Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 5, pp. 1275–1290
Online ISSN: 1735-8515

CONVERGENCE ANALYSIS OF SPECTRAL TAU METHOD FOR FRACTIONAL RICCATI DIFFERENTIAL EQUATIONS

P. MOKHTARY* AND F. GHOREISHI

(Communicated by Mohammad Asadzadeh)

Abstract. In this paper, a spectral Tau method for solving fractional Riccati differential equations is considered. This technique describes converting of a given fractional Riccati differential equation to a system of nonlinear algebraic equations by using some simple matrices. We use fractional derivatives in the Caputo form. Convergence analysis of the proposed method is given and rate of convergence is established in the weighted L^2-norm. Numerical results are presented to confirm the high accuracy of the method.
Keywords: Fractional Riccati differential equations, Caputo derivative, spectral Tau method.
MSC(2010): Primary: 34A08; Secondary: 65L60.

Archive of SID

1. Introduction

Fractional differential equations are suitable to describe some physical phenomena such as damping laws, rheology, diffusion processes, and so on [3, 13, 18, 22]. Recently, linear fractional differential equations with variable coefficients have been solved by adapting various analytical and numerical methods [4–6, 8, 9]. Nowadays, applications have included

Article electronically published on October 27, 2014.
Received: 8 February 2013, Accepted: 22 September 2013.
*Corresponding author.

©2014 Iranian Mathematical Society

www.SID.ir

some classes of nonlinear fractional differential equations, and this results in considering numerical methods for solving these equations. Recently, several numerical methods have been proposed for solving nonlinear fractional differential equations; see [2, 7, 11, 12, 14, 17, 23].

Spectral Tau method was proposed by Ortiz and it was applied to ordinary differential equations [19], eigenvalue problems [20], partial differential equations [21], integral equations [10], linear multi order fractional differential equations [9], nonlinear fractional integro differential equations [16] etc. In this method approximations are defined by truncated series expansions, such that residual which should be exactly equal to zero, is forced to be zero only in an approximate sense. This approach has two main advantages. First, it reduces the given problems to those of solving a system of algebraic equations and approximate representation of a smooth function converges exponentially; see [24].

In this paper, we use spectral Tau method for the numerical solution of fractional Riccati differential equation

$$\begin{cases} D^{\alpha}(u(t)) = u(t) + A(t)u^2(t) + B(t), \\ u^{(i)}(0) = u_i, \quad i = 0, 1, ..., m-1, \end{cases} \tag{1.1}$$

where α is fractional derivative order. m is an integer satisfying $m-1 < \alpha \leq m$. Coefficients $A(t), B(t)$ are known real functions and $u(t)$ is the exact solution. $u_i,\ i = 0, 1, ..., m-1$ are constants and D^{α} is the Caputo fractional derivative which is given by(see [3, 13, 18, 22])

$$D^{\alpha}u(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-\alpha-1} u^{(m)}(s) ds. \tag{1.2}$$

The organization of this paper is as follows: Section 2 is devoted to solve (1.1) by adopting the spectral Tau method. In Section 3, we prove convergence of the Tau solution of (1.1). In Section 4, we illustrate some examples to demonstrate the efficiency of the proposed method.

2. Numerical approach

In this section, we present a numerical solution of (1.1) by using the spectral Tau method based on shifted Jacobi basis functions on $I = [0, 1]$.

Archive of SID

www.SID.ir

Define $u_N(t)$ as an approximate solution of (1.1) as

(2.1)
$$u_N(t) = \sum_{i=0}^{N} a_i V_i^{\rho,\sigma}(t) = \underline{a}\ \underline{V}^{\rho,\sigma} = \underline{a} V^{\rho,\sigma} \underline{X_t},\ \underline{a} = [a_0, a_1, ..., a_N, 0, ...],$$

where $\underline{V}^{\rho,\sigma} := [V_0^{\rho,\sigma}(t), V_1^{\rho,\sigma}(t), ..., V_N^{\rho,\sigma}(t), ...]^T = V^{\rho,\sigma}\underline{X_t}$ with parameters $\rho, \sigma \in (-1, 1)$ showing arbitrary shifted Jacobi polynomial bases. $V^{\rho,\sigma}$ is infinitely nonsingular lower triangular coefficient matrix. $\underline{X_t} = [1, t, t^2, ..., t^N, ...]^T$ is the standard basis and $V_i^{\rho,\sigma}(t)$, for $i = 0, 1, \dots$, are arbitrary shifted Jacobi polynomials with degree at most i which are orthogonal with respect to the weight function $w^{\rho,\sigma}(t) = 2^{\rho+\sigma} t^{\sigma}(1-t)^{\rho}$ on I.

Substituting (2.1) in (1.1) we get

$$D^{\alpha}(\underline{a}\ \underline{V}^{\rho,\sigma}) = \underline{a}\ \underline{V}^{\rho,\sigma} + A(t)\Big(\underline{a}\ \underline{V}^{\rho,\sigma}\Big)^2 + B(t). \tag{2.2}$$

Now, we find suitable matrix forms for $D^{\alpha}(\underline{a}\ \underline{V}^{\rho,\sigma})$ and $\Big(\underline{a}\ \underline{V}^{\rho,\sigma}\Big)^2$. First, we consider $D^{\alpha}(\underline{a}\ \underline{V}^{\rho,\sigma})$. To this end, we have

(2.3)
$$D^{\alpha}(\underline{a}\ \underline{V}^{\rho,\sigma}) = D^{\alpha}\Big(\underline{a}V^{\rho,\sigma}\underline{X_t}\Big) = \underline{a}V^{\rho,\sigma}D^{\alpha}\Big(\underline{X_t}\Big) = \underline{a}V^{\rho,\sigma}D^{\alpha}[1, t, t^2, ..., t^N, ...]^T.$$

By using the relation(see [3, 13, 18, 22])

$$D^{\alpha}t^i = \begin{cases} \frac{i!}{\Gamma(i-\alpha+1)}\, t^{i-\alpha}, & i \in \mathbb{N} \text{ and } i \geq m \ or\ i \notin \mathbb{N}\ and\ i > m, \\ 0, & i \in \mathbb{N} \text{ and } i < m, \end{cases}$$

we rewrite (2.3) as

(2.4)
$$\begin{aligned} D^{\alpha}(\underline{a}\ \underline{V}^{\rho,\sigma}) &= \underline{a}V^{\rho,\sigma}[\underbrace{0, 0, ..., 0}_{m-1},\ \frac{m!}{\Gamma(m-\alpha+1)}t^{m-\alpha}, ...,\ \frac{N!}{\Gamma(N-\alpha+1)}t^{N-\alpha}, ...]^T \\ &= \underline{a}V^{\rho,\sigma}[\underbrace{0, 0, ..., 0}_{m-1},\ \frac{m!}{\Gamma(m-\alpha+1)}\sum_{j=0}^{\infty}\delta_{m,j}V_j^{\rho,\sigma}(t), ..., \\ &\qquad \frac{N!}{\Gamma(N-\alpha+1)}\sum_{j=0}^{\infty}\delta_{N,j}V_j^{\rho,\sigma}(t), ...]^T \\ &= \underline{a}V^{\rho,\sigma}\Omega\underline{V}^{\rho,\sigma} \end{aligned}$$

Archive of SID

www.SID.ir

where

$$\Omega = \begin{bmatrix} 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots \\ \frac{m!}{\Gamma(m-\alpha+1)}\delta_{m,0} & \cdots & \frac{m!}{\Gamma(m-\alpha+1)}\delta_{m,N} & \cdots \\ \frac{(m+1)!}{\Gamma(m-\alpha+2)}\delta_{m+1,0} & \cdots & \frac{(m+1)!}{\Gamma(m-\alpha+2)}\delta_{m+1,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{N!}{\Gamma(N-\alpha+1)}\delta_{N,0} & \cdots & \frac{N!}{\Gamma(N-\alpha+1)}\delta_{N,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

with

$$\delta_{i,j} = \frac{(t^{i-\alpha}, V_j^{\rho,\sigma})_{\rho,\sigma}}{(V_j^{\rho,\sigma}, V_j^{\rho,\sigma})_{\rho,\sigma}}, \quad \begin{matrix} i \geq m \\ j = 0, \ldots, N. \end{matrix}$$

The symbol $(.,.)_{\rho,\sigma}$ stands for the weighted inner product defined by $(f,g)_{\rho,\sigma} = \int_I f(t)g(t)w^{\rho,\sigma}(t)dt$.

Now we obtain a matrix form for $\left(\underline{a}\ \underline{V}^{\rho,\sigma}\right)^2$. To this end, we have

$$\begin{aligned} (2.5)\quad \left(\underline{a}\ \underline{V}^{\rho,\sigma}\right)^2 &= \left(\underline{a}V^{\rho,\sigma}\underline{X_t}\right)^2 = \underline{a}V^{\rho,\sigma}\Big(\underline{X_t} \times \left(\underline{a}V^{\rho,\sigma}\underline{X_t}\right)\Big) \\ &= \underline{a}V^{\rho,\sigma}\Big(\underline{X_t} \times \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} a_r v_{r,s}^{\rho,\sigma} t^s\Big) \\ &= \underline{a}V^{\rho,\sigma}\Big[\sum_{s=0}^{\infty}\Big(\sum_{r=0}^{\infty} a_r v_{r,s}^{\rho,\sigma}\Big)t^{s+i}\Big]_{i=0}^{\infty} \\ &= \underline{a}V^{\rho,\sigma}\Upsilon\underline{X_t}, \end{aligned}$$

where

$$\Upsilon = \begin{bmatrix} a\tilde{V}_0^{\rho,\sigma} & a\tilde{V}_1^{\rho,\sigma} & a\tilde{V}_2^{\rho,\sigma} & \cdots \\ \mathbf{0} & a\tilde{V}_0^{\rho,\sigma} & a\tilde{V}_1^{\rho,\sigma} & \cdots \\ \mathbf{0} & \mathbf{0} & a\tilde{V}_0^{\rho,\sigma} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with $\tilde{V}_i^{\rho,\sigma} = \{v_{j,i}^{\rho,\sigma}\}_{j=0}^{\infty}$.

putting (2.4) and (2.5) into (2.2) we get

$$(2.6)\qquad \underline{a}V^{\rho,\sigma}\Omega\underline{X_t} = \underline{a}V^{\rho,\sigma}\underline{X_t} + A(t)\Big(\underline{a}V^{\rho,\sigma}\Upsilon\underline{X_t}\Big) + B(t).$$

Archive of SID

www.SID.ir

If we take $c(t) = \underline{c}\underline{X_t}$ where $\underline{c} = [c_0, c_1, ..., c_N, ...]$ to be a polynomial and

$$A(t) = \underline{A}\ \underline{X_t}, \quad \underline{A} = [A_0, A_1, ...], \tag{2.7}$$

we can write

$$A(t)c(t) = \underline{c}A(\mu)\underline{X_t},$$

where μ is a nonsingular matrix with only nonzero elements $\mu_{i+1,i} = 1,\ i = 1, 2, ...$; see [19].

Using (2.7) and assuming $B(t) = \underline{B}\underline{V}^{\sigma,\rho},\ \underline{B} = [B_0, B_1, ...]$ we can rewrite (2.6) as

$$\underline{a}V^{\rho,\sigma}\Omega\underline{X_t} = \underline{a}V^{\rho,\sigma}\underline{X_t} + \underline{a}V^{\rho,\sigma}\Upsilon A(\mu)\underline{X_t} + \underline{B}V^{\rho,\sigma}\underline{X_t}. \tag{2.8}$$

Since $\underline{X_t} = \left(V^{\rho,\sigma}\right)^{-1}V^{\rho,\sigma}\underline{X_t} = \left(V^{\rho,\sigma}\right)^{-1}\underline{V}^{\rho,\sigma}$, from (2.8) we can conclude that

$$\underline{a}V^{\rho,\sigma}\Big(\Omega - Id - \Upsilon\Big)\Big(V^{\rho,\sigma}\Big)^{-1}\underline{V}^{\rho,\sigma} = \underline{B}\ \underline{V}^{\rho,\sigma}, \tag{2.9}$$

where Id is the infinite identity matrix.

To obtain algebraic form of the spectral Tau discretization for (1.1) it is sufficient to obtain a matrix form for the initial conditions $u_N^{(i)}(0) = u_i,\ i = 0, 1, ..., m-1$. Using the relation

$$u_N^{(i)}(t) = \underline{a}V^{\rho,\sigma}\eta^i\underline{X_t},$$

we can obtain the following matrix form for initial conditions

$$\underline{a}\Phi = \underline{u},\ \underline{u} = [u_0, u_1, ..., u_{m-1}]^T,\ \Phi = \Big[\Phi_i\Big]_{i=0}^{m-1}, \tag{2.10}$$

where $\Phi_i = V^{\rho,\sigma}\eta^i e_1$ is the ith column of Φ. η is a nonsingular matrix with only non zero elements $\eta_{i+1,i} = i, i = 1, 2, ...$;(see [19]).

Following (2.9) and (2.10) we obtain the system of algebraic equations

$$\begin{cases} \underline{a}V^{\rho,\sigma}\Big(\Omega - Id - \Upsilon\Big)\Big(V^{\rho,\sigma}\Big)^{-1}\underline{V}^{\rho,\sigma} = \underline{B}\ \underline{V}^{\rho,\sigma}, \\ \underline{a}\Phi = \underline{u}. \end{cases} \tag{2.11}$$

If we set $\Pi = V^{\rho,\sigma}\Big(\Omega - Id - \Upsilon\Big)\Big(V^{\rho,\sigma}\Big)^{-1}$, then, because of the orthogonality of $\Big[V_i^{\rho,\sigma}(t)\Big]_{i=0}^{\infty}$ projecting (2.11) onto $\Big[V_i^{\rho,\sigma}(t)\Big]_{i=0}^{N}$ we get

$$\underline{a}\Pi_j = B_j,\ j = 0, 1, 2, ...$$

Archive of SID

www.SID.ir

where Π_j is the jth column of Π. By setting

$$\tilde{\Pi} = [\Phi_0, \Phi_1, ..., \Phi_{m-1}, \Pi_0, \Pi_1, ..., \Pi_N], \tilde{B} = [u_0, u_1, ..., u_{m-1}, B_0, B_1, ..., B_N],$$

we obtain $\underline{a}\tilde{\Pi} = \tilde{B}$. We restrict this nonlinear system to its first $N+1$ columns and solve resulted system to find the unknown vector $[a_0, a_1, ...a_N]$.

3. Convergence analysis

The purpose of this section is to analyze convergence of the proposed method to the numerical solution of (1.1). From [3] we can see that some derivatives of the solutions of fractional differential equations have discontinuity at the origin. As discussed in [15] the solutions of fractional differential equations belong to the space $C^m_{-1}(I)$, where

$$C^m_{-1}(I) := \Big\{u | u \in C^{m-1}(I),\ u^{(m)}(t) = t^p \tilde{u}(t); \quad p > -1,\ \tilde{u} \in C(I)\Big\}.$$

Previously, some authors provided convergence analysis for the numerical solution of fractional differential equations using spectral methods but their convergence theorems are based on the restrictive assumptions on the exact solution. Ghoreishi and Mokhtary [8] proposed spectral Collocation method based on the Jacobi polynomials for solving linear fractional differential equations and error analysis of the proposed method was discussed. To recover the spectral rate of convergence, authors changed the main equation into a new equation which has smooth solution and Collocation scheme with spectral rate of convergence for the new equation was presented. Ghoreishi and Yazdani [9], provided a convergence analysis for approximation of linear multi order fractional differential equations with smooth solution using the Tau method. In [16], Mokhtary and Ghoreishi proved that if solutions of nonlinear fractional Integro differential equations were sufficiently smooth then their Tau solutions have exponential rate of convergence.

Thus providing a suitable convergence analysis to the spectral solutions of fractional differential equations with the exact solutions belong to the space $C^m_{-1}(I)$ is important and a new problem in the literature.

In this section, we study convergence behaviour of the spectral Tau method to the numerical solutions of (1.1) with the exact solutions belong to the space $C^m_{-1}(I)$. In this section C denotes a generic positive constant that is independent of N.

Archive of SID

www.SID.ir

$B^k_{\rho,\sigma}(I)$ denotes the non-uniform Jacobi-Sobolev space of all functions $u(t)$ on I such that $u(t)$ and its derivatives of order l are in $L^2_{\rho+l,\sigma+l}(I)$ for $0 \le l \le k$, where $L^2_{\rho,\sigma}(I)$ is the weighted L^2 space of all functions $u(t) : I \to \mathbb{R}$ with $||u(t)||_{\rho,\sigma} < \infty$, and

$$||u(t)||^2_{\rho,\sigma} = \int_I u^2(t)\ w^{\rho,\sigma}(t)dt.$$

We define the following norm for the space $B^k_{\rho,\sigma}(I)$

$$||u||_{B^k_{\rho,\sigma}(I)} = \Big(\sum_{l=0}^{k} ||u^{(l)}||^2_{\rho+l,\sigma+l}\Big)^{\frac{1}{2}}.$$

We recall that the L^∞-norm of a function u over I is defined as follows

$$\|u\|_\infty = \sup_{x\in I} |u(t)|.$$

The following estimates hold(see [24], [1])

$$\|u(t) - u_N(t)\|_{B^l_{\rho,\sigma}(I)} \le CN^{l-k}\|u^{(k)}\|_{\rho+k,\sigma+k}, \tag{3.1}$$

for $u \in B^k_{\rho,\sigma}(I)$ and $0 \le l \le k \le N+1$, as well as

$$\|u(t) - u_N(t)\|_\infty \le C\Big(1 + \Lambda_N\Big)N^{-k}\|u^{(k)}\|_\infty, \tag{3.2}$$

for $k \ge 0$, $\|u^{(k)}\|_\infty < \infty$ and

$$\Lambda_N = \begin{cases} \log(N), & -1 < \rho, \sigma \le -\frac{1}{2}, \\ N^{\gamma+\frac{1}{2}}, & \gamma = \max(\rho,\sigma), \text{otherwise}. \end{cases} \tag{3.3}$$

An important fact is that the shifted Jacobi polynomials $\Big\{V_i^{\rho,\sigma}(t)\Big\}_{i\ge 0}$ form a complete orthogonal system in $L^2_{\rho,\sigma}(\Lambda)$. Thus we define

$$P_N^{\rho,\sigma} := \text{span}\Big\{V_0^{\rho,\sigma}, V_1^{\rho,\sigma}, ..., V_N^{\rho,\sigma}\Big\},$$

and consider the orthogonal projection $\mathcal{P}_N^{\rho,\sigma} := L^2_{\rho,\sigma}(\Lambda) \to P_N^{\rho,\sigma}$ defined by

$$\Big(u - \mathcal{P}_N^{\rho,\sigma}u, v_N\Big)_{\rho,\sigma} = 0 \quad \forall v_N \in P_N^{\rho,\sigma}.$$

To prove the error estimate in L^2-norm, we need the generalized Hardy's inequality:

Archive of SID

www.SID.ir

Lemma 3.1. *(generalized Hardy's inequality [9])For all measurable function $f \geq 0$, the following generalized Hardy's inequality*

$$(\int_a^b |(\lambda f)(t)|^q w_1(t)dt)^{1/q} \leq C(\int_a^b |f(t)|^p w_2(t)dt)^{1/p},$$

holds if and only if

$$\sup_{a<t<b} \left(\int_t^b w_1(t)dt\right)^{1/q} \left(\int_a^t w_2^{1-p'}(t)\right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1},$$

for $1 < p \leq q < \infty$. Here, λ is an operator of the form

$$(\lambda f)(t) = \int_a^t k(t,s)f(s)ds,$$

with $k(t,s)$ a given kernel, w_1, w_2 weight functions, and $-\infty \leq a < b \leq \infty$.

Now we state and prove the main result of this section regarding the error estimate of the proposed method for the numerical solution of (1.1).

Theorem 3.2. *Assume the approximated solution $u_N(t)$ of the form (2.2) is given by the proposed spectral Tau scheme in the previous section. If $A(t)$ is continuous and $B(t) \in L^2_{\rho,\sigma}(I)$, then there exist parameters $\rho, \sigma \in (-1,1)$ such that for sufficiently large N, we have*

$$\|e_N\|^2_{\rho,\sigma} \leq C_1 N^{-s} \|u^{(s)}\|_{\rho+s,\sigma+s}\Big(\mathcal{F}_0(u) + \mathcal{F}_1(u) + \mathcal{F}_2(u)\Big).$$

where $u(t) \in B^s_{\rho,\sigma}(I)$ for $s \geq m$, $e_N(t) = u(t) - u_N(t)$ and

$$\begin{aligned}
\mathcal{F}_0(u) &= C_2 N^{-s}\Big(3 + \Lambda_N\Big)\|A(t)\|_\infty \|u\|_\infty \|u^{(s)}\|_{\rho+s,\sigma+s},\\
\mathcal{F}_1(u) &= C_4 N^{m-s}\|u^{(s)}\|_{\rho+s,\sigma+s},\\
\mathcal{F}_2(u) &= C_5\Big(\Big(\|D^\alpha u\|_{\rho,\sigma} + \mathcal{F}_1(u)\Big) + \Big(\|u\|_{\rho,\sigma} + N^{-s}\|u^{(s)}\|_{\rho+s,\sigma+s}\Big)\\
&+ \|A\|_\infty\Big((1+\Lambda_N)\|u\|_\infty\Big)^2 + \|B\|_{\rho,\sigma}\Big).
\end{aligned}$$

Proof. Consider the residual function

$$R_N(t) = D^\alpha u_N(t) - u_N(t) - A(t)u_N^2(t) - B(t). \tag{3.4}$$

Archive of SID

www.SID.ir

According to the proposed method we have

$$\Big(R_N(t), V_i^{\rho,\sigma}(t)\Big)_{\rho,\sigma} = 0, \quad 0 \le i \le,$$

which yields

$$\text{(3.5)} \qquad \mathcal{P}_N^{\rho,\sigma}(R_N) = \sum_{i=0}^{N} \frac{\Big(R_N, V_i^{\rho,\sigma}\Big)_{\rho,\sigma}}{\|V_i^{\rho,\sigma}\|_{\rho,\sigma}^2} V_i^{\rho,\sigma}(t) = 0.$$

Subtracting (1.1) from (3.5), we get

$$\text{(3.6)} \qquad D^\alpha u(t) - u(t) - A(t)u^2(t) - B(t) - \mathcal{P}_N^{\rho,\sigma}(R_N) = 0.$$

By some simple calculation we can rewrite (3.6) as

$$\text{(3.7)} \quad \begin{aligned} |e_N(t)| &= |D^\alpha e_N - A(t)\Big(u^2(t) - u_N^2(t)\Big) - e_{\mathcal{P}_N^{\rho,\sigma}}(R_N)| \\ &\le |D^\alpha e_N| + |A(t)\Big(u^2(t) - u_N^2(t)\Big)| + |e_{\mathcal{P}_N^{\rho,\sigma}}(R_N)|, \end{aligned}$$

where $e_{\mathcal{P}_N^{\rho,\sigma}}(R_N) = R_N(t) - \mathcal{P}_N^{\rho,\sigma}(R_N)$ is the truncation error of a Jacobi series.

Since $\mathcal{P}_N^{\rho,\sigma}(R_N) = 0$, we have

$$\text{(3.8)} \qquad \begin{cases} e_{\mathcal{P}_N^{\rho,\sigma}}(R_N) = R_N(t) - \mathcal{P}_N^{\rho,\sigma}(R_N) = R_N(t), \\ u^2(t) - u_N^2(t) = (u - u_N)(u + u_N) = 2ue_N - e_N^2. \end{cases}$$

Putting (3.8) into (3.7) we get

$$\text{(3.9)} \qquad |e_N(t)| \le |D^\alpha e_N| + |A(t)\Big(2ue_N - e_N^2\Big)| + |R_N|.$$

Now we multiply two sides of (3.9) by $|e_N(t)|w^{\rho,\sigma}(t)$ and integrate over I. Thus

$$\text{(3.10)} \quad \begin{aligned} \Big(|e_N|, |e_N|\Big)_{\rho,\sigma} \le \; & \Big(|D^\alpha e_N|, |e_N|\Big)_{\rho,\sigma} + \Big(|A(t)\Big(2ue_N - e_N^2\Big)|, |e_N|\Big)_{\rho,\sigma} \\ & + \Big(|R_N|, |e_N|\Big)_{\rho,\sigma}. \end{aligned}$$

Using $(|e_N|, |e_N|)_{\rho,\sigma} = \|e_N\|_{\rho,\sigma}^2$ and the weighted Cauchy-Schwartz inequality we rewrite (3.10) as

(3.11)

$$\|e_N\|_{\rho,\sigma}^2 \le \Big(\|D^\alpha e_N\|_{\rho,\sigma} + \|R_N\|_{\rho,\sigma}\Big)\|e_N\|_{\rho,\sigma} + \Big(|A(t)\Big(2ue_N - e_N^2\Big)|, |e_N|\Big)_{\rho,\sigma}.$$

Archive of SID

www.SID.ir

On the other hand we have

(3.12)
$$\Big(|A(t)\Big(2ue_N - e_N^2\Big)|, |e_N|\Big)_{\rho,\sigma} = \int_I \Big(|A(t)\Big(2ue_N - e_N^2\Big)||e_N|\Big) w^{\rho,\sigma}(t)dt$$
$$\leq \int_I |A(t)|\Big(|2u(t)||e_N|^2 + |e_N|^3\Big) w^{\rho,\sigma}(t)dt$$
$$\leq \|A(t)\|_\infty \Big(2\|u\|_\infty + \|e_N\|_\infty\Big)\|e_N\|^2_{\rho,\sigma}.$$

By inserting (3.12) in (3.11) and using (3.1) with $l = 0$ for $\|e_N\|_{\rho,\sigma}$ and applying (3.2) with $k = 0$ for $\|e_N\|_\infty$ we conclude

$$\text{(3.13)}\quad \|e_N\|^2_{\rho,\sigma} \leq C_1 N^{-s}\|u^{(s)}\|_{\rho+s,\sigma+s}\Big(\|D^\alpha e_N\|_{\rho,\sigma} + \mathcal{F}_0(u) + \|R_N\|_{\rho,\sigma}\Big),$$

where

$$\mathcal{F}_0(u) = C_2 N^{-s}\Big(3 + \Lambda_N\Big)\|A(t)\|_\infty \|u\|_\infty \|u^{(s)}\|_{\rho+s,\sigma+s}.$$

Now, it is sufficient to derive suitable bounds for $\|D^\alpha e_N\|_{\rho,\sigma}$ and $\|R_N\|_{\rho,\sigma}$. First, we consider $\|D^\alpha e_N\|_{\rho,\sigma}$. To this end, if $\rho, \sigma \in (-1, 1)$, using Hardy inequality(Lemma 3.1) with $w_1(t) = 1$, $w_2(t) = w^{\rho-\alpha,\sigma-\alpha}(t)$ and $\hat{K}(t, s) = (t - s)^{m-\alpha-1}(w^{\frac{m+\alpha}{2},\frac{m+\alpha}{2}}(s))^{-1}$ we may write

$$\text{(3.14)}\quad \|D^\alpha e_N\|_{\rho,\sigma} = \|\frac{1}{\Gamma(m-\alpha)}\int_0^t \hat{K}(t, s)e_N^{(m)}(s)w^{\frac{m+\alpha}{2},\frac{m+\alpha}{2}}(s)ds\|_{\rho,\sigma}$$
$$\leq C_3\|w^{\frac{m+\alpha}{2},\frac{m+\alpha}{2}} e_N^{(m)}(t)\|_{\rho-\alpha,\sigma-\alpha}$$
$$\leq C_3\|e_N^{(m)}(t)\|_{\rho+m,\sigma+m} \leq C_3\|e_N\|_{B^m_{\rho,\sigma}(I)}.$$

Putting (3.1) in (3.14) we get

$$\text{(3.15)}\qquad \|D^\alpha e_N\|_{\rho,\sigma} \leq \mathcal{F}_1(u), \quad \mathcal{F}_1(u) = C_4 N^{m-s}\|u^{(s)}\|_{\rho+s,\sigma+s}.$$

By inserting (3.15) in (3.13), we obtain

$$\text{(3.16)}\qquad \|e_N\|^2_{\rho,\sigma} \leq C_1 N^{-s}\|u^{(s)}\|_{\rho+s,\sigma+s}\Big(\mathcal{F}_1(u) + \mathcal{F}_0(u) + \|R_N\|_{\rho,\sigma}\Big).$$

Next, we find a suitable bound for $\|R_N\|_{\rho,\sigma}$. From (3.4) we have

$$\|R_N\|_{\rho,\sigma} \leq \|D^\alpha u_N\|_{\rho,\sigma} + \|u_N\|_{\rho,\sigma} + \|A(t)u_N^2\|_{\rho,\sigma} + \|B(t)\|_{\rho,\sigma}.$$

Archive of SID

www.SID.ir

Since $u_N(t) = u(t) - e_N(t)$, we can rewrite the above equation as

$$\|R_N\|_{\rho,\sigma} \leq \Big(\|D^\alpha u\|_{\rho,\sigma} + \|D^\alpha e_N\|_{\rho,\sigma}\Big) + \Big(\|u\|_{\rho,\sigma} + \|e_N\|_{\rho,\sigma}\Big)$$
$$+ \quad \|A(t)\Big(u - e_N\Big)^2\|_{\rho,\sigma} + \|B(t)\|_{\rho,\sigma}. \tag{3.17}$$

Using the relation

$$\|\Big(u - e_N\Big)^2\|_{\rho,\sigma} \leq \|\Big(u - e_N\Big)^2\|_\infty \leq \Big(\|u\|_\infty + \|e_N\|_\infty\Big)^2,$$

in (3.17) we get

$$\|R_N\|_{\rho,\sigma} \leq \Big(\|D^\alpha u\|_{\rho,\sigma} + \|D^\alpha e_N\|_{\rho,\sigma}\Big) + \Big(\|u\|_{\rho,\sigma} + \|e_N\|_{\rho,\sigma}\Big)$$
$$+ \quad \|A(t)\|_\infty\Big(\|u\|_\infty + \|e_N\|_\infty\Big)^2 + \|B(t)\|_{\rho,\sigma}. \tag{3.18}$$

Finally, by applying (3.15) in $\|D^\alpha e_N\|_{\rho,\sigma}$ and using (3.1) in $\|e_N\|_{\rho,\sigma}$ with $l = 0$ and adopting (3.2) in $\|e_N\|_\infty$ with $k = 0$, we rewrite (3.18) as

$$\|R_N\|_{\rho,\sigma} \leq \mathcal{F}_2(u), \tag{3.19}$$

where

$$\mathcal{F}_2(u) = C_5\Big(\Big(\|D^\alpha u\|_{\rho,\sigma} + \mathcal{F}_1(u)\Big) + \Big(\|u\|_{\rho,\sigma} + N^{-s}\|u^{(s)}\|_{\rho+s,\sigma+s}\Big)$$
$$+ \quad \|A\|_\infty\Big((1 + \Lambda_N)\|u\|_\infty\Big)^2 + \|B\|_{\rho,\sigma}\Big).$$

The desired result can be achieved by inserting (3.19) into (3.16). □

Remark 3.3. *It can be easily seen that if $u(t) \in C^s_{-1}(I)$, $s \geq m$, there exist parameters $\rho, \sigma \in (-1, 1)$, such that $u(t) \in B^s_{\rho,\sigma}(I)$. As a sequence, the convergence of (1.1) with exact solutions $u(t) \in C^s_{-1}(I)$, $s \geq m$ can be concluded from Theorem 3.2. It is trivial that for larger values of s we can expect the higher rate of convergence.*

4. Numerical results

In this section, we report numerical results of two examples, selected through (1.1), solved by the proposed method using Chebyshev($\rho, \sigma = -\frac{1}{2}$) and Legendre ($\rho, \sigma = 0$) polynomials. the numerical results obtained, confirm the theoretical prediction of Theorem 3.2. All calculations were performed on a PC running Mathematica software. In tables

Archive of SID

www.SID.ir

"Numerical Error" always refers to the weighted L^2-norm of the obtained error function. In all cases, any non-polynomial coefficient was replaced by its orthogonal expansion.

Example 4.1. *Consider fractional Riccati differential equation*

$$D^{\frac{3}{2}}u(t) = \cot(t)u^2(t) + B(t), \quad u(0) = 0, u'(0) = 1$$

in which $B(t) = -\frac{4t^{\frac{3}{2}} F_{1,2}\left(1;\{\frac{5}{4},\frac{7}{4}\};-\frac{t^2}{4}\right)}{3\sqrt{\pi}} - \frac{1}{2}\sin(2t)$. $F_{p,q}(\{a1,...,a_p\};\{b1,...,bq\};z)$ *is the generalized Hypergeometric function. The exact solution is* $u(t) = \sin t$.

The numerical results obtained are given in Table 1 and Figure 1. The results confirm the exponential rate of convergence, as it has been proved in Theorem 3.2.

Table 1: The Tau approximation errors of example 4.1.

N	Numerical Error	
	Chebyshev bases	Legendre bases
5	1.76×10^{-6}	2.24×10^{-6}
7	1.37×10^{-9}	2.49×10^{-9}
9	9.48×10^{-13}	1.47×10^{-12}
11	2.44×10^{-15}	3.77×10^{-15}
13	1.33×10^{-15}	2.33×10^{-15}
15	2.99×10^{-16}	3.22×10^{-16}

Example 4.2. *Consider fractional Riccati differential equation*

$$D^{\frac{1}{2}}u(t) = u(t) + \sqrt{t}u^2(t) + B(t), \quad u(0) = 0,$$

in which $B(t) = -t^q(1 + t^{q+\frac{1}{2}}) + \frac{qt^{q-\frac{1}{2}}\Gamma(q)}{\Gamma(q+\frac{1}{2})}$. *The exact solution is* $u(t) = t^q, q > 0$.

Numerical results obtained for example 4.2 with the Chebyshev and the Legendre polynomials are presented in Table 2 and Figure 2. Figure 2, shows the rate of convergence for various values of q. Each part of the figure contains numerical errors for several values of N, which are plotted for a special value of $q = 1.2, 1.4, 1.6, 1.8$ in the weighted L^2 norm. Similar prediction in Theorem 3.2, numerical results obtained by applying the proposed method show that when q tends to the $q = 2$ (smooth

Archive of SID

www.SID.ir

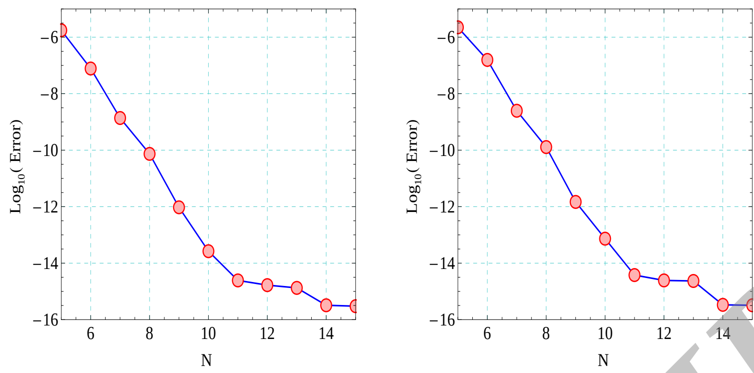

Figure 1. We display the errors of the example 4.1 for various values of N. The left and right hand side figures show numerical errors concerning the shifted Chebyshev and Legendre Tau method on I respectively.

solution) the rate of convergence increases. For $q = 2$, numerical results have not been presented, since the exact solution is obtained. As we see from Table 2, we have accurate numerical solutions, when q tends to 2.

Table 2: The Tau approximation errors for example 4.2 using two classical bases and different values of q and N.

N	Chebyshev bases			
	$q = 1.2$	$q = 1.4$	$q = 1.6$	$q = 1.8$
5	5.28×10^{-2}	3.53×10^{-2}	1.57×10^{-2}	4.68×10^{-3}
8	8.06×10^{-3}	4.46×10^{-3}	1.66×10^{-3}	4.07×10^{-4}
11	2.68×10^{-3}	1.29×10^{-3}	4.21×10^{-4}	9.01×10^{-5}
14	1.17×10^{-3}	5.11×10^{-4}	1.49×10^{-4}	2.91×10^{-5}
17	6.01×10^{-4}	2.43×10^{-4}	6.57×10^{-5}	1.17×10^{-5}
20	3.44×10^{-4}	1.31×10^{-4}	3.29×10^{-5}	5.51×10^{-6}

N	Legendre bases			
	$q = 1.2$	$q = 1.4$	$q = 1.6$	$q = 1.8$
5	7.05×10^{-2}	4.19×10^{-2}	2.81×10^{-2}	1.59×10^{-2}
8	1.53×10^{-2}	9.32×10^{-3}	3.81×10^{-3}	1.03×10^{-3}
11	5.52×10^{-3}	2.96×10^{-3}	1.06×10^{-3}	2.55×10^{-4}
14	2.51×10^{-3}	1.22×10^{-3}	3.98×10^{-4}	8.65×10^{-5}
17	1.32×10^{-3}	5.94×10^{-4}	1.79×10^{-4}	3.62×10^{-5}
20	7.68×10^{-4}	3.25×10^{-4}	9.23×10^{-5}	1.74×10^{-5}

Archive of SID
www.SID.ir

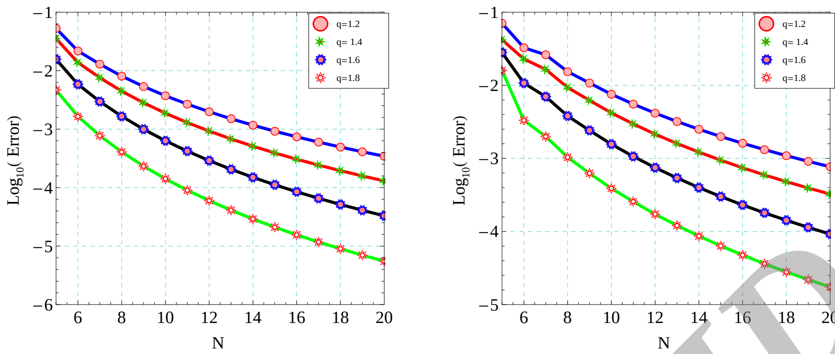

FIGURE 2. An illustration of the rate of convergence for the Tau method with various q. We display the errors of example 4.2 using Chebyshev (left) and Legendre bases (right).

5. Conclusion

The spectral Tau approximation was introduced to discuss the numerical solution of the fractional Riccati differential equation (1.1). We proved the convergence of the proposed method and obtained the error estimates in the weighted L^2-norm. These results were confirmed by two numerical examples. Results show that, this methodology is powerful in finding the numerical solutions of the fractional Riccati differential equations (1.1).

Archive of SID

References

1. C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, Spectral Methods, Fundamentals in Single Domains, Springer-Verlag, Berlin, 2006.
2. J. Cang, Y. Tan, H. Xu and S. Liao, Series solutions of non-linear Riccati differential equations with fractional order, *Chaos Solitons Fractals* **40** (2009), no. 1, 1–9.
3. K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, 2010.

www.SID.ir

4. K. Diethelm and G. Walz, Numerical solution of fractional order differential equations by extrapolation, *Numer. Algorithms* **16** (1997), no. 3-4, 231–253.
5. A. El-sayed and M. Gaber, The Adomian decomposition method for solving partial differential equations of fractional order in finite domains, *Phys. Lett. A* **359** (2006), no. 3, 175–182.
6. L. Galeone and R. Garrappa, On multistep methods for differential equations of fractional order, *Mediterr. J. Math.* **3** (2006), no. 3-4, 565–580.
7. A. Ghorbani, Toward a new analytical method for solving nonlinear fractional differential equations, *Comput. Methods Appl. Mech. Engrg.* **197** (2009), no. 49-50, 4173–4179.
8. F. Ghoreishi and P. Mokhtary, Spectral collocation method for multi-order fractional differential equations, *Int. J. Comput. Methods*, DOI: 10.1142/S0219876213500722, 23 pages.
9. F. Ghoreishi and S. Yazdani, An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis, *Comput. Math. Appl.* **61** (2011), no. 1, 30–43.
10. F. Ghoreishi and M. Hadizadeh, Numerical computation of the tau approximation for the Volterra-Hammerstein integral equations, *Numer. Algorithms* **52** (2009), no. 4, 541–559.
11. H. Jafari and S. Seifi, Solving a system of nonlinear fractional partial differential equations using homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simul.* **14** (2009), no. 5, 1962–1969.
12. H. Jafari and V. Daftardar-Gejji, Solving a system of nonlinear fractional differential equations using adomian decomposition, *J. Comput. Appl. Math.* **196** (2006), no. 2, 644–651.
13. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and application of fractional differential equations, Elsevier Science B.V., Amsterdam, 2006.
14. R. Lin and F. Liu, Fractional high order methods for the nonlinear fractional ordinary differential equation, *Nonlinear Anal.* **66** (2007), no. 4, 856–869.
15. Y. Luchko and R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, *Acta Math. Vietnam.* **24** (1999), no. 2, 207–233.
16. P. Mokhtary and F. Ghoreishi, The L^2-convergence of the legendre spectral tau matrix formulation for nonlinear fractional integro differential equations, *Numer. Algorithms* **58** (2011), no. 4, 475–496.
17. S. Momani and N. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, *Appl. Math. Comput.* **182** (2006), no. 2, 1083–1092.
18. K. B. Oldham and J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York-London, 1974.
19. E. Ortiz, H. Samara, An Operational approach to the Tau method for the numerical solution of nonlinear differential equations, *Computing* **27** (1981), no. 1, 15–25.
20. E. Ortiz and H. Samara, Numerical solution of differential eigenvalue problems with an operational approach to the Tau method, *Computing* **31** (1983), no. 2, 95–103.

Archive of SID

www.SID.ir

21. E. Ortiz and H. Samara, Numerical solution of partial differential equations with variable coefficients with an operational approach to the Tau method, *Comput. Math. Appl.* **10** (1984), no. 1, 5–13.
22. I. Podlubny, Fractional Differential Equations, Academic press, Inc., San Diego, 1999.
23. N. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, *Appl. Math. Comput.* **131** (2002), no. 2-3, 517–529.
24. J. Shen, T. Tang and L. L. Wang, Spectral Methods, Algorithms, Analysis and Applications, Springer, Heidelberg, 2011.

(Payam Mokhtary) Department of Mathematics, Sahand University of Technology, Tabriz, Iran
E-mail address: mokhtary.payam@gmail.com, mokhtary@sut.ac.ir

(Farideh Ghoreishi) Department of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
E-mail address: ghoreishif@kntu.ac.ir

Archive of SID

www.SID.ir