\big(P_*^{[s,K_m]}\lambda - P_*^{[s,K_m]}\nu\big)\big\|_1 \\ &\leq \|P_*^{[s,K_m]}\lambda - P_*^{[s,K_m]}\nu\|_1 \leq 2\prod_{i=0}^{m}\gamma_{s+i} \end{aligned} \tag{3.16}$$

According to (3.5) one can find $m \in \mathbb{N}$ such that $\prod_{j=0}^{m}\gamma_{s+i} < \varepsilon/2$. Then it follows from (3.16) that

$$\|P_*^{[s,t]}\lambda - P_*^{[s,t]}\nu\|_1 < \varepsilon \quad \text{for all} \ \ t \geq K_m$$

which implies the L^1-weak ergodicity.

Now, we consider the implication (iii)$\Rightarrow$ (i). Fix $1 > \varepsilon > 0$. Then given $s \in \mathbb{R}_+$ and $\lambda, \mu_0 \in \mathfrak{M}$, (here μ_0 is fixed) one has

$$\|P_*^{[s,t]}\lambda - P_*^{[s,t]}\mu_0\|_1 \to 0 \ \text{ as } \ n \to \infty.$$

Take any increasing sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$. Then one can find a sequence $\{Y_n\} \subset \mathcal{F}$ such that $\mu(X \setminus Y_n) \to 0$, as $n \to \infty$, and

$$\|(P_*^{[s,t_n]}\lambda - P_*^{[s,t_n]}\mu_0)1_{Y_n}\|_\infty \to 0 \ \text{ as } \ n \to \infty.$$

Therefore, there exists an $n_k \in \mathbb{N}$ such that $\mu(X \setminus Y_{t_{n_k}}) < \varepsilon$ and

$$\|(P_*^{[s,s+t_{n_k}]}\lambda - P_*^{[s,s+t_{n_k}]}\mu_0)1_{Y_{n_k}}\|_\infty < \frac{\varepsilon}{2} \tag{3.17}$$

Let $\nu_s = P_*^{[s,s+t_{n_k}]}\mu_0$. Hence, from (3.17) we get

$$\begin{aligned} P_*^{[s,s+t_{n_k}]}\lambda &\geq P_*^{[s,s+t_{n_k}]}\lambda 1_{Y_{n_k}} \\ &\geq \nu_s 1_{Y_{n_k}} - \frac{\varepsilon}{2}1_{Y_{n_k}} \\ &\geq \mu_s 1_{Y_{n_k}}, \end{aligned}$$

where

$$\mu_s = \frac{1}{2}\nu_s 1_{A_k}, \quad A_s = \left\{x \in X : \ \nu_s(x) \geq \frac{\varepsilon}{2}\right\}.$$

Since ν_s is a probability measure, therefore, we have $0 < \|\mu_s\|_1 \leq 1/2$, so

$$1 - \frac{\|\mu_s\|_1}{2} \geq \frac{3}{4}.$$

This completes the proof. □

From the proof of the previous theorem we can estimate the rate of convergence whenever the process satisfies condition (B). Namely, one has the following

Corollary 3.5. *Let $P^{[s,t]}(x,A)$ be a NHCTMP given on $(X, \mathcal{F}, \mu)$. Assume that $P^{[s,t]}(x,A)$ satisfies condition (B) with*

$$\alpha := \liminf_{s\to\infty} \|\mu_s\|_1 > 0.$$

Then for each $s \in \mathbb{R}_+$ and $\lambda, n \in \mathfrak{M}$ one can find $N(s,\lambda,\nu) \in \mathbb{R}_+$ such that

$$\|P_*^{[s,t]}\lambda - P_*^{[s,t]}\nu\|_1 \leq C\left(1 - \frac{\alpha}{2}\right)^{(t-s)/N(s,\lambda,\nu)} \|\lambda - \nu\|_1, \tag{3.18}$$

where C is a some constant.

Moreover, the last inequality is equivalent to the L_1-weak ergodicity.

Proof. The proof immediately follows from the estimation (3.16). □

We note that if the number $N(s,\lambda,\nu)$ in (3.18) does not depend on λ and ν, then the process satisfies the weak ergodicity.

Now, let us turn to a nonhomogeneous version of the condition (A_0). Namely, we say that a NHCTMP $P^{[s,t]}(x,A)$ given on $(X,\mathcal{F},\mu)$ satisfies *condition (A)* if for each $s\in\mathbb{R}_+$ there exists a positive measure $\mu_s\in L^1$, $\|\mu_s\|_1\neq 0$, and for every $\lambda\in\mathfrak{M}$ one can find a sequence $\{X_n^{(s)}\}\subset\mathcal{F}$ with $\mu(X\setminus X_t^{(s)})\to 0$, as $t\to\infty$, and $t_0(\lambda,k)\geq 1$ such that for all $t\geq t_0(\lambda,s)$ one has

$$P_*^{[s,t]}\lambda\geq\mu_s 1_{X_t^{(s)}}. \tag{3.19}$$

It is easy to see that condition (A) implies condition (B), hence by Theorem 3.4 we immediately conclude that condition (A) with (3.3) is sufficient for the L_1-weak ergodicity. On the other hand, if one looks at the homogeneous Markov process which satisfies the L_1-weak ergodicity, then in [17] it has been proved that such process also satisfies condition (A). In another words, for homogeneous Markov process condition (A) is necessary and sufficient for the L_1-weak ergodicity. Therefore, we can formulate the following:

Problem. Is condition (A) with (3.3) necessary for the L_1-weak ergodicity of NHCTMP?

4. **Applications**

In this section we apply of condition (B) to a couple of concrete cases.

4.1. **Discrete case.** Let us consider a countable state space NHCTMP. Namely, let $X=\mathbb{N}$ and μ be the Poisson measure. Then the process can be given in form of stochastic matrices $\{p_{i,j}^{[s,t]}\}_{i,j\in\mathbb{N}}$.

Theorem 4.1. *Let $\{p_{i,j}^{[s,t]}\}_{i,j\in\mathbb{N}}$ be a NHCTMP. If there exists a function $\lambda(s)$, $s\in[1,\infty)$ $(l(s)\in[0,1))$ satisfying*

$$\liminf_{s\to\infty}\lambda(s)>0 \tag{4.1}$$

and such that for some sequence of states $\{n_s\}$

$$p_{i,n_s}^{[s-1,s]}\geq\lambda(s)\quad\text{for all}\quad i\in\mathbb{N},\ s\geq 1, \tag{4.2}$$

then the process satisfies the L_1-weak ergodicity.

Proof. We now show that the process satisfies condition (B). For each $s \in \mathbb{R}_+$ we first define a measure $\mu^{(s)}$ on X as follows:

$$\mu_i^{(s)} = \begin{cases} \lambda(s), & i = n_s \\ 0, & i \neq n_s \end{cases}$$

It is clear that $\liminf_{s\to\infty} \|\mu^{(s)}\|_1 0$. From (4.2) it follows that

$$p_{i,j}^{[s-1,s]} \geq \mu_j^{(s)}, \quad \text{for all} \quad i, j \in \mathbb{N}. \tag{4.3}$$

For any given $\nu \in \mathfrak{M}$ and $s \in \mathbb{R}_+$ we take $X_s = X$, then from (4.3) one finds

$$P_*^{[s-1,s]}\nu \geq \mu^{(s)} \quad \text{for all} \quad s \geq 1.$$

Hence, condition (B) is satisfied. So, taking into account (4.1), from Theorem 3.4 we get the desired assertion. □

We observe that the proved theorem extends results of [4, 16].

4.2. **Continuous case.** Let $(X, \mathcal{F}, \mu)$ be a probability space and $P^{[s,t]}(x, A)$ be a nonhomogeneous Markov process on this space.

Theorem 4.2. *Let $P^{[s,t]}(x, A)$ be a NHCTMP on $(X, \mathcal{F}, \mu)$. If for every $s \in \mathbb{R}_+$ there exists a set $A_s \in \mathcal{F}$ and a function $0 \leq \alpha(s) < 1$ such that*

$$P^{[s-1,s]}(x, A_s) \geq \alpha(s) \text{ for all } x \in X, \ s \geq 1, \tag{4.4}$$

where

$$\liminf_{s\to\infty} \alpha(s) > 0, \tag{4.5}$$

then the process satisfies the L_1-weak ergodicity.

Proof. To prove the statement it is enough to establish that the process satisfies condition (B). Indeed, for each $s \in \mathbb{R}_+$ let us define

$$\nu_s(A) = \bigwedge_{x\in X} P^{[s-1,s]}(x, A \cap A_s), \quad A \in \mathcal{F}$$

Due to Theorem IV.7.5 [3] the defined mapping ν_s is a measure on X, and moreover, one has $\nu_s(A_s) \geq \alpha(s)$. Now let us put

$$\mu_s(A) = \frac{\nu_s(A \cap A_s)}{\nu_k(A_s)}, \quad A \in \mathcal{F}.$$

Then one can see that

$$P_*^{[s-1,s]}\delta_x \geq \alpha(s)\mu_s \text{ for all } x \in X, s \geq 1. \tag{4.6}$$

It is clear that $\liminf_{s\to\infty} \|\mu_s\|_1 > 0$. Now given any $\lambda \in \mathfrak{M}$ and each $s \in \mathbb{R}_+$ we put $X_s = X$, then using the standard density argument from (4.6) we obtain

$$P_*^{[s-1,s]}\lambda \geq \alpha(s)\mu_s. \tag{4.7}$$

Hence, condition (B) is satisfied. So, taking into account (4.11), from Theorem 3.4 we get the desired assertion.

Define

$$\gamma = \limsup_{s\to\infty}(1 - \|\mu_s\|_1/2).$$

It is clear that $0 < \gamma < 1$. From the proof of Theorem 3.4 we conclude that

$$\|P_*^{[s,t]}\lambda - P_*^{[s,t]}\mu\|_1 \leq 2\gamma^{m+1},$$

where $t = s + m + r$.

Hence, one gets

$$\|P_*^{[s,t]}\lambda - P_*^{[s,t]}\mu\|_1 \leq 2\gamma^{t-s},$$

for any $\lambda, \mu \in \mathfrak{M}$. □

4.3. **Quadratic stochastic processes.** In this section we apply the obtained results to quadratic stochastic processes. Note that such kind of processes are related to quadratic operators as well as Markov processes with linear operators (see [5] for review).

Let $(X, \mathcal{F}, \mu)$ be a probability space. We recall that a family of functions $\{Q^{[s,t]}(x, y, A)\}$ defined for $s + 1 \leq t$ $(s, t \in \mathbb{R}_+)$ for all $x, y \in X$, $A \in \mathcal{F}$, is called *quadratic stochastic process (QSP)* if the following conditions are satisfied:

(i) $Q^{[s,t]}(x, y, A) = Q^{[s,t]}(y, x, A)$ for any $x, y \in X$ and $A \in \mathcal{F}$;

(ii) $Q^{[s,t]}(x, y, \cdot) \in \mathfrak{M}$ for any fixed $x, y \in X$;

(iii) $Q^{[s,t]}(x, y, A)$ as a function of x and y is measurable on $(X \times X, \mathcal{F} \otimes \mathcal{F})$ for any $A \in \mathcal{F}$;

(iv) (Analogue of the Chapman-Kolmogorov equation) for the initial measure $\mu \in \mathfrak{M}$ and arbitrary $s < \tau < t$ with $\tau - s \geq 1$, $t - \tau \geq 1$ we have either (iv)$_A$

$$Q^{[s,t]}(x,y,A)=\int_X\int_X Q^{[s,\tau]}(x,y,du)Q^{[\tau,t]}(u,v,A)\mu_\tau(dv),$$

where measure μ_τ on $(X,\mathcal{F})$ is defined by

$$\mu_\tau(B)=\int_X\int_X Q^{[0,\tau]}(x,y,B)\mu(dx)\mu(dy),$$

for any $B\in\mathcal{F}$, or

(iv)$_B$

$$Q^{[s,t]}(x,y,A) = \int_X\int_X\int_X\int_X Q^{[s,\tau]}(x,z,du)Q^{[s,\tau]}(y,v,dw)Q^{[\tau,t]}(u,w,A)\mu_s(dz)\mu_s(dw).$$

If the condition $(iv)_A$ (respectively $(iv)_B$) holds, then QSP is called of *type (A)* (respectively (B)).

The process $Q^{[s,t]}(x,y,A)$ can be interpreted as the probability of the following event: if x and y in X interact at time s, then one of the elements of the set $A\in\mathcal{F}$ will be realized at time t. All phenomena in physics, chemistry, and biology develop along non-zero finite time intervals. Therefore, we assume that the maximum of these values of time is equal to 1. Hence, $Q^{[s,t]}(x,y,A)$ is defined for $t-s\geq 1$ (we refer the reader to [5] for more information).

By $\mathfrak{M}^2$ we denote the set of all probability measures on $X\times X$ which are absolutely continuous w.r.t. $\mu\otimes\mu$, i.e., $\mathfrak{M}^2$ can be considered as a subset of $L^1(X\times X,\mathcal{F}\otimes\mathcal{F},\mu\otimes\mu)$. Given QSP $Q^{[s,t]}(x,y,A)$ one can define

$$(Q_*^{[s,t]}\tilde{\nu})(A)=\int_X\int_X Q^{[s,t]}(x,y,A)d\tilde{\nu}(x,y),\tilde{\nu}\in L^1(X\times X,\mu\otimes\mu). \tag{4.8}$$

We recall that a QSP $Q^{[s,t]}(x,y,A)$ is said to satisfy the L_1-*weak ergodicity* (or *ergodic principle*) if for any probability measures $\tilde{\lambda},\tilde{\nu}\in\mathfrak{M}^2$ and $s\in\mathbb{R}_+$ one has

$$\lim_{t\to\infty}\|Q_*^{[s,t]}\tilde{\lambda}-Q_*^{[s,t]}\tilde{\nu}\|_1=0;$$

Let $Q^{[s,t]}(x,y,A)$ be a given QSP. Now define the following transition probability

$$P_Q^{[s,t]}(x,A) = \int_X Q^{[s,t]}(x,y,A)d\mu_s(y). \tag{4.9}$$

In [14] it has been proved the following

Theorem 4.3. *Let $Q^{[s,t]}(x,y,A)$ be a given QSP on $(X,\mathcal{F},\mu)$. Then the following statements hold true:*

(i) *the defined $P_Q^{[s,t]}(x,A)$ is a NHCTMP on $(X,\mathcal{F},\mu)$;*
(ii) *the process $P_Q^{[s,t]}(x,A)$ satisfies the L_1-weak ergodicity if and only if $Q^{[s,t]}(x,y,A)$ satisfies the L_1-weak ergodicity.*

This theorem allows us to prove the following result.

Theorem 4.4. *Let $Q^{[s,t]}(x,y,A)$ be a given QSP on $(X,\mathcal{F},\mu)$. If for every $s \in \mathbb{R}_+$ there exists a set $A_s \in \mathcal{F}$ and a function $0 \le \alpha(s) < 1$ such that*

$$Q^{[s-1,s]}(x,y,A_s) \ge \alpha(s) \text{ for all } x,y \in X,\ s \ge 1, \tag{4.10}$$

where

$$\liminf_{s\to\infty} \alpha(s) > 0, \tag{4.11}$$

then the QSP is the L_1-weak ergodic.

Proof. Consider the process $P_Q^{[s,t]}(x,A)$. Then from (4.9) and (4.3) one finds

$$P_Q^{[s-1,s]}(x,A_s) = \int_X Q^{[s-1,s]}(x,y,A_s)d\mu_s(y) \ge \alpha(s) \text{ for all } x \in X,\ s \ge 1.$$

Hence, the Markov process $P_Q^{[s,t]}(x,A)$ satisfies the conditions of Theorem 4.2, so it is weak ergodic. Therefore, Theorem 4.3 implies that QSP $Q^{[s,t]}(x,y,A)$ satisfies the L_1-weak ergodicity. □

Note that the last theorem improves the result of [17].

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