Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 5, pp. 1243–1262
Online ISSN: 1735-8515

ON FINITE X-DECOMPOSABLE GROUPS FOR $X = \{1, 2, 3, 4\}$

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(Communicated by Ali Reza Ashrafi)

Abstract. Let $\mathcal{N}_G$ denote the set of all proper normal subgroups of a group G and A be an element of $\mathcal{N}_G$. We use the notation $ncc(A)$ to denote the number of distinct G-conjugacy classes contained in A and also $\mathcal{K}_G$ for the set $\{ncc(A) \mid A \in \mathcal{N}_G\}$. Let X be a non-empty set of positive integers. A group G is said to be X-decomposable, if $\mathcal{K}_G = X$. In this paper we give a classification of finite X-decomposable groups for $X = \{1, 2, 3, 4\}$.

Keywords: n-decomposable, X-decomposable, G-conjugacy classes.
MSC(2010): Primary: 20D10; Secondary: 20D20.

1. Introduction

All groups in this paper are finite. The relation between the structure of a group and the cardinality of its conjugacy classes has already been extensively studied (see, e.g., [7–9, 12, 19]). Let G be a group and N be a normal subgroup of G. Then N is a union of G-conjugacy classes contained in N, and some authors hope to investigate the structure of a normal subgroup if it is a union of a *small* number of G-conjugacy classes (see, e.g., [1, 13, 16]). Furthermore, some authors hope to determine the structure of a group if every non-trivial normal subgroup is a union of a given number of G-conjugacy classes (see, e.g., [2, 3, 10]).

Article electronically published on October 27, 2014.
Received: 28 March 2013, Accepted: 16 September 2013.
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Let n be a positive integer. Recall that a normal subgroup N of a group G is called n-decomposable if it is a union of n distinct G-conjugacy classes, and a group G is called an n-decomposable group if it is not simple and its every non-trivial normal subgroup is n-decomposable. Up to now, 2-, 3-, 7-, 8-, 9- and 10-decomposable normal subgroups have been investigated (see [5, 6, 17] and [18]) and the authors in [2] give some properties for finite n-decomposable groups. Furthermore, they classify finite n-decomposable groups for $n = 2, 3, 4$ in the same paper.

Let G be a group. For convenience, we use $\mathcal{N}_G$ to denote the set of all proper normal subgroups of G. If A is an element of $\mathcal{N}_G$, then we use $ncc(A)$ to denote the number of distinct G-conjugacy classes contained in A. Furthermore, suppose that X is a non-empty set of positive integers and $\mathcal{K}_G = \{ncc(A) \mid A \in \mathcal{N}_G\}$. A group G is said to be X-decomposable if $\mathcal{K}_G = X$. A. R. Ashrafi in [3] raised the following question:

Question. [3, Question 2.7] *Suppose that X is a finite subset of positive integers containing 1. Is there a finite X-decomposable group G?*

Now X-decomposable groups have been classified for $X = \{1, 2, 3\}$, $\{1, 3, 4\}$ and $\{1, 2, 4\}$. They are as follows:

Theorem A. [4] *Let G be a finite non-perfect $\{1, 2, 3\}$-decomposable group. Then G is isomorphic to Z_6, D_8, Q_8, S_4, SmallGroup(20, 3) or SmallGroup(24, 3).*

Theorem B. [3] *Let G be a finite non-perfect $\{1, 3, 4\}$-decomposable group. Then G is isomorphic to SmallGroup(36, 9), a metabelian group of order $2^n(2^{\frac{n-1}{2}} - 1)$, in which n is an odd positive integer and $2^{\frac{n-1}{2}} - 1$ is a Mersenne prime or a metabelian group of order $2^n(2^{\frac{n}{3}} - 1)$ where $3|n$ and $2^{\frac{n}{3}} - 1$ is a Mersenne prime.*

Theorem C. [10] *Let G be a finite non-perfect $\{1, 2, 4\}$-decomposable group. Then G is isomorphic to Q_{12}, $Z_2 \times A_4$ or $G = \langle a, b, c \mid a^{11} = b^5 = c^2 = 1, b^{-1}ab = a^4, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle$.*

We note here that SmallGroup(n, i) in Theorem A and Theorem B is the i^{th} group of order n in the small group library of GAP (see [15]).

In this paper, we continue to study the above question for the case $X = \{1, 2, 3, 4\}$ and give the classification of non-perfect $\{1, 2, 3, 4\}$-decomposable groups. Our main result is as follows.

Main Theorem. *Let G be a finite non-perfect $\{1, 2, 3, 4\}$-decomposable group. Then G is one of the following groups:*

(1) $|G| = 216$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^3 = f^3 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd, e^a = f^2, e^b = e^2f, e^c = f^2, e^d = e^2, f^a = ef^2, f^b = ef, f^c = e, f^d = f^2\rangle$.

(2) $|G| = 600$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^5 = f^5 = 1, b^2 = c^2 = d, b^a = bc, c^a = b, c^b = cd, e^a = ef^3, e^b = e^3f^3, e^c = e^3, e^d = e^4, f^a = e^4f^3, f^b = f^2, f^c = e^4f^2, f^d = f^4\rangle$.

(3) $|G| = 42$ *and* $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^5\rangle$.

(4) $G = D_{12}$.

Let G be a finite group. Throughout this paper, $G', \Phi(G), Z(G)$ and $exp(G)$ denotes the derived subgroup, the Frattini subgroup, the center and the exponent of G, respectively. A group G is said to be non-perfect if $G' \neq G$. If x is an element in G, then $x^G = \{x^g \mid g \in G\}$ is the G-conjugacy class containing x. Furthermore, Z_n denotes the cyclic group of order n, $E(p^n)$ denotes the elementary abelian group of order p^n and $d(n)$ denotes the set of all positive divisors of n. We always assume that $X = \{1, 2, 3, 4\}$ in the next two sections.

2. Preliminaries

In this section, we list some fundamental facts which are useful in the sequel.

Example 2.1. [3, Example 2.5] *Let* $G = \langle a, b \mid a^6 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ *be the dihedral group of order 12. Then* $\mathcal{N}_G = \{1, H = \langle a^2, b\rangle, K = \langle a^2, ab\rangle, \langle a\rangle, \langle a^2\rangle, \langle a^3\rangle\}$. *It is easy to see that* $\langle a^2\rangle$ *and* $\langle a^3\rangle$ *are 2-decomposable,* H *and* K *are 3-decomposable and* $\langle a\rangle$ *is 4-decomposable. Therefore,* G *is* X*-decomposable.*

Lemma 2.2. [10, Example 2.1] *Let* G *be an abelian group of order* n *and* $Y = d(n) - \{n\}$. *Then* G *is* Y*-decomposable.*

Corollary 2.3. *There is no finite abelian* X*-decomposable group.*

Lemma 2.4. *There is no finite* X*-decomposable group of prime power order.*

Proof. Suppose that there is a prime p such that G is a p-group. Then $p = 2$ by [17, Theorem 1(3)]. Assume that $|G| = 2^n$ for some integer n. There is a chief series

$$1 = G_0 < G_1 < \cdots < G_{n-1} < G_n = G$$

in G such that $|G_i| = 2^i$ for $i = 1, 2, \cdots, n$. As G is X-decomposable, we have $n = 4$.

Since $Z(G) \neq 1$ and G is non-abelian by Corollary 2.3, $Z(G)$ can not be 4-decomposable in G. Furthermore, if $Z(G)$ is 3-decomposable in G, then $|Z(G)| = 3$, contradicting that G is a 2-group. Therefore, $Z(G)$ is 2-decomposable in G, and thus $|Z(G)| = 2$. Let H be a 3-decomposable normal subgroup of G. As $Z(G) \cap H \neq 1$ and $Z(G)$ is a minimal subgroup of G, we have that $Z(G) < H$ and $|H| = 4$. Suppose that $H = Z(G) \cup x^G$. Then $|x^G| = 2$ and thus $|C_G(x)| = 8$. So $C_G(x)$ is normal in G. Since $\langle Z(G), x\rangle \leq Z(C_G(x))$, $C_G(x)$ is abelian. Therefore,

$$1 < Z(G) < H < C_G(x) < G$$

is a chief series of G. Let $C_G(x) = H \cup y^G$. Then $|y^G| = |C_G(x)| - |H| = 4$, and thus $|C_G(y)| = 4$, which contradicts the fact that $C_G(x)$ is abelian and $y \in C_G(x)$. □

Lemma 2.5. *Let G be a finite X-decomposable group such that G' is a Sylow 2-subgroup of G. Suppose that G' is 4-decomposable in G and that $Z(G) = Z(G')$ is of order 2. Then $Z(G)$ is contained in every non-trivial normal subgroup of G.*

Proof. As G' is a Sylow 2-subgroup of G, then it is solvable.

Let N be a non-trivial proper normal subgroup of G. We claim that $N \leq G'$. In fact, by the hypothesis, one can see that G' is a maximal subgroup of G. If $N \nleq G'$, then $G = G'N$ and thus $(G/N)' = G/N$, which contradicts that G/N is solvable.

Now, it is easy to see that $Z(G) = Z(G') \leq N$ since $N \cap Z(G') \neq 1$ and $|Z(G')| = 2$.

□

Lemma 2.6. [10, Lemma 2.1] *Suppose that p and q are primes and n is a positive integer such that $p^n = 1 + 3q$. Then $p = 7, n = 1$, and $q = 2$ or $p = 2, n = 4$, and $q = 5$.*

Lemma 2.7. *If n is a positive integer and $n \geq 2$, then there is no odd prime q such that $q^2 = 2^n - 1$.*

Proof. Suppose that there exists an odd prime q such that $q^2 = 2^n - 1$. Then there exist positive integers l and t such that $q = 2^l \cdot t + 1$. Therefore, $2^n = q^2 + 1 = (2^l \cdot t + 1)^2 + 1 = 2^{2l} \cdot t^2 + 2^{l+1} \cdot t + 2$. It follows that $2^{2l-1} \cdot t^2 + 2^l \cdot t + 1 = 2^{n-1}$, which is a contradiction.

□

Lemma 2.8. [10, Lemma 2.2] *There is no prime p such that $2p + 1$ is also a prime and that $2p^2 + p + 1 = 2^n$ for some positive integer n.*

3. Proof of the main theorem

In this section, we will give the proof of our main theorem. We have shown in Corollary 2.3 and Lemma 2.4 that G is neither an abelian group nor a group of prime power order if G is an X-decomposable group. Also an X-decomposable group must be of even order by [17, Theorem 1(3)], and we will use these facts frequently in the proofs.

We first give the following three theorems.

Theorem 3.1. *Let G be a finite non-perfect X-decomposable group. If G' is 4-decomposable in G, then G is one of the following two groups:*

(1) $|G| = 216$ and $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^3 = f^3 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd, e^a = f^2, e^b = e^2 f, e^c = f^2, e^d = e^2, f^a = ef^2, f^b = ef, f^c = e, f^d = f^2\rangle$.

(2) $|G| = 600$ and $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^5 = f^5 = 1, b^2 = c^2 = d, b^a = bc, c^a = b, c^b = cd, e^a = ef^3, e^b = e^3 f^3, e^c = e^3, e^d = e^4, f^a = e^4 f^3, f^b = f^2, f^c = e^4 f^2, f^d = f^4\rangle$.

Proof. Since G' is 4-decomposable in G, G' must be one of the following groups by [13, Theorem 1] and [13, Theorem 2]:

1) $G' \cong A_5$, the alternating group of degree 5, and $G/C_G(G') \cong S_5$.

2) G' is a p-group for some prime p and $G''' = 1$.

3) G' is a group of order $p^n q^b$, where p and q are distinct primes, and n and b are positive integers.

Furthermore, if G' is of type 3), then G' has the following three possibilities.

(A) G' is the direct product of its elementary abelian Sylow p- and q-subgroups.

(B) G' is a Frobenius group with kernel N and $G'/N \cong Z_q$ or Z_{q^2} or Q_8, where N is 2-decomposable in G.

(C) G' is a Frobenius group with kernel N and $G'/N \cong Z_q$, where N is 3-decomposable in G.

Case 1. $G' \cong A_5$ and $G/C_G(G') \cong S_5$.

As A_5 is centerless, $G' \cap C_G(G') = Z(G') \cong Z(A_5) = 1$. If $C_G(G') \neq 1$, then $G = G'C_G(G')$ since G' is a maximal normal subgroup of G. Therefore, $S_5 \cong G/C_G(G') \cong G' \cong A_5$, which is a contradiction. If $C_G(G') = 1$, then $G \cong S_5$. However, S_5 does not contain a 2-decomposable normal subgroup. Hence this case is impossible.

Case 2. G' is a p-group for some prime p and $G''' = 1$.

Assume that $|G'| = p^n$ for some positive integer n. As G' is a maximal subgroup of G and G is not of prime power order, there is a prime $q \neq p$ such that $|G| = p^n q$. It follows that G is solvable. Arguing similarly as in the proof of Lemma 2.5, we see that every proper normal subgroup of G is contained in G'. Recall that G is non-abelian, $Z(G)$ can not be maximal in G, so we conclude that $Z(G) < G'$.

If G' is abelian, then $G' \leq C_G(x)$ for every $x \in G'$, and so $|x^G| = 1$ or q. It follows that $p^n = 1+1+1+q$ or $p^n = 1+1+q+q$ or $p^n = 1+q+q+q$. If $p^n = 1+1+1+q$, then $|Z(G)| = 3$ and $p = 3$. Therefore, $q = 2$ as G is of even order, and thus $3^n = 5$, which is impossible. If $p^n = 1+1+q+q$, then $|Z(G)| = 2$ and so $p = 2$. Let Q be a Sylow q-subgroup of G. Then Q acts on the abelian group G', and thus $G' = Z(G) \times [G', Q]$. It is clear that $[G', Q]Q$ is a normal subgroup of G and so $G = Z(G) \times [G', Q]Q$. It follows that $G' = [G', Q]$, leading that $Z(G) = 1$, which is a contradiction. If $p^n = 1+q+q+q$, then $p = 7, n = 1$, and $q = 2$ or $p = 2, n = 4$, and $q = 5$ by Lemma 2.6. If $p = 7, n = 1$ and $q = 2$, then $|G| = 14$. It is easy to see that G has no 2-decomposable normal subgroup. If $p = 2, n = 4$ and $q = 5$, then we can choose H to be a 2-decomposable normal subgroup of G. As $H \leq G'$, we may assume that $|H| = 2^t$, then $2^t = 1+q = 6$, which is impossible. Consequently, we conclude that G' is non-abelian.

If $1 < G'' < Z(G') < G'$, then there exist positive integers $1 < s < t$ such that $|G''| = p^s$ and $|Z(G')| = p^t$. Let $Z(G') = G'' \cup x^G$. Then $G' = C_G(x)$, and thus $|x^G| = q$. It follows that $p^t = p^s + q$, which gives the contradiction that p divides q.

If $1 < Z(G') < G'' < G'$, then $G'' = \Phi(G')$ and there exist positive integers $1 < s < t$ such that $|Z(G')| = p^s$ and $|G''| = p^t$. Let $Z(G') = 1 \cup x^G, G'' = Z(G') \cup y^G, G' = G'' \cup z^G$. Assume that $Z(G) = 1$. If $p = 2$, then $|x^G| = 2^s - 1 = q$ and $|z^G| = 2^t(2^{n-t} - 1)$. As G' is non-abelian and $|G'/\Phi(G')| = 2^{n-t}$, we have $n - t > 1$. Therefore, $q = 2^{n-t} - 1$ and $n - t = s$. It follows that $|C_G(z)| = 2^{n-t} = 2^s$, which is contrary to that $Z(G') < \langle Z(G'), z \rangle \leq C_G(z)$. Hence $q = 2$. Then $p^s = 1 + |x^G| = 3$, $|y^G| = 3(3^{t-1} - 1)$ and $|z^G| = 3^t(3^{n-t} - 1)$. Therefore, $t = 2, n = 3$ and $|G| = 54$. Recall that $G' = G'' \cup z^G$, then $|C_G(z)| = 3$, which contradicts $Z(G') \neq 1$. So $Z(G) \neq 1$. Note that $Z(G) \leq Z(G')$, implies $Z(G) = Z(G')$ is 2-decomposable in G, and thus $Z(G) = Z(G')$ is of order 2. Now consider the factor group $\overline{G} = G/Z(G)$. It is easy to see that $\overline{G}$ is $\{1, 2, 3\}$-decomposable. Then $|G| = 40$ or 48 by Theorem A, Corollary 2.3 and Lemma 2.4. First, suppose that

$|G| = 40$, then $\Phi(G') = G''$. It follows that $|G'/\Phi(G')| = 2$, which implies the contradiction that G' is abelian. Suppose that $|G| = 48$. By arguing similarly as in the former case, we see that $|G'/G''| \neq 2$, hence $|G'/G''| = 4$. Suppose that $G'' = Z(G') \cup u^G$. Then $|u^G| = 2$, and thus $C_G(u) \trianglelefteq G$. It follows that $G' \leq C_G(u)$, which contradicts that $u \notin Z(G')$.

From the above two paragraphs we conclude that if $Z(G') \neq G''$, then $Z(G')G''$ is a 4-decomposable normal subgroup of G, and thus $G' = Z(G')G''$, leading the contradiction that $G'' = G''' = 1$. Hence $Z(G') = G''$.

If $|Z(G)| = 3$, then $p = 3, q = 2$ and $Z(G) = Z(G')$. Let T be a 2-decomposable normal subgroup of G. Keeping in mind that $T \leq G'$, we have $T \cap Z(G') \neq 1$. However, as we have seen that $|Z(G')| = 3$, then $Z(G') \leq T$, which is a contradiction.

If $|Z(G)| = 2$, then $p = 2$. If $Z(G) \neq Z(G')$, then $Z(G') = G''$ is 3-decomposable in G. Set $|Z(G')| = 2^s$. Then $2 + q = 2^s$ and therefore $q = 2 = p$, which is a contradiction. So $Z(G) = Z(G') = G''$ and thus $Z(G') = G'' = \Phi(G')$. Therefore, G' is an extraspecial 2-group and $|G'| = 2^n = 2^{2m+1}$ for some positive integer m. Consider the factor group $\overline{G} = G/Z(G)$. Then $\overline{G}$ is $\{1, 2, 3\}$-decomposable and $|G| = 40$ or 48 by Theorem A, Corollary 2.3 and Lemma 2.4. If $|G| = 40$, then $|G'| = 8$. Let K be a 3-decomposable normal subgroup of G. Then $Z(G') \leq K \leq G'$. Suppose $K = Z(G') \cup u^G$. Then $|u^G| = 2$ and thus $G' \leq C_G(u)$, which contradicts that $u \notin Z(G')$. If $|G| = 48$, then $|G'| = 16 = 2^4 = 2^{2m+1}$, another contradiction.

Therefore, $Z(G) = 1$. Suppose $|G''| = p^s$ for some positive integer s. If $G'' = Z(G')$ is 3-decomposable in G, then $1 + 2q = p^s$. Recall that $p = 2$ or $q = 2$, we have $q = 2$ and $p^s = 5$. Since $|G'| - |Z(G')| = 5^n - 5$ divides $|G| = 5^n \cdot 2$, we see that $5^{n-1} - 1$ divides 2, which is impossible. Hence, $G'' = Z(G')$ is 2-decomposable in G. Let $1 \neq g_1 \in G''$. Then $C_G(g_1) = G'$, and so $q = p^s - 1$. Suppose that $G' = Z(G') \cup g_2^G \cup g_3^G$. Then $C_G(g_i) \geq \langle g_i, Z(G') \rangle$ for $i = 2$ and 3. Therefore, $|C_G(g_2)| = p^{s+t_1} \geq p^{s+1}$ and $|C_G(g_3)| = p^{s+t_2} \geq p^{s+1}$. Hence, $p^n - p^s = \frac{p^n q}{p^{s+t_1}} + \frac{p^n q}{p^{s+t_2}}$. It follows that

$$p^s(p^{n-s} - 1) = q(p^{n-s-t_1} + p^{n-s-t_2}).$$

As G/G'' is non-abelian, the length of G/G''-conjugacy classes of each non-trivial element in G'/G'' is q. Hence $q + q = p^{n-s} - 1$ or $q = p^{n-s} - 1$. If $q + q = p^{n-s} - 1$, then by $p^s(p^{n-s} - 1) = q(p^{n-s-t_1} + p^{n-s-t_2})$, we have $2p^s q = q(p^{n-s-t_1} + p^{n-s-t_2})$. Therefore, $p^{n-2s-t_1} + p^{n-2s-t_2} = 2$, whence

$p^{n-2s-t_1} = p^{n-2s-t_2} = 1$. It follows that $t_1 = t_2$ and $n = 2s + t_1$. Since $q + q$ is even, we see that $p \neq 2$, and thus $q = 2$, $p = 5$ and $n - s = 1$. Therefore, $s + t_1 = n - s = 1$, which is a contradiction. If $q = p^{n-s} - 1$, then $n = 2s$. It follows from $p^s(p^{n-s} - 1) = q(p^{n-s-t_1} + p^{n-s-t_2})$ that $p^{n-2s-t_1} + p^{n-2s-t_2} = 1$. Hence $p = 2$ and $t_1 = t_2 = 1$. In this case, we have $2^n = |G'| = 2^s + 2^{s+1} + 2^{s+1} = 2^s \cdot 5$, which is impossible.

Case 3. G' is a group of order $p^n q^b$.

In this case, there exists a prime r such that $|G| = p^n q^b r$. It follows that G is solvable as G' is solvable. Arguing similarly as in the proof of Lemma 2.5, we have that every proper normal of G is contained in G'. We discuss the three possibilities (A), (B) and (C) described in the beginning of the proof of this theorem.

(A) $G' = P_1 \times Q_1$ with P_1 and Q_1 its elementary abelian Sylow p- and q-subgroups, respectively.

If $r = p$, then $Z(P) \cap P_1 \neq 1$ for every Sylow p-subgroup P of G. Write $K = Z(P) \cap P_1$. Then $K \leq Z(G)$ and thus $K \trianglelefteq G$. If P_1 or Q_1 is 3-decomposable in G, then $P_1 \times Q_1$ has at least 5 G-conjugacy classes, which is a contradiction. Therefore, P_1 is 2-decomposable in G, and thus $P_1 = K \leq Z(G)$. So $|P_1| = 2$ and $|G| = 4q^b$. It follows that G/Q_1 is of order 4 and thus it is abelian, which implies the contradiction that $G' \leq Q_1$. By arguing similarly, we have $r \neq q$. Therefore, $q \neq r \neq p$. If P_1 is 3-decomposable in G, then there are more that 4 G-conjugacy classes in $P_1 \times Q_1$, which is a contradiction. Therefore, P_1 is 2-decomposable in G. Similarly, we have that Q_1 is 2-decomposable in G. If $P_1 \leq Z(G)$, then $|P_1| = 2$ and $|G| = 2q^b r$. Let K be a subgroup of G with order $q^b r$. Then $K \trianglelefteq G$ and thus $G' \leq K$ since G/K is abelian of order 2, which is a contradiction. Therefore, $P_1 \nleq Z(G)$. Similarly, we have that $Q_1 \nleq Z(G)$. Therefore, $p^n - 1 = r = q^b - 1$, and thus $p = q$, another contradiction.

(B) $G' = N \rtimes H$ is a Frobenius group with kernel N and $G'/N \cong Z_q$ or Z_{q^2} or Q_8, where N is 2-decomposable in G.

(i) $G'/N \cong Z_{q^2}$.

If $r = p$, then $|G| = p^{n+1}q^2$. Let $P \in \mathrm{Syl}_p(G)$ and $N = \{1\} \cup x^G$. As $Z(P) \cap N \neq 1$, without loss of generality we may assume that $x \in Z(P)$. Therefore, $C_G(x) = P$ and $q^2 = p^n - 1$. Since, $p = 2$ or $q = 2$, we have $q = 2$ and $p^n = 5$ by Lemma 2.7. It follows that $|G| = 100$. By Sylow's Theorem, a Sylow 5-subgroup of G is normal in G, so $P \trianglelefteq G$. Since G/P

is abelian of order 4, we have that $G' \leq P$, which is a contradiction by order consideration.

If $r = q$, then $|G| = p^n q^3$. As $|G/N| = q^3$, we may choose K/N to be a normal subgroup of G/N such that $|K/N| = q$. Then $|K| = p^n q$ and $|G/K| = q^2$. It follows that G/K is abelian, and thus $G' \leq K$, which is again a contradiction by order consideration.

Now, we conclude that $p \neq r \neq q$. By the fact that N is 2-decomposable in G and G' is a Frobenius group, we have $|N| - 1 = p^n - 1 = q^2$ or $q^2 r$. If $p^n - 1 = q^2$, then $p = 2$ or $q = 2$. By Lemma 2.7, we have $q = 2$ and $p^n = 5$. Therefore, $|G| = 5{\cdot}2{\cdot}r$. As $r \neq 2$, there exists a normal subgroup K of G of order $5r$. It follows that G/K is abelian of order 2, leading to the contradiction that $G' \leq K$. Now, suppose that $p^n - 1 = q^2 r$. Let K be a 3-decomposable normal subgroup of G. Since $K \leq G'$, $N \leq K$ by [14, Exercise 8.5.7]. It follows that $|K| = p^n q$. If $q \neq 2$, then $q = 3$ and $r = 2$ as $|G'| - |K| = p^n q(q-1)$ divides $p^n q^2 r$. Therefore, $|G| = 342$. Let $G' = K \cup w^G$. Then $|w^G| = 114$ and thus $|C_G(w)| = 3$. The fact that G' has abelian Sylow 3-subgroups gives $|C_{G'}(w)| \geq 9$, which is a contradiction. If $q = 2$, then $r \neq 2$ and $|G/K| = 2r$. Let T/K be a subgroup of G/K of order r. Then T is a normal subgroup of G of index 2, and thus $G' \leq T$, contrary to that $|G'| = p^n 2^b$ and $|T| = p^n 2^{b-1} r$.

(ii) $G'/N \cong Z_q$.

Let $Q \in \mathrm{Syl}_q(G')$. Then $G = G' N_G(Q) = N N_G(Q)$ by the Frattini's argument. As G' is a Frobenius group, $N \cap N_G(Q) = 1$. Therefore, $G/N \cong N_G(Q)$ and $N_G(Q)$ is non-abelian. Hence $r \neq q$. If $r = p$ and $P \in \mathrm{Syl}_p(G)$, then $N \leq P$ since N is a normal p-subgroup of G. It follows that $P = P \cap G = P \cap N N_G(Q) = N(P \cap N_G(Q))$ and thus $|P \cap N_G(Q)| = p$. If $1 \neq x \in Z(P) \cap G'$, then $C_G(x) = P$ since G' is a Frobenius group and thus $|G : C_G(x)| = q$. If $N \nleq Z(P)$, then there exists $y \in N - Z(P)$. In this case, $|G : C_G(y)| > q$, and therefore x and y are not conjugate in G, which contradicts the fact that N is 2-decomposable in G. Hence $N \leq Z(P)$ and P is abelian. Since N is 2-decomposable in G, Q acts transitively on $N - \{1\}$. So $p^n - 1 = q$. For every $1 \neq x \in Q$, we have $C_G(x) = Q$ and $|x^G| = p^{n+1}$. As G' is 4-decomposable, $p^{n+1} + p^{n+1} = qp^n - p^n$. It follows that $q = 2p + 1$. Therefore, $p^n = 2(p + 1)$, and it is easy to see that $p = 2$ and $2^n = 6$, which is impossible. Hence, $q \neq r \neq p$.

Let $R \in \mathrm{Syl}_r(G)$. If R acts trivially on G', then $R \leq Z(G) \leq G'$, which is a contradiction. If R acts on N trivially, then $p^n - 1 = q$ and $p^n r + p^n r = qp^n - p^n$. It follows that $p = 2, q = 2^n - 1$ and $r = 2^{n-1} - 1$.

If $n \geq 4$, then either n or $n-1$ is even. Without loss of generality we may assume that $n = 2k$ for some positive integer $k > 1$. Then $q = 2^n - 1 = (2^k - 1)(2^k + 1)$, which is a contradiction since neither $2^k - 1$ nor $2^k + 1$ is equal to 1. Therefore, $n \leq 3$. It follows that $q = 7$ and $r = 3$. So $|C_G(N)| = 24$ and $G/C_G(N)$ is abelian. It follows that $G' \leq C_G(N)$ and $N \leq Z(G') = 1$, which is a contradiction. Therefore, R does not act trivially on N. As N is 2-decomposable in G, we have $p^n - 1 = qr$. By arguing similarly we have that $p^n r + p^n r = qp^n - p^n$. In this case, $q = 2r + 1$ and $p^n = 2r^2 + r + 1$. If $r \neq 2$, then $p = 2$. However, the equation $2^n = 2r^2 + r + 1$ does not have any solution by Lemma 2.8. So $r = 2$. In this case, $q = 5, p = 11$, $n = 1$ and $|G| = 110$. Let H be a 3-decomposable subgroup of G. As $H \leq G'$, we have $N \leq H$ by [14, Exercise 8.5.7], which contradicts the fact that N is a maximal subgroup of G'.

(iii) $G'/N \cong Q_8$

In this case, $N < G''$ by [14, Exercise 8.5.7]. Therefore, $|G'| = 8p^n$ and $|G''/N| = |Q_8'| = 2$, and thus $|G''| = 2p^n$. Notice that $1 < N < G'' < G' < G$, then both $|N| - 1 = p^n - 1$ and $|G'| - |G''| = 6p^n$ divide $|G| = 8p^n r$. It follows that $r = 3$ and that $p^n - 1$ divides 24. Recall that G' is a Frobenius group, we have $p \neq 2$ and write $2 \nmid |C_G(x)|$ for every $1 \neq x \in N$. Consequently, $|G| = 600$ or 216. In both cases, we write $\overline{G} = G/N$. Then $\overline{G}$ is a $\{1, 2, 3\}$-decomposable group. It follows from Theorem A that $\overline{G}$ is isomorphic S_4 or $SmallGroup(24, 3)$. Since the derived subgroup of S_4 has index 2 in S_4, $\overline{G}$ is isomorphic to $SmallGroup(24, 3)$. Noticing that $N = F(G)$ is a minimal normal subgroup of G and that G is solvable, we have $\Phi(G) < F(G)$, whence $\Phi(G) = 1$. So there exists $H \leq G$ such that $G = NH$ and $N \cap H = 1$. Therefore, $H \cong G/N$ is isomorphic to $SmallGroup(24, 3)$. If $|G| = 216$, then we may assume that $H = \langle a, b, c, d \mid a^3 = d^2 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd\rangle$ and that $N = \langle e, f \mid e^3 = f^3 = 1, [e, f] = 1\rangle$. As N is normal in G, we may assume that $e^a = e^i f^j, f^a = e^k f^l, e^b = e^s f^t, f^b = e^m f^n, e^c = e^u f^v, f^c = e^w f^x$, where $i, j, k, l, s, t, m, n, u, v, w, x \in \{0, 1, 2\}$. Then $e^{a^3} = e^{b^4} = e^{c^4} = e, f^{a^3} = f^{b^4} = f^{c^4} = f, e^{b^2} = e^{c^2} = e^d \neq e, f^{b^2} = f^{c^2} = f^d \neq f, e^{ba} = e^{acd}, f^{ba} = f^{acd}, e^{ca} = e^{abc}, f^{ca} = f^{abc}, e^{cb} = e^{bcd}$ and $f^{cb} = f^{bcd}$. Therefore, the integers $i, j, k, l, s, t, m, n, u, v, w, x$ must satisfy all of the following congruence equations:

$s^2 + mt - u^2 - vw \equiv 0 \ (mod\ 3)$

$st + tn - uv - vx \equiv 0 \ (mod\ 3)$

$sm + mn - uw - wx \equiv 0 \ (mod\ 3)$

$tm + n^2 - wv - x^2 \equiv 0\ (mod\ 3)$
$i^3 + 2ijk + jkl \equiv 1\ (mod\ 3)$
$ijk + 2jkl + l^3 \equiv 1\ (mod\ 3)$
$i^2j + kj^2 + ijl + l^2j \equiv 0\ (mod\ 3)$
$i^2k + ikl + kl^2 \equiv 0\ (mod\ 3)$
$s^4 + 2stmn + m^2t^2 + mn^2t \equiv 1\ (mod\ 3)$
$s^2tm + 2smnt + t^2m^2 + n^4 \equiv 1\ (mod\ 3)$
$u^4 + 2uvwx + v^2w^2 + wx^2v \equiv 1\ (mod\ 3)$
$u^2vw + 2uvwx + v^2w^2 + x^4 \equiv 1\ (mod\ 3)$
$usi + umj + wti + wnj - iu - kv \equiv 0\ (mod\ 3)$
$vsi + vmj + xti + xnj - ju - lv \equiv 0\ (mod\ 3)$
$usi + umj + wti + wnj - iw - kx \equiv 0\ (mod\ 3)$
$vsi + vmj + xti + xnj - jw - lx \equiv 0\ (mod\ 3)$
$u^3v + 2uv^2w + 2v^2wx + u^2xv + uvx^2 + x^3v \equiv 0\ (mod\ 3)$
$u^3w + u^2wx + 2uvw^2 + uwx^2 + 2vw^2x + wx^3 \equiv 0\ (mod\ 3)$
$s^3t + 2st^2m + 2t^2mn + s^2nt + stn^2 + n^3t \equiv 0\ (mod\ 3)$
$s^3m + s^2mn + 2stm^2 + smn^2 + 2tm^2n + mn^3 \equiv 0\ (mod\ 3)$
$s^3u + s^2wt + mtus + mwt^2 + s^2mv + smxt + mnvs + mnst - su - mv \equiv 0\ (mod\ 3)$
$s^2um + s^2wn + tum^2 + mtwn + sm^2v + smxn + m^2vn + mxn^2 - sw - mx \equiv 0\ (mod\ 3)$
$stum + stwn + tnum + twn^2 + tvm^2 + tmxn + n^2vm + xn^3 - tw - nx \equiv 0\ (mod\ 3)$
$s^2ut + swt^2 + tnus + t^2wn + tmvs + mxt^2 + n^2vs + n^2xt - tu - nv \equiv 0\ (mod\ 3)$
$s^2ui + s^2wj + mtui + mtwj + smvi + smxj + mnvi + mnxj - is - lt \equiv 0\ (mod\ 3)$
$stui + stwj + tnui + tnwj + tmvi + tmxj + n^2vj + n^2xj - js - lt \equiv 0\ (mod\ 3)$
$s^2uk + s^2wl + mtuk + mtwl + smvk + smxl + mnvk + mnxl - im - kn \equiv 0\ (mod\ 3)$
$stuk + stwl + tnuk + tnwl + tmvk + tmxl + n^2vk + n^2xl - jm - ln \equiv 0\ (mod\ 3)$.

Noting that $e^{b^2} \neq e$ and $f^{b^2} \neq f$, we have neither

$$\begin{cases} s^2 + mt \equiv 1\ (mod\ 3) \\ st + tn \equiv 0\ (mod\ 3) \end{cases} \tag{3.1}$$

nor

$$(3.2) \qquad \begin{cases} sm + mn \equiv 0 \ (mod\ 3) \\ tm + n^2 \equiv 1 \ (mod\ 3) \end{cases}$$

happens.

According to the calculation, we have the following 8 solutions for the above conditions:

1. $i = 0, j = 1, k = 2, l = 2, s = 0, t = 2, m = 1, n = 0, u = 1, v = 2, w = 2, x = 2$,
2. $i = 0, j = 2, k = 1, l = 2, s = 0, t = 1, m = 2, n = 0, u = 1, v = 1, w = 1, x = 2$,
3. $i = 2, j = 2, k = 1, l = 0, s = 1, t = 1, m = 1, n = 2, u = 2, v = 1, w = 1, x = 1$,
4. $i = 2, j = 1, k = 2, l = 0, s = 1, t = 1, m = 1, n = 2, u = 0, v = 1, w = 2, x = 0$,
5. $i = 2, j = 2, k = 1, l = 0, s = 1, t = 2, m = 2, n = 2, u = 0, v = 2, w = 1, x = 0$,
6. $i = 0, j = 1, k = 2, l = 2, s = 2, t = 1, m = 1, n = 1, u = 1, v = 1, w = 1, x = 2$,
7. $i = 0, j = 2, k = 1, l = 2, s = 2, t = 1, m = 1, n = 1, u = 0, v = 2, w = 1, x = 0$,
8. $i = 2, j = 2, k = 1, l = 0, s = 0, t = 1, m = 2, n = 0, u = 2, v = 2, w = 2, x = 1$.

Let $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$ be the groups corresponding to the above solutions, respectively. In group G_2, we may replace f by f^2, then G_2 has the same representation as G_1, and thus $G_2 \cong G_1$. Similarly, in groups $G_3, G_4, G_5, G_6, G_7, G_8$, we may replace e, f by e^2, ef, replace e by ef^2, replace e, f by ef, f^2, replace e, f by f^2, ef, replace e, f by ef^2, e^2, replace e, f by f, e, respectively, then all of them are isomorphic to G_1. Therefore, we may define G as in (1) of this Theorem. If $|G| = 600$, with the help of Program 1 in the Appendix, we have that $G \cong SmallGroup(600, 150)$, and thus we may assume that G is defined as in (2) of this Theorem.

(C) G' is a Frobenius group with kernel N and $G'/N \cong Z_q$, where N is 3-decomposable in G.

As $|G'| - |N| = p^n q - p^n$ divides $p^n qr$, we see that $q - 1$ divides qr. So $q = 2$, and $r \neq 2$ or $q = 3$, and $r = 2$.

Suppose $r \neq p$. If $q = 2$ and $r \neq 2$, then $|G| = 2p^n r$. We can choose T to be a subgroup of G of order $p^n r$. Then $T \trianglelefteq G$ and $G' \leq T$ as G/T

is abelian of order 2, which is a contradiction by order consideration. If $q = 3$ and $r = 2$, then $|G| = 6p^n$. Suppose that N is abelian. Then $p^n = 1 + 2qr$ or $p^n = 1 + 2q$ or $p^n = 1 + qr + q$. It follows that $|G| = 78$ or 42. Since every normal subgroup of G is contained in G', it is easy to see that there is no 2-decomposable normal subgroup in G in each case. Therefore, N is non-abelian, and thus $Z(N)$ is 2-decomposable in G. Suppose $|Z(N)| = p^t$ for some integer t. Then $p^n = p^t + p^s qr$ or $p^n = p^t + p^s q$ for some integer s. If $p^n = p^t + p^s qr$, then $p = 7$ and $n - t = 1$. Note that $Z(N)$ is 2-decomposable, we have $p^t = 1 + qr$. Therefore, $t = 1, n = 2$ and $|G| = 294$. Suppose $N = Z(N) \cup u^G$. Then $|C_G(u)| = 7$, which is contrary to the fact that $Z(N) \neq 1$. If $p^n = p^t + p^s q$, then $p = 2 = r$, contrary to our assumption.

Therefore, $r = p$. In this case, we have $|G| = p^{n+1}q$. If $q = 2$, we can choose T to be a subgroup of G of order p^{n+1}. Then G/T is abelian and thus $G' \leq T$, which is a contradiction. If $p = 2$, then $q = 3$ and $|G| = 2^{n+1} \cdot 3$. Let K be a 2-decomposable normal subgroup of G. Then $K \leq G'$, and thus $K \leq N$ by [14, Exercise 8.5.7]. If $t = 1$, then $K = Z(G)$, which gives $Z(G') \neq 1$, and this contradicts the fact that G' is a Frobenius group. Since both $|K| - 1 = 2^t - 1$ and $|N| - |K| = 2^n - 2^t$ divide $|G| = 2^{n+1} \cdot 3$, we have $t = 2$, and $n = 3$ or $n = 4$. Therefore, $|G| = 48$ or 96. First suppose that $|G| = 48$. Then $\overline{G} = G/K$ is $\{1, 2, 3\}$-decomposable and $|\overline{G}| = 12$, which contradicts Theorem A. Now, suppose that $|G| = 96$. Then $|N| = 16$. It is easy to see that $N' = Z(N)$ is an elementary abelian 2-group of order 4 and that $exp(N) = 4$. However, by investigating the structures of non-abelian 2-groups of order 16 with exponent 4, we find that there does not exist a group satisfying this condition. Therefore, there is no X-decomposable group in this case.

□

Theorem 3.2. *There is no finite non-perfect X-decomposable group G such that G' is 3-decomposable in G.*

Proof. Since G' is 3-decomposable, G' must be one of the following groups by [16]:

1) $|G'| = p^n$ for some prime p and some integer n and G' is metabelian.

2) $|G'| = p^n$ for some prime p and some integer n and G' is elementary abelian.

3) G' is a Frobenius group and $G' = \{1\} \cup g^G \cup h^G$, with $h^{-1} \in h^G$ and $(|h|, |g|) = 1$.

Furthermore, if G' is of type 3), then $|G'| = 2^n p$, where $p = 2^n - 1$ is a prime by [2, Lemma 1].

We see that in all cases, G is solvable as G' is solvable. Let N be an arbitrary normal subgroup of G. We claim that $G' \leq N$ or $N \leq G'$. For otherwise, since G is X-decomposable and G' is 3-decomposable in G, there are more than 4 G-conjugacy classes in $G'N$. It follows that $G = G'N$, and thus $(G/N)' = G/N$, which is a contradiction.

Case A. $|G'| = p^n$ for some prime p and some integer n and G' is metabelian.

If $Z(G) \nleq G'$, then $G' < Z(G)$ by the above paragraph, which gives the contradiction that G is abelian. Therefore, $Z(G) \leq G'$.

(i) If G has at least two distinct 4-decomposable normal subgroups K_1 and K_2, then $K_1 \cap K_2 = G'$ and $G = K_1K_2$. Furthermore, there exist primes r_1 and r_2 such that $|G/K_1| = |K_2/G'| = r_1$ and $|G/K_2| = |K_1/G'| = r_2$, and thus $|G| = p^n r_1 r_2$. On the other hand, since $|K_1| - |G'|$ and $|K_2| - |G'|$ divide $|G|$, we have that both $r_1 - 1$ and $r_2 - 1$ divide $r_1 r_2$. It is easy to see that $|G| = 4p^n$ or $|G| = 6p^n$.

Suppose $|G| = 4p^n$. Then $p \neq 2$. Let K be a 2-decomposable normal subgroup of G. Then $K \leq G'$, and so there exists a positive integer $t < n$ such that $|K| = p^t$. Then both $|K| - 1 = p^t - 1$ and $|G'| - |K| = p^n - p^t$ divide $4p^n$. It is easy to see that $p^n = 9$ or 25. Suppose that $p^n = 9$ and that $G' = K \cup x^G$ for some $x \in G'$. Then $|x^G| = 9 - 3 = 6$ and $|C_G(x)| = 6$. On the other hand, we have that $G' \leq C_G(x)$ as G' is abelian, and thus $|C_G(x)| \geq 9$, which is a contradiction. If $p^n = 25$, by arguing similarly as for $p^n = 9$, we can get a contradiction.

Now, suppose $|G| = 6p^n$. Let H be a 2-decomposable normal subgroup of G. Then $H \leq G'$. If $H \leq Z(G)$, then $|H| = 2$ and $p = 2$. Therefore, $|G'| - |H| = 2^n - 2$ divides $2^n \cdot 6$. It follows that $|G| = 24$ or 48. First suppose $|G| = 24$. If G has normal Sylow 3-subgroup Q, then $Z(G) \times Q$ is 4-decomposable in G, and thus $G' \leq Z(G) \times Q$, which is a contradiction. Therefore, a Sylow 3-subgroup of G is not normal and $G/Z(G) \cong A_4$ by [11, Theorem 4.3.4], which is a contradiction. Now, suppose $|G| = 48$. Then $|G'| = 8$ and we can choose a 4-decomposable normal subgroup of G, say K_1, such that $|K_1| = 24$. If $K_1 = G' \cup w^G$, then $|w^G| = 16$ and thus $|C_G(w)| = 3$, contrary to that $Z(G) \neq 1$.

Therefore, $H \nleq Z(G)$. Recall that $H \leq G'$, so there exists a positive integer i such that $|H| = p^i$. Then both $|H| - 1 = p^i - 1$ and $|G'| - |H| = p^n - p^i$ divides $6p^n$. Note that $|H| \neq 2$ as $H \nleq Z(G)$. We conclude that $|G|$ =48, 54, 96 or 294. If $|G| = 48$. Let K_1 be a 4-decomposable normal subgroup such that $|K_1| = 24$ and $K_1 = G' \cup w^G$. Then $|C_G(w)| = 3$. On the other hand, write $G' = H \cup v^G$. Then $|C_G(v)| = 12$, which is a contradiction. If $|G| = 54$, then we can choose K_1 to be a 4-decomposable normal subgroup such that $|K_1| = 27$ and that $K_1 = G' \cup u^G$. It follows that K_1 is a Sylow 3-subgroup of G and $Z(K_1) \neq 1$, which contradicts the fact that $|C_G(u)| = 3$. If $|G| = 96$. We can choose K_2 to be a 4-decomposable subgroup of G such that $|K_2| = 32$ and $K_2 = G' \cup k^G$. Then $|C_G(k)| = 6$, which contradicts that $Z(G') \neq 1$. Finally, suppose $|G| = 294$. Let $G' = H \cup h^G$. Then $|C_G(h)| = 7$, contrary to that G' is abelian.

(ii) There is exactly one 4-decomposable normal subgroup in G. Then there exists a prime $q \neq p$ such that G/G' is a cyclic group and $|G/G'| = q^2$. Let H/G' be a normal subgroup of G/G' of order q. Then $|H| = p^n q$ and H is 4-decomposable in G. Therefore, $|H| - |G'| = p^n(q-1)$ divides $|G| = p^n q$. It follows that $q = 2$ and $|G| = 4p^n$. By arguing similarly as in (i), we conclude that there is no X-decomposable group in this case.

Case B. $|G'| = p^n$ for some prime p and some integer n and G' is elementary abelian.

We can similarly have $Z(G) \leq G'$ as in Case A.

(i) There are at least two distinct 4-decomposable normal subgroups in G. By arguing similarly as in Case A(i), we have $|G| = 4p^n$ or $|G| = 6p^n$.

Suppose $|G| = 4p^n$. Then $p \neq 2$. If $Z(G) \neq 1$, then $|Z(G)| = 3$ and $G' = Z(G)$ as G' is 3-decomposable in G. It follows that G is abelian, which is a contradiction. Therefore $Z(G) = 1$, and thus G' is the only minimal normal subgroup of G by [2, Theorem 1(i)], so G does not have a 2-decomposable normal subgroup, which is a contradiction.

Now suppose $|G| = 6p^n$. If $Z(G) = 1$, by arguing similarly as in the above paragraph, we can get a contradiction. Therefore, $Z(G) \neq 1$. If $|Z(G)| = 2$, then $|G'| - |Z(G)| = 2^n - 2$ divides $|G| = 6 \cdot 2^n$. It follows that $|G| = 24$ or 48. If $|G| = 24$, then $|G'| = 4$. Let K_1 be a 4-decomposable normal subgroup of G such that $|K_1| = 12$ and let $K_1 = G' \cup x^G$. Then $|C_G(x)| = 3$, which contradicts the fact that $Z(G) \neq 1$. If $|G| = 48$, by arguing similarly as for $|G| = 24$, we arrive at a contradiction. Therefore, $Z(G) = G'$ is of order 3. Consequently, we conclude that $|G| = 18$ and $G/Z(G) = G/G'$ is a cyclic group of order 6, which gives the contradiction that G is abelian.

(ii) There is exactly one 4-decomposable normal subgroup in G. By arguing similarly as in Case A(ii), we have that $|G| = 4p^n$. By (i) of this case, we see that there is no X-decomposable group in this case.

Case C. G' is a Frobenius group of order $2^n p$, where $p = 2^n - 1$ is a prime and $G' = \{1\} \cup g^G \cup h^G$, with $h^{-1} \in h^G$ and $(|h|, |g|) = 1$.

Let H be a 4-decomposable normal subgroup of G. Then $G' \leq H$ by the beginning of this theorem. We see that H is a Frobenius group by [13, Theorem 2]. Let M be the Frobenius kernel of H. Then M is nilpotent, and thus $M \leq G'$ by [14, Exercise 8.5.7]. It follows that M is the Frobenius kernel of G'. So $|H| = 2^n p^b$ and $|G| = 2^n p^b r$ for some prime r. As $|H| - |G'| = 2^n(p^b - p)$ divides $2^n p^b r$, we have that $p^{b-1} - 1$ divides r. Therefore, $p^{b-1} - 1 = r$ since $p = 2^n - 1$. It is easy to see that $r = 2, p = 3, b = 2$ and $n = 2$. Let $H = G' \cup w^G$. Then $|w^G| = 24$ and thus $|C_G(w)| = 3$, which contradicts the fact that G has abelian Sylow 3-subgroups. □

Theorem 3.3. *Let G be a finite non-perfect X-decomposable group. If G' is 2-decomposable in G, then G is one of the following two groups:*

(1) $|G|$=42 and $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^5 \rangle$.

(2) $G = D_{12}$.

Proof. As G' is 2-decomposable in G, there is a prime p such that G' is an elementary abelian p-group by [17, Theorem 1]. Suppose $|G'| = p^n$ for some positive integer n.

If $G' \leq \Phi(G)$, then G is nilpotent. As $Z(G)$ can not be maximal in G, $Z(G)$ is 2- or 3-decomposable in G. However, $|Z(G)|$ is divided by at least two primes since G is not of prime power order, which is a contradiction. Therefore, $G' \nleq \Phi(G)$. In this case, there exists a maximal subgroup M of G such that $G' \nleq M$. So $G = G'M$ and $G' \cap M = 1$. Moreover, $M \cong G/G'$ is abelian. For $1 \neq x \in M$, the maximality of M implies that $C_G(x) = M$ or $C_G(x) = G$.

If $C_G(x) = M$ for every $1 \neq x \in M$, then G is a Frobenius group with kernel G' and a complement M. By the structure of the Frobenius complements, M is a cyclic group. Take K to be an arbitrary non-trivial subgroup of M. Then $G'K \trianglelefteq G$ and so $G'K$ is 3- or 4-decomposable in G. For every $1 \neq y \in G'K \backslash G'$, y must be a p'-element and there exists a Hall p'-subgroup M_1 of G such that $y \in M_1$. Noticing that M_1 and M are conjugate, we conclude that M_1 is also abelian and thus $|y^G| = \frac{|G|}{|M_1|} = |G'|$. If $G'K$ is 3-decomposable in G, then $|G'||K| =$

$|G'K| = 2|G'|$ and $|K| = 2$. If $G'K$ is 4-decomposable in G, then $|G'||K| = |G'K| = 3|G'|$ and $|K| = 3$. Therefore, M is a cyclic group of order 6. On the other hand, M acts transitively and fixed-point freely on $G'\backslash\{1\}$, so $|G'| - 1 = p^n - 1 = 6$. It follows that $|G'| = 7$ and there exists $i \in \{2, 3, 4, 5, 6\}$ such that $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^i\rangle$. It is easy to see that $i = 5$ and G is the first group described in this theorem.

Now, suppose that there exists $1 \neq x \in M$ such that $C_G(x) = G$. Then $Z(G) \neq 1$. If $G' \leq Z(G)$, then $|G'| = 2$ and $|G : M| = |G'| = 2$. So $G' \leq M$, which is a contradiction. Therefore, $G' \nleq Z(G)$. The minimality of G' implies that $G' \cap Z(G) = 1$. Let $H = G' \times Z(G)$. Then H is abelian and thus $H < G$. In this case, $Z(G)$ must be 2-decomposable and so H is 4-decomposable in G. Therefore, there exists a prime q such that $|G/H| = q$. Since $|H| = 2p^n$, we have $|G| = 2p^nq$. If $p = 2$, then $q \neq 2$ as G is not a 2-group. As G' is 2-decomposable and $H \leq C_G(G')$, we have that $|G'| - 1 = 2^n - 1 = q$. Let Q be a Sylow q-subgroup of G and $K = G'Q$. Then K is normal in G and $|K| = 2^nq$. If K is 3-decomposable in G, then $|K| - |G'| = 2^n(q - 1)$ divides $|G| = 2^{n+1}q$. It follows that $q = 3, n = 2$ and $|G| = 24$. Since $Q \ntrianglelefteq G$ and $Z(G) \neq 1$, $G/Z(G) \cong A_4$ by [11, Theorem 4.3.4], which is a contradiction. If K is 4-decomposable in G, then K is a Frobenius group by [13, Theorem 2]. So all elements of order q in K form two G-conjugacy classes. Let $y \in K$ be an element of order q. We can see that $|C_G(y)| = 2q$, and thus $|y^G| = 2^n$. Now we have $2^nq = |K| = 2^n + 2^n + 2^n$ and thus $q = 3, n = 2$ and $|K| = 12$. It is easy to see that $K \cong A_4$ and $K \cap Z(G) = 1$. Therefore, $G \cong A_4 \times Z_2$. However, G is $\{1, 2, 4\}$-decomposable by Theorem C, which is a contradiction. If $p \neq 2$, then there exist elements in H of order 2, p and $2p$. So all elements of order p in H form one G-conjugacy class. Noticing that H is abelian, we conclude that $p^n - 1 = q$ and thus $p^n = 3$ and $q = 2$. In this case, $|G| = 12$ and G is an extension of a cyclic group H of order 6 by a cyclic group of order 2. Suppose that $H = \langle a\rangle$ and let $1 \neq b \in G\backslash H$. Then $b^{-1}ab = a^{-1}$ since $b^{-1}ab \neq a$. On the other hand, $b^2 \in H$ since $|G/H| = 2$. If $b^2 = a^2$ or $b^2 = a^4$, then b is of order 6. It is easy to see that $|\langle a\rangle \cap \langle b\rangle| = 3$, and thus $|Z(G)| \geq 3$, which contradicts to that $|Z(G)| = 2$. If $b^2 = a^3$, then $G = \langle a, b \mid a^6 = 1, b^2 = a^3, b^{-1}ab = a^{-1}\rangle \cong Q_{12}$. However, G is $\{1, 2, 4\}$-decomposable by Theorem C. Therefore, $b^2 = 1$ and $G = \langle a, b \mid a^6 = b^2 = 1, b^{-1}ab = a^{-1}\rangle \cong D_{12}$, and G is X-decomposable by Example 2.1.

□

Now, from the above three theorems, we come to our main theorem.

Theorem 3.4 (**Main theorem).** *Let G be a finite non-perfect X-decomposable group. Then G is one of the following groups:*

(1) $|G| = 216$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^3 = f^3 = 1, b^2 = c^2 = d, b^a = cd, c^a = bc, c^b = cd, e^a = f^2, e^b = e^2 f, e^c = f^2, e^d = e^2, f^a = ef^2, f^b = ef, f^c = e, f^d = f^2\rangle$.

(2) $|G| = 600$ *and* $G = \langle a, b, c, d, e, f \mid a^3 = d^2 = e^5 = f^5 = 1, b^2 = c^2 = d, b^a = bc, c^a = b, c^b = cd, e^a = ef^3, e^b = e^3 f^3, e^c = e^3, e^d = e^4, f^a = e^4 f^3, f^b = f^2, f^c = e^4 f^2, f^d = f^4\rangle$.

(3) $|G| = 42$ *and* $G = \langle a, b \mid a^7 = b^6 = 1, b^{-1}ab = a^5\rangle$.

(4) $G = D_{12}$.

Appendix

Program 1 : A Magma Program

```
SetLogFile("nnn..txt");
   P:=SmallGroupProcess(600);
   repeat
   G:=Current(P);
   "..................................................................group ";
   CurrentLabel(P);
   M:=NormalSubgroups(G); m:=0;
   for j in [1..♯M] do
   N:=M[j]‘subgroup;
   S:=[n:n in N—Order(n) ge 1];
   while ♯S gt 1 do
   h:=1;X:=S[1];Remove( S,1);
            for k in [♯S..1 by -1] do
                     if IsConjugate(G, X, S[k]) then
                            h:=h+1;Remove( S,k);
                     end if;
            end for;
   "c,",h;
   end while; "c,",♯S; N;m:=m+1; "......................................." ,m;
   if ♯N eq 1 then "1"; end if;
   end for;
   "......................................................................";
   Advance( P);
   until IsEmpty(P);
```

Acknowledgments

The research is partially supported by the National Natural Science Foundation of China(11371237) and Shanghai Leading Academic Discipline Project(J50101). The authors wish to thank the referees for their helpful suggestions.

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