

Steering Control of an Underwater Vehicle

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Abstract—This paper presents a simple and systematic approach to steer an underwater vehicle model by considering two different cases: (i) when all actuators are functional, and (ii) when one actuator is not working. In first case, the model of an underwater vehicle is steered by employing a Lie bracket extension of the original system and the resulting feedback law is as a composition of a standard stabilizing feedback control for the extended system and a periodic continuation of a parameterized solution to an open loop, finite horizon control problem stated in the logarithmic coordinates of flows. In second case (which represents a physical example where second level Lie bracket is necessary for controllability), the original system is decomposed into two subsystems; one subsystem, which is fifth dimensional, steered by a similar approach used in case (i) and the second subsystem, which is one dimensional, steered by using sinusoidal inputs. The mixture of both type of control is utilized to steer the actual system. The synthesis method is general, in that it applies to a large class of drift free, completely controllable systems, for which the associated controllability Lie algebra is locally nilpotent.

Index Terms—Feedback stabilization, systems with drift, nonholonomic systems, nilpotent Lie algebra, locally nilpotent, Lyapunov function.

I. INTRODUCTION

THIS PAPER presents a simple solution to the steering problem for an underwater vehicle which represents a nonholonomic control system. Also an underwater vehicle model presents a physical example where second level Lie brackets are necessary for controllability. This type of vehicle is expected to perform a key role in automation of underwater missions for oceanographic observations, and in oil and mineral explorations, which motivates our interest.

A kinematics model of an underwater vehicle, as described by [1], involves six configuration variables and four inputs (velocities), of which three are the angular velocity components, and the fourth represents the forward velocity of the vehicle. If the body-fixed translational y and z velocities are assumed to be un-actuated, the vehicle exhibits nonholonomic behavior, for details see [2].

Feedback control of the underwater vehicle with this type of nonholonomic constraint was studied in [1]-[3]. In [4], Yoerger and Slotine applied sliding modes to trajectory control of such a vehicle. Due to the presence of the nonholonomic constraint, the kinematics model of the vehicle belongs to the class of systems which cannot be stabilized by continuous static feedback, see [5]. As demonstrated in [6], for this class of systems, the dependence of the stabilizing control law on time is

essential. Synthesis approaches have been presented in [7], [8] but rely heavily on the existence of suitable time-varying Lyapunov functions, which are often difficult to find.

In this article we present a simple and systematic approach for steering an underwater vehicle model by considering two different cases: (i) when all actuators are functional, and (ii) one actuator is not working. In case (i), the model of an underwater vehicle is steered by employing a Lie bracket extension of the original system (see [9], [10]) and an arbitrary Lyapunov function is used to construct a closed loop steering control for the extended system. This classical static feedback is then combined with a periodic continuation of a parameterized solution to an open loop steering problem for the comparison of flows of the original and extended systems. Since the controllability Lie algebra associated with this system is locally nilpotent, the latter can be recast as an open loop control problem for a finite set of the logarithmic coordinates of flows, see [3], [11]. In combination with the static, time invariant feedback for the extended system, the solution to this open loop problem delivers a time varying control which provides for periodic intersection of the trajectories of the controlled extended system and the original system. For steering the original system, the extended system trajectory serves as a reference.

In case (ii), the original system is decomposed into two subsystems. One subsystem, which is fifth dimensional, steered by a similar approach as used in case (i) and the second subsystem, which is one dimensional, is steered by using sinusoidal inputs, which are similar as given in [12]. The mixture of both type of control is used to steer the original system. The synthesis method is general, in that it applies to a large class of drift free, completely controllable systems, for which the associated controllability Lie algebra is locally nilpotent. The approach does not necessitate the conversion of the system model into a “chained form”, and thus does not rely on any special transformation techniques. By introducing approximate models often permits significant simplification of the differential equations describing the evolution of the logarithmic coordinates in the open-loop problem formulation (which are usually difficult to solve analytically).

II. A KINEMATICS MODEL OF UNDERWATER VEHICLE

In the derivation of the model of the underwater vehicle, two frames of reference are considered, as shown in Fig. 1 (for detail see [1]). The $o-XYZ$ is the inertial frame, while the local frame, $c-xyz$, is attached to the vehicle at its center of mass c . Six coordinates are used to describe the orientation. The $Z-Y-X$ Euler angles are denoted by (ϕ, θ, ψ) . When the angles are small, ϕ corresponds to

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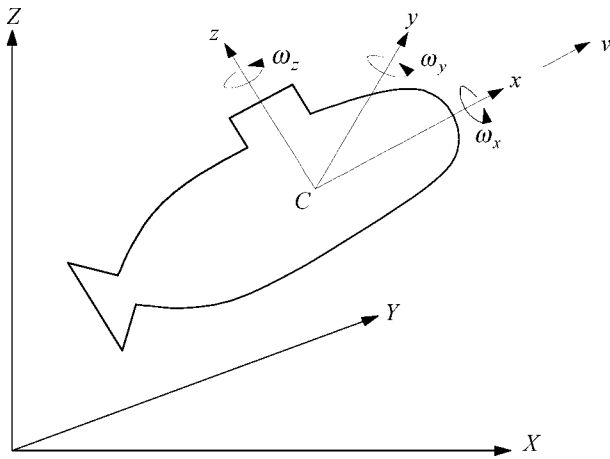


Fig. 1. Model of an underwater vehicle.

what is commonly called the roll motion, while θ and ψ correspond to the pitch and yaw motions, respectively.

As given in [1], it is assumed that the vehicle is moving with velocity v , whose direction is the $c-x$ axis in the local frame, so the components of this velocity along the x , y , and z axes are given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v \cos\psi \cos\theta \\ v \sin\psi \cos\theta \\ -v \sin\theta \end{bmatrix} \quad (1)$$

The relation between the time rate of the Euler angles and the angular velocity in the local frame, $\omega = (\omega_x, \omega_y, \omega_z)^T$, is given by, (see [1]):

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad (2)$$

Combining (1) and (2), and introducing a new set of state and control variables:

$$(z_1, z_2, z_3, z_4, z_5, z_6) = (x, y, z, \phi, \theta, \psi)$$

$$(u_1, u_2, u_3, u_4) = (v, \omega_x, \omega_y, \omega_z),$$

yields a kinematics model for the vehicle :

$$S1: \dot{z} = Z_1(z)u_1 + Z_2(z)u_2 + Z_3(z)u_3 + Z_4(z)u_4 \quad (3)$$

where

$$Z_1(z) \stackrel{def}{=} \begin{bmatrix} \cos z_6 \cos z_5 \\ \sin z_6 \cos z_5 \\ -\sin z_5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Z_2(z) \stackrel{def}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$Z_3(z) \stackrel{def}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin z_4 \tan z_5 \\ \cos z_4 \\ \sin z_4 \sec z_5 \end{bmatrix}, \quad Z_4(z) \stackrel{def}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos z_4 \tan z_5 \\ -\sin z_4 \\ \cos z_4 \sec z_5 \end{bmatrix}.$$

A. The Control Problem

Given a desired set point $z_{des} \in \mathfrak{R}^6$, construct a feedback strategy in terms of the controls $u_i: \mathfrak{R}^6 \rightarrow \mathfrak{R}, i=1, 2, \dots, 4$ such that the desired set point z_{des} is an attractive set for (3), so that there exists an $\varepsilon > 0$, such that $z(t; t_0, z_0) \rightarrow z_{des}$, as $t \rightarrow \infty$ for any initial condition $(t_0, z_0) \in \mathfrak{R}^+ \times B(z_{des}; \varepsilon)$.

Without the loss of generality, it is assumed that $z_{des} = 0$, which can be achieved by a suitable translation of the coordinate system.

B. Properties of the Kinematics Model (When All Actuators Are Functional)

The kinematics model of an underwater vehicle is given by (3) when all actuators are working and has the following important properties:

- [P1] The vector fields Z_1, Z_2, Z_3 & Z_4 are real analytic, and it can be shown that solutions to (3) exist for all times.
- [P2] The system defined by (3) is completely controllable on the manifold $M = \{z = (z_1, \dots, z_6) \in \mathfrak{R}^6 : |z_5| < \frac{\pi}{2}\}$ as it satisfies the LARC (Lie algebraic rank condition) for controllability on M , in that the Lie algebra, $L(Z_1, Z_2, Z_3, Z_4)$ spans \mathfrak{R}^6 at each point $z \in M$.
- [P3] The Lie algebra $L(Z_1, Z_2, Z_3, Z_4)$, generated by the vector fields Z_1, Z_2, Z_3 and Z_4 , is not nilpotent.

To verify property P2, it is sufficient to calculate the following Lie brackets:

$$Z_5(z) \stackrel{def}{=} [Z_1, Z_3](z) = \begin{bmatrix} \sin z_5 \cos z_6 \cos z_4 + \sin z_6 \sin z_4 \\ \sin z_6 \sin z_5 \cos z_4 - \cos z_6 \sin z_4 \\ \cos z_5 \cos z_4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Z_6(z) \stackrel{def}{=} [Z_1, Z_4](z) = \begin{bmatrix} -\sin z_5 \cos z_6 \sin z_4 + \sin z_6 \cos z_4 \\ -\sin z_6 \sin z_5 \sin z_4 - \cos z_6 \cos z_4 \\ -\cos z_5 \sin z_4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

It is then a straightforward task to verify that, if the motion of the system is restricted to the manifold M , then $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$ are linearly independent, which demonstrates the satisfaction of the LARC condition, in that:

$$\text{span}\{Z_1(z), \dots, Z_6(z)\} = \mathfrak{R}^6 \quad \forall z \in M \quad (4)$$

The Lie bracket multiplication table for $L(Z_1, Z_2, Z_3, Z_4)$:

$$\begin{aligned} [Z_1, Z_2] &= 0 & [Z_2, Z_3] &= Z_4 & [Z_2, Z_4] &= -Z_3 \\ [Z_3, Z_4] &= Z_2 & [Z_1, Z_5] &= [Z_1, Z_6] &= 0 \\ [Z_2, Z_5] &= Z_6 & [Z_2, Z_6] &= -Z_5 & [Z_3, Z_5] &= Z_1 \\ [Z_3, Z_6] &= [Z_4, Z_5] &= [Z_4, Z_6] &= [Z_5, Z_6] &= 0 \end{aligned}$$

shows that the Controllability Lie algebra $L(Z_1, Z_2, Z_3, Z_4)$ is finite dimensional but not nilpotent i.e. we cannot find an integer m such that

$$L(Z_1, Z_2, Z_3, Z_4) = L_m(Z_1, Z_2, Z_3, Z_4),$$

where $L_m(Z_1, Z_2, Z_3, Z_4)$ is Lie algebra containing all Lie brackets of level less than or equal to m .

The Lie algebra $L(Z_1, Z_2, Z_3, Z_4)$ is called the locally nilpotent if $L(Y_1, Y_2, Y_3, Y_4)$ is nilpotent, where Y_i is linearized form of the vector field Z_i , for $i=1, \dots, 4$.

III. APPROXIMATE MODEL

An approximation to system $S1$ is considered which gives nilpotent Controllability Lie algebra. Such an approximation is obtained as follows:

Linearize the nonlinear terms in the expression of the vector field Z_1 by using truncated Taylor series of order one i.e. substituting $\sin z \approx z$ and $\cos z \approx 1$.

Linearize the nonlinear terms in the expression of the vector fields Z_3 and Z_4 by using truncated Taylor series of order zero i.e. each term is evaluated at zero or substituting $\sin z \approx 0$ and $\cos z \approx 1$

$$\hat{S}1: \dot{z} = \sum_{i=1}^m Y_i(z) u_i(z), \quad z \in \mathfrak{R}^6 \quad (5)$$

$$Y_1(z) \stackrel{\text{def}}{=} \begin{bmatrix} 1 \\ z_6 \\ -z_5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Y_2(z) \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad Y_3(z) \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad Y_4(z) \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The approximate system $\hat{S}1$ is controllable since the LARC condition is satisfied as:

$$\text{span}\{Y_1(z), \dots, Y_6(z)\} = \mathfrak{R}^6, \quad \forall z \in \mathfrak{R}^6 \quad (6)$$

where the vector fields $Y_5(z)$ and $Y_6(z)$ are given by

$$Y_5(z) \stackrel{\text{def}}{=} [Y_1, Y_3](z) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Y_6(z) \stackrel{\text{def}}{=} [Y_1, Y_4](z) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The Lie brackets multiplication table for $L(Y_1, Y_2, Y_3, Y_4)$ is given by:

$$[Y_1, Y_3] = Y_5 \quad [Y_1, Y_4] = Y_6$$

$$[Y_j, Y_2] = 0, \quad j=1, \dots, 4$$

$$[Y_j, Y_5] = [Y_j, Y_6] = 0, \quad j=1, \dots, 6$$

so that $L(Y_1, Y_2, Y_3, Y_4)$ is nilpotent and hence $L(Z_1, Z_2, Z_3, Z_4)$ is locally nilpotent.

A. Extended Systems of $S1$ and $\hat{S}1$

The extended system of the system $S1$ as defined in [9], [10] is:

$$\dot{z} = \sum_{i=1}^4 Z_i(z) v_i(z) + \sum_{i=5}^6 Z_i(z) v_i(z), \quad z \in \mathfrak{R}^6 \quad (7)$$

where, Z_i $i=5, 6$ are the Lie brackets involve in $L(Z_1, Z_2, Z_3, Z_4)$. Similarly the extended system of approximate system $\hat{S}1$ is defined as:

$$\dot{z} = \sum_{i=1}^4 Y_i(z) v_i(z) + \sum_{i=5}^6 Y_i(z) v_i(z), \quad z \in \mathfrak{R}^6 \quad (8)$$

where, Y_i $i=5, 6$ are Lie brackets involve in $L(Y_1, Y_2, Y_3, Y_4)$.

THEOREM 1:

The extended system (8) can be made (locally) asymptotically stable by introducing the following feedback control:

$$v_i(z) \stackrel{\text{def}}{=} -L_{Y_i} V(z), \quad i=1, \dots, 6 \quad (9)$$

Proof: Let $V: \mathfrak{R}^6 \rightarrow \mathfrak{R}$ be any smooth, positive definite, decrescent and radially unbounded function with the origin as a unique stationary point. One simple choice is: $V(z) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=1}^6 z_i^2$, then along the controlled extended system trajectories we have

$$\frac{d}{dt} V(z) = \sum_{i=1}^6 L_{Y_i} V(z) v_i(z) = - \sum_{i=1}^6 \{L_{Y_i} V(z)\}^2 < 0 \quad \text{for } z \neq 0$$

$$\frac{d}{dx} V(z) = 0 \quad \text{for } z = 0$$

which is due to the fact that $\text{span}\{Y_1, Y_2, \dots, Y_6\} = \mathfrak{R}^6$. This completes the proof.

The discretization of the above control in time, with sufficiently high sampling frequency ($1/T$), does not prejudice stabilization in that if the feedback control (9) is substituted by the discretized control:

$$v_i^T(z(t)) \stackrel{\text{def}}{=} v_i^T(z(nT)), \quad t \in [nT, (n+1)T], \quad (10)$$

$$n=0, 1, \dots, \quad i=1, \dots, 6$$

then the latter also stabilizes the system if T is small enough. This leads to a parameterized, asymptotically stable, controlled extended system:

$$\dot{z} = \sum_{i=1}^6 Y_i(z) a_i \quad (11)$$

where $a_i = v_i^T(z(t))$, $i=1, \dots, 6$ are constant over each interval $[nT, (n+1)T]$.

THEOREM 2:

Suppose the controlled extended system (8) is exponentially stable. Then, for any compact region $R \subset M$ which contains the origin, there exists a constant $T > 0$ such that the discretized controlled extended system (11) is exponentially stable with region of attraction R (see [13]).

IV. THE TRAJECTORY INTERCEPTION PROBLEM

Find control functions $m_i(a, t)$, $i=1, 2, 3, 4$, in the class of functions which are continuous in $a = [a_1, a_2, \dots, a_6]$, and piece-wise continuous and locally bounded in t , such that for any initial condition $z(0) = z_0$ the trajectory

$z^a(t; z_0, 0)$ of the extended, parameterized system (11) intersects the trajectory $z^m(t; z_0, 0)$ of the approximate system (5) with controls $m_i, i = 1, 2, 3, 4$ i.e.

$$\dot{z} = \sum_{i=1}^4 Y_i(x) m_i(a, t) \tag{12}$$

precisely at time T , so that

$$z^a(T; z_0, 0) = z^m(T; z_0, 0) \tag{13}$$

This problem will be hereafter referred to as TIP problem.

THEOREM 3:

Suppose that a solution to the TIP problem can be found. Then, there exists an admissible time horizon T_{max} and a neighborhood of the origin R such that for any $T < T_{max}$, the time-varying feedback controls (see [13]):

$$u_i(t) \stackrel{def}{=} m_i(v^T(x), t), \quad i = 1, 2, 3, 4, \quad v^T \stackrel{def}{=} [v_1^T, \dots, v_6^T]$$

are asymptotically stabilizing the approximate system (6) with the region of attraction R .

A. The TIP in Logarithmic Coordinates of Flows

To solve the TIP; as the algebra $L(Y_1, Y_2, Y_3, Y_4)$ is nilpotent, it is possible to employ the formalism of [11] and consider a formal equation for the evolution of flows for the approximate model (6):

$$\dot{U}(t) = U(t) \sum_{i=1}^6 Y_i w_i, \quad U(0) = I \tag{14}$$

where the solution of (14) is known to represent the flow of the dynamic system

$$\dot{z}(t) = \sum_{i=1}^6 Y_i w_i$$

whose controllability Lie algebra $L(Y_1, Y_2, \dots, Y_m)$ is nilpotent. Such solution can be expressed locally as (see [11]):

$$U(t) = \prod_{i=1}^6 e^{\gamma_i(t) Y_i} \tag{15}$$

where the functions $\gamma_i, i = 1, 2, \dots, 6$ are the logarithmic coordinates for this flow and can be computed as follows.

Equation (15) is first substituted into (14) which yields:

$$\begin{aligned} Y_1 a_1 + Y_2 a_2 + \dots + Y_6 a_6 &= \dot{\gamma}_1 Y_1 + \dot{\gamma}_2 (e^{\gamma_1 ad Y_1}) Y_2 \\ &+ \dot{\gamma}_3 (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2}) Y_3 \\ &+ \dot{\gamma}_4 (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3}) Y_4 \\ &+ \dot{\gamma}_5 (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} e^{\gamma_4 ad Y_4}) Y_5 \\ &+ \dot{\gamma}_6 (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} e^{\gamma_4 ad Y_4} e^{\gamma_5 ad Y_5}) Y_6 \end{aligned} \tag{16}$$

where $(e^{ad X}) Y \stackrel{def}{=} e^X Y e^{-X}$ and $(ad X) Y \stackrel{def}{=} [X, Y]$.

Employing the Campbell-Baker-Hausdorff formula:

$$(e^{ad X}) Y = e^X Y e^{-X} = Y + [X, Y] + [X, [X, Y]]/2! + \dots$$

which gives

$$\begin{aligned} (e^{\gamma_1 ad Y_1}) Y_2 &= e^{\gamma_1 Y_1} Y_2 e^{-\gamma_1 Y_1} = Y_2 + (\gamma_1 / 1!) [Y_1, Y_2] \\ &+ (\gamma_1^2 / 2!) [Y_1, [Y_1, Y_2]] + \dots = Y_2 \end{aligned} \tag{17}$$

Similarly

$$\begin{aligned} (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2}) Y_3 &= e^{\gamma_1 ad Y_1} (e^{\gamma_2 ad Y_2} Y_3) \\ &= e^{\gamma_1 ad Y_1} (Y_3) = Y_3 + \gamma_1 [Y_1, Y_3] = Y_3 + \gamma_1 Y_5 \end{aligned} \tag{18}$$

$$\begin{aligned} (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3}) Y_4 &= e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} (e^{\gamma_3 ad Y_3} Y_4) \\ &= e^{\gamma_1 ad Y_1} (e^{\gamma_2 ad Y_2} Y_4) = e^{\gamma_1 ad Y_1} (Y_4) \\ &= Y_4 + \gamma_1 [Y_1, Y_4] = Y_4 + \gamma_1 Y_6 \end{aligned} \tag{19}$$

$$\begin{aligned} (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} e^{\gamma_4 ad Y_4}) Y_5 &= e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} e^{\gamma_4 ad Y_4} (Y_5) \\ &= e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} (Y_5) \\ &= e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} (Y_5) = e^{\gamma_1 ad Y_1} (Y_5) = Y_5 \end{aligned} \tag{20}$$

and

$$\begin{aligned} (e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} e^{\gamma_4 ad Y_4} e^{\gamma_5 ad Y_5}) Y_6 &= e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} e^{\gamma_4 ad Y_4} (e^{\gamma_5 ad Y_5} Y_6) \\ &= e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} e^{\gamma_3 ad Y_3} (Y_6) \\ &= e^{\gamma_1 ad Y_1} e^{\gamma_2 ad Y_2} (Y_6) = e^{\gamma_1 ad Y_1} (Y_6) = Y_6 \end{aligned} \tag{21}$$

Substituting (17)-(21) in (14) and comparing the coefficients of $Y_i, i = 1, 2, \dots, 6$ yields the following equations for the evaluation of the logarithmic coordinates $\gamma_i, i = 1, 2, \dots, 6$:

$$\begin{aligned} \dot{\gamma}_1 &= w_1 \\ \dot{\gamma}_2 &= w_2 \\ \dot{\gamma}_3 &= w_3 \\ \dot{\gamma}_4 &= w_4 \\ \dot{\gamma}_5 &= -\gamma_1 w_3 + w_5 \\ \dot{\gamma}_6 &= -\gamma_1 w_4 + w_6 \\ \text{with } \gamma_i(0) &= 0, \quad i = 1, 2, \dots, 6 \end{aligned} \tag{22}$$

The TIP in logarithmic coordinates now takes the form of a trajectory interception problem for the following two control systems

$$\begin{array}{ll} CS1: \dot{\gamma}_1 = m_1 & CS2: \dot{\gamma}_1 = a_1 \\ \dot{\gamma}_2 = m_2 & \dot{\gamma}_2 = a_2 \\ \dot{\gamma}_3 = m_3 & \dot{\gamma}_3 = a_3 \\ \dot{\gamma}_4 = m_4 & \dot{\gamma}_4 = a_4 \\ \dot{\gamma}_5 = -\gamma_1 m_3 & \dot{\gamma}_5 = -\gamma_1 a_3 + a_5 \\ \dot{\gamma}_6 = -\gamma_1 m_4 & \dot{\gamma}_6 = -\gamma_1 a_4 + a_6 \end{array}$$

with initial conditions with $\gamma_i(0) = 0, i = 1, 2, \dots, 6$.

Complete controllability of CS1 and CS2 guarantees existence of solutions to the TIP.

One such solution can be calculated as follows. Motivated by the fact that a flow of $\dot{z} = [g_1, g_2]$ can be approximated by the flow of $\dot{z} = c g_1 \sin(2\pi t/T) + c g_2 \cos(2\pi t/T)$, where c is some constant, we seek the controls $m_i(a, t), i = 1, \dots, 4$ in the form

$$\begin{aligned} m_1 &= (c_1 + c_5 \sin \frac{2\pi t}{T} + c_6 \cos \frac{2\pi t}{T}), \quad m_2 = c_2, \\ m_3 &= (c_3 + c_5 \cos \frac{2\pi t}{T}), \quad m_4 = (c_4 + c_6 \sin \frac{2\pi t}{T}) \end{aligned} \tag{23}$$

where $c_i, i = 1, 2, \dots, 6$ are some unknown coefficients.

The above are substituted into CS1, and the systems

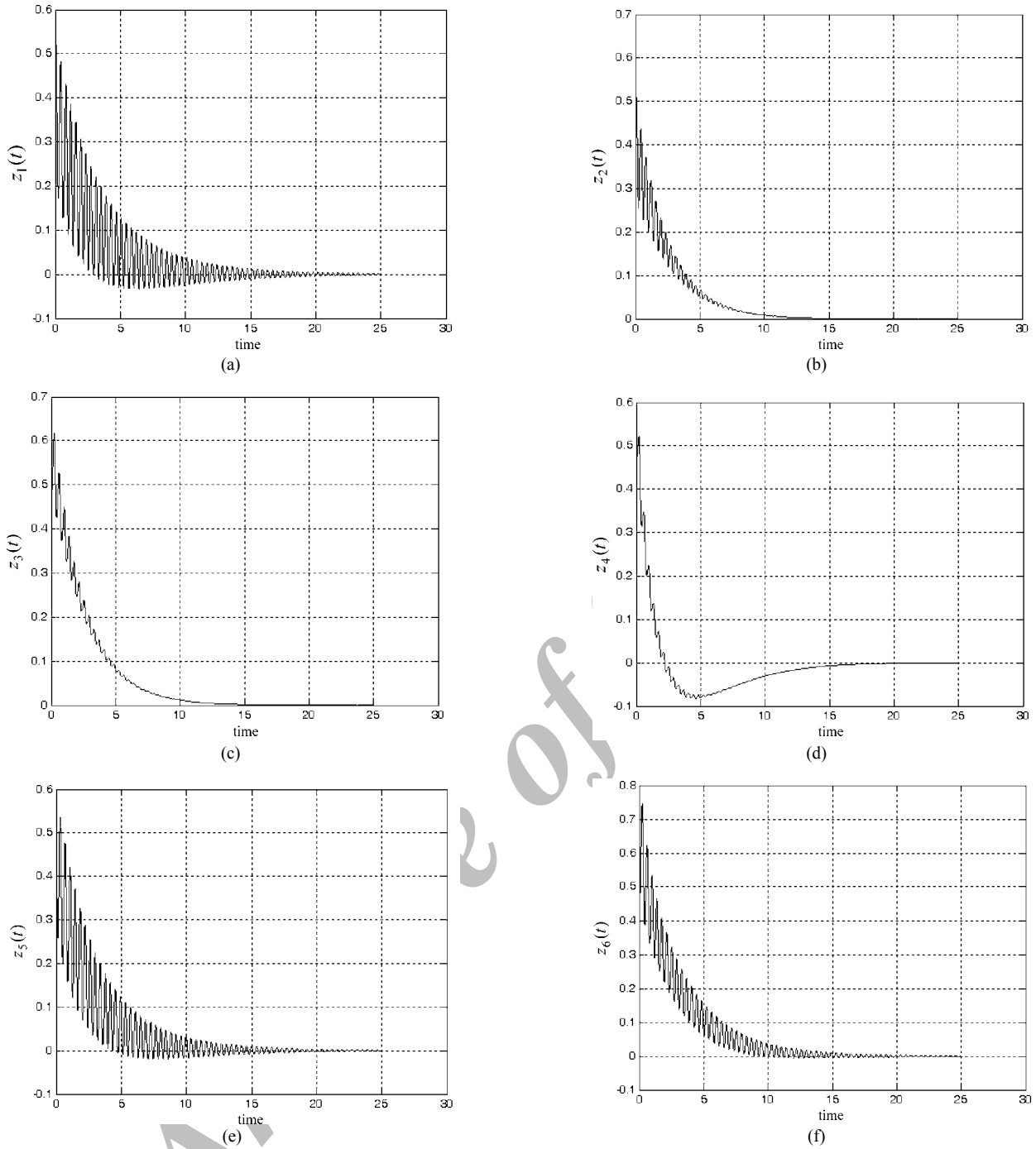


Fig. 2. Underwater vehicle model 1: plots of the controlled state trajectories $t \mapsto (z_1(t), \dots, z_6(t))$ versus time.

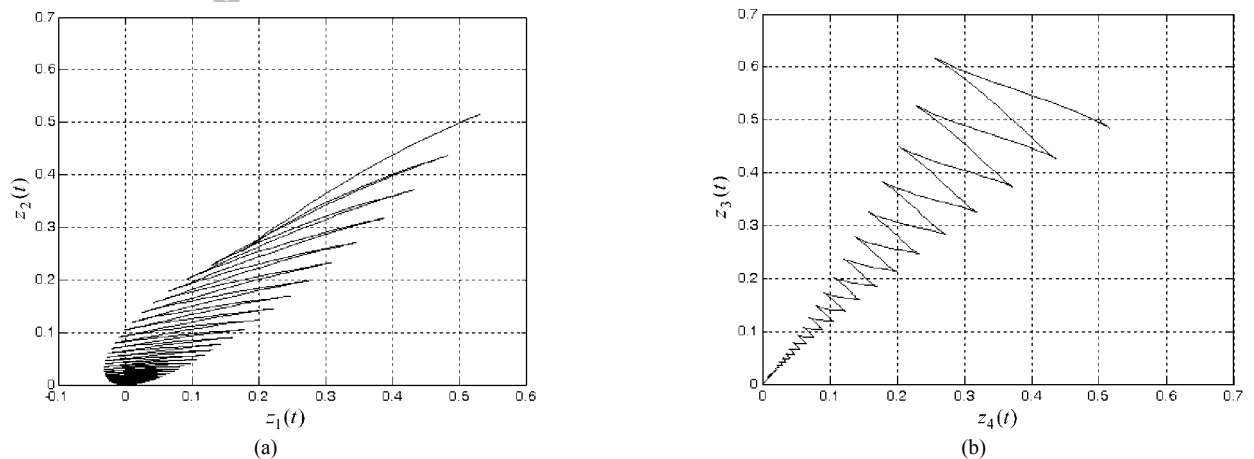


Fig. 3. Underwater vehicle model 1: plots of the controlled state trajectories $z_2(t)$ versus $z_1(t)$, and $z_3(t)$ versus $z_4(t)$.

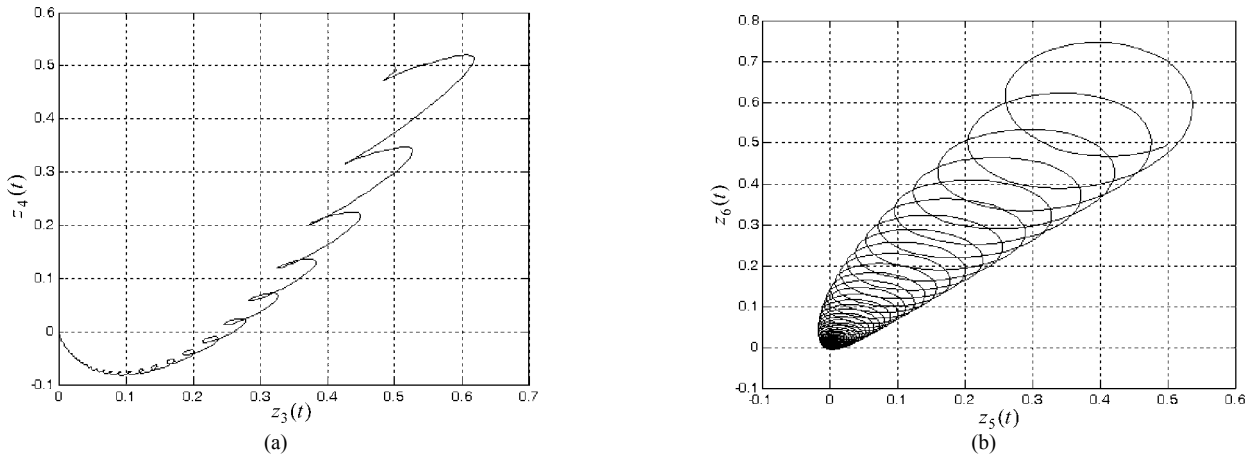


Fig. 4. Underwater vehicle model 1: plots of the controlled state trajectories, $z_4(t)$ versus $z_3(t)$ and $z_6(t)$ versus $z_5(t)$.

CS1 and CS2 are integrated symbolically, using Mathematica[®], to yield respective solutions $\gamma^a(t)$ and $\gamma^m(t)$ in terms of parameters a and c . The equation $\gamma^m(T) = \gamma^a(T)$ is then also solved symbolically to deliver the values for the unknown coefficients c_i in terms of their counterparts a_i :

$$\begin{aligned} c_1 &= a_1, & c_2 &= a_2, & c_3 &= a_3, & c_4 &= a_4 \\ c_5 &= \pm 3.54491 \sqrt{\frac{a_5}{T}}, \\ c_6 &= (2a_1 T^2 \pm \sqrt{(-50.2655a_6 T^3 + 4a_1^2 T^4)}) / 2T^2 \end{aligned}$$

This reflects that two solutions are found. In simulation we used the positive values of c_i .

Therefore by TIP the following control stabilize the system S1:

$$\begin{aligned} u_1 &= (c_1 + c_5 \sin \frac{2\pi t}{T} + c_6 \cos \frac{2\pi t}{T}), & u_2 &= c_2, \\ u_3 &= (c_3 + c_5 \cos \frac{2\pi t}{T}), & u_4 &= (c_4 + c_6 \sin \frac{2\pi t}{T}) \end{aligned} \quad (24)$$

The controls given in (24) can be utilized to stabilize the system S1 by just replacing a_i to b_i , where $b_i = \hat{v}_i^T(z(t))$, and $\hat{v} = -L_z V(x)$.

$$\begin{aligned} u_1 &= (b_1 + d_5 \sin \frac{2\pi t}{T} + d_6 \cos \frac{2\pi t}{T}), & u_2 &= b_2, \\ u_3 &= (b_3 + d_5 \cos \frac{2\pi t}{T}), & u_4 &= (b_4 + d_6 \sin \frac{2\pi t}{T}) \end{aligned} \quad (25)$$

where

$$\begin{aligned} d_5 &= \pm 3.54491 \sqrt{\frac{b_5}{T}}, \\ d_6 &= (2b_1 T^2 \pm \sqrt{(-50.2655b_6 T^3 + 4b_1^2 T^4)}) / 2T^2. \end{aligned}$$

COROLLARY:

If the controlled extended system possesses a sufficiently wide stability margin, the controls given in (24) and (25) provide an asymptotically stabilizing feedback control for the approximate model S1 and exact model S1, respectively (see [13]).

The controls given in (25), as applied to the model of the underwater vehicle (3), result in controlled trajectories depicted in Figs. 2 to 5.

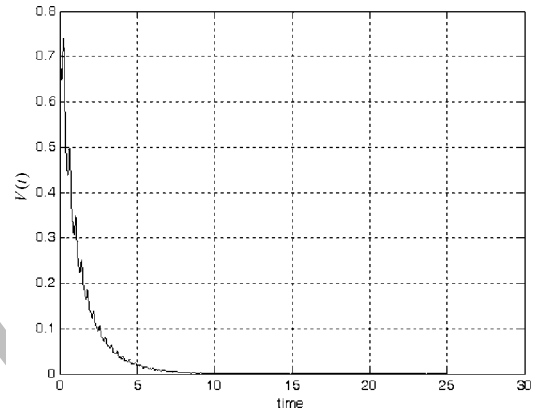


Fig. 5. Underwater vehicle model 1: plot of the Lyapunov function $V(z(t)) = \frac{1}{2} \sum_{i=1}^6 z_i^2(t)$ versus time.

V. CASE II: (WHEN ONE ACTUATOR IS NOT WORKING)

A model of an underwater vehicle (3) is considered in which the actuator corresponding to control u_4 fails to be operational. The model of the underwater vehicle with such reduced number of controls, is referred to be Model 2 which is an example of a physical system where second level Lie brackets are necessary for controllability. By using the transformations

$$x = (x_1, x_2, x_3, x_4, x_5, x_6) \stackrel{def}{=} (z_5, z_4, z_1, z_6, z_3, z_2)$$

and $(v_1, v_2, v_3, 0) = (u_3, u_2, u_1, u_4)$ in (3) gives:

$$\begin{aligned} \text{Model 2: } \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} &= \begin{bmatrix} \cos x_2 \\ \sin x_2 \tan x_1 \\ 0 \\ \sin x_2 \sec x_1 \\ 0 \\ 0 \end{bmatrix} v_1 + \\ &v_2 + \begin{bmatrix} 0 \\ 0 \\ \cos x_4 \cos x_1 \\ 0 \\ -\sin x_1 \\ \sin x_4 \cos x_1 \end{bmatrix} v_3 \end{aligned} \quad (26)$$

$$\stackrel{def}{=} Z_1(x) v_1 + Z_2(x) v_2 + Z_3(x) v_3$$

where

$$Z_1(x) \stackrel{\text{def}}{=} \begin{bmatrix} \cos x_2 \\ \sin x_2 \tan x_1 \\ 0 \\ \sin x_2 \sec x_1 \\ 0 \\ 0 \end{bmatrix}, Z_2(x) \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, Z_3(x) \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ 0 \\ \cos x_4 \cos x_1 \\ 0 \\ -\sin x_1 \\ \sin x_4 \cos x_1 \end{bmatrix},$$

Computing the following Lie brackets:

$$Z_4(x) \stackrel{\text{def}}{=} [Z_1, Z_2](x) = \begin{bmatrix} -\sin x_2 \\ \cos x_2 \tan x_1 \\ 0 \\ \cos x_2 \sec x_1 \\ 0 \\ 0 \end{bmatrix},$$

$$Z_5(x) \stackrel{\text{def}}{=} [Z_1, Z_3](x) = \begin{bmatrix} 0 \\ 0 \\ \sin x_1 \cos x_2 \cos x_4 + \sin x_2 \sin x_4 \\ 0 \\ \cos x_1 \cos x_2 \\ \sin x_1 \sin x_4 \cos x_2 - \sin x_2 \cos x_4 \end{bmatrix},$$

$$Z_6(x) \stackrel{\text{def}}{=} [Z_2, [Z_1, Z_3]](x) = \begin{bmatrix} 0 \\ 0 \\ -\sin x_1 \sin x_2 \cos x_4 + \cos x_2 \sin x_4 \\ 0 \\ -\cos x_1 \sin x_2 \\ -\sin x_1 \sin x_2 \sin x_4 - \cos x_2 \cos x_4 \end{bmatrix}$$

which demonstrates that, if the motion is restricted to the manifold:

$$N = \{x \in \mathbb{R}^6 : |x_1| < \frac{\pi}{2}\}$$

then LARC condition is satisfied:

$$\text{span}\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}(x) = \mathbb{R}^6, \forall x \in N \quad (27)$$

The reasoning behind this transformation is just to convert the system states in an order such that each state variable x_i can be steered along the vector field $Z_i(x)$ for $i = 1, 2, \dots, 6$.

A. Decomposition of the System into Two Subsystems

Decompose the original system (26) into two subsystems such as: one subsystem is consist of first five state variables which can be steered along the original vector fields and all independent Lie brackets with level one, and other subsystem is consist of one state variable which can be steered along the Lie bracket with level two. Evaluating all vector fields in (27) at zero will indicate that which state variable is related to which vector fields. Then we have the following decomposition:

$$T1: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} \cos x_2 \\ \sin x_2 \tan x_1 \\ 0 \\ \sin x_2 \sec x_1 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_2 + \begin{bmatrix} 0 \\ 0 \\ \cos x_4 \cos x_1 \\ 0 \\ -\sin x_1 \end{bmatrix} v_3 \quad (28)$$

$$T2: \dot{x}_6 = \sin x_4 \cos x_1 v_3 \stackrel{\text{def}}{=} f(x) v_3 \quad (29)$$

By defining $y = (x_1, x_2, x_3, x_4, x_5)$, the subsystem $T1$ can be written as:

$$\dot{y} = X_1(y)v_1 + X_2(y)v_2 + X_3(y)v_3, \quad y \in \mathbb{R}^5 \quad (30)$$

where,

$$X_1(y) = \begin{bmatrix} \cos x_2 \\ \sin x_2 \tan x_1 \\ 0 \\ \sin x_2 \sec x_1 \\ 0 \end{bmatrix}, X_2(y) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, X_3(y) = \begin{bmatrix} 0 \\ 0 \\ \cos x_4 \cos x_1 \\ 0 \\ -\sin x_1 \end{bmatrix}$$

Subsystem $T1$ is controllable as it satisfies the LARC condition:

$$\text{span}\{X_1(y), X_2(y), \dots, X_5(y)\} = \mathbb{R}^5 \quad \forall y \in \hat{N}, \text{ where}$$

$$\hat{N} \stackrel{\text{def}}{=} \{y = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : |x_1| < \frac{\pi}{2}\},$$

$$X_4(y) \stackrel{\text{def}}{=} [X_1, X_2](y) = \begin{bmatrix} -\sin x_2 \\ \cos x_2 \tan x_1 \\ 0 \\ \cos x_2 \sec x_1 \\ 0 \end{bmatrix},$$

$$X_5(y) \stackrel{\text{def}}{=} [X_1, X_3](y) = \begin{bmatrix} 0 \\ 0 \\ \sin x_1 \cos x_2 \cos x_4 + \sin x_2 \sin x_4 \\ 0 \\ \cos x_1 \cos x_2 \end{bmatrix}$$

It can be easily verified that the Lie algebra $L(X_1, X_2, X_3)$ is not nilpotent. The approximation to subsystem $T1$ is considered in such a way that the controllability Lie algebra $L(X_1, X_2, X_3)$ is locally nilpotent:

$$\hat{T}1: \dot{y} = Y_1(y)v_1 + Y_2(y)v_2 + Y_3(y)v_3, \quad y \in \mathbb{R}^5 \quad (31)$$

where,

$$Y_1(y) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ x_2 \\ 0 \end{bmatrix}, Y_2(y) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, Y_3(y) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -x_1 \end{bmatrix},$$

$$Y_4(y) \stackrel{\text{def}}{=} [Y_1, Y_2](y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, Y_5(y) \stackrel{\text{def}}{=} [Y_1, Y_3](y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

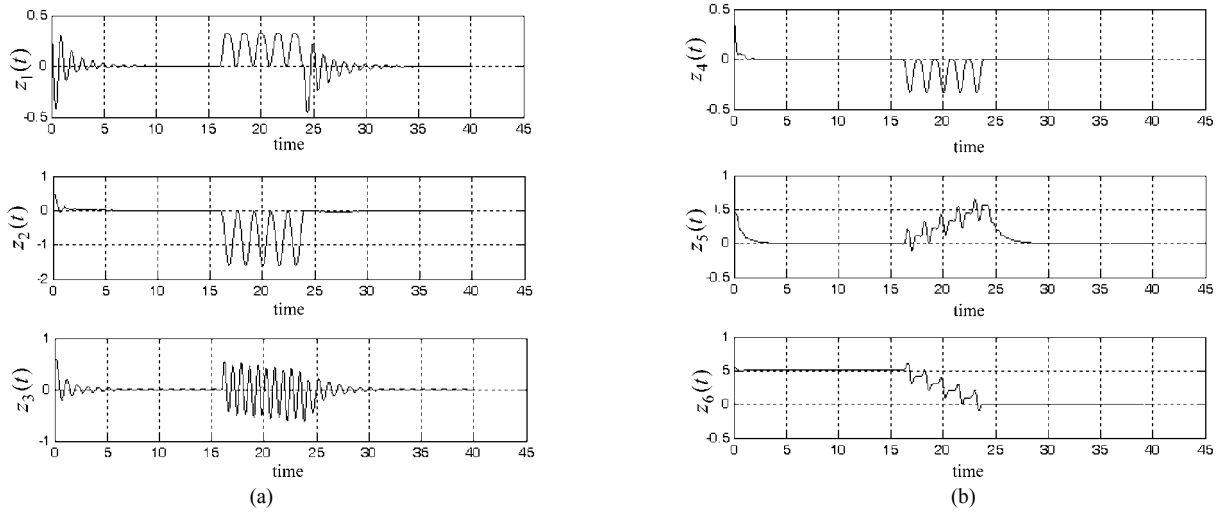


Fig. 6. Underwater vehicle Model 2: plots of the controlled state trajectories $t \mapsto (z_1(t), \dots, z_6(t))$ versus time.

gives,

$$\text{span}\{Y_1(y), Y_2(y), \dots, Y_5(y)\} = \mathfrak{R}^5 \quad \forall y \in \mathfrak{R}^5.$$

The Lie brackets multiplication table for $L(Y_1, Y_2, Y_3)$:

$$[Y_1, Y_2] = Y_4 \quad [Y_1, Y_3] = Y_5 \quad [Y_2, Y_3] = 0$$

$$[Y_i, Y_4] = [Y_i, Y_5] = 0, \quad i = 1, 2, 3$$

shows that the controllability algebra $L(Y_1, Y_2, Y_3)$ is nilpotent.

The extended system for $\hat{T}1$ is given by:

$$\dot{y} = Y_1(y)v_1 + Y_2(y)v_2 + Y_3(y)v_3 + Y_4(y)v_4 + Y_5(y)v_5 \quad (32)$$

where $v_i \stackrel{\text{def}}{=} -L_{Y_i}W(y), i = 1, \dots, 5,$ & $W(y) = \frac{1}{2} \sum_{i=1}^5 x_i^2$.

The discretized form of system (32) is:

$$\dot{y} = Y_1(y)a_1 + Y_2(y)a_2 + Y_3(y)a_3 + Y_4(y)a_4 + Y_5(y)a_5 \quad (33)$$

The logarithmic coordinates for $\hat{T}1$ satisfy the following differential equations:

$$\dot{\gamma}_1 = a_1$$

$$\dot{\gamma}_2 = a_2$$

$$\dot{\gamma}_3 = a_3$$

$$\dot{\gamma}_4 = -\gamma_1 a_2 + a_4$$

$$\dot{\gamma}_5 = -\gamma_1 a_3 + a_5, \quad \text{with } \gamma_i(0) = 0, \quad i = 1, 2, \dots, 5$$

Therefore by TIP the following control stabilize the system $\hat{T}1$:

$$u_1(x) = a_1 + (c_4 + c_5) \sin \frac{2\pi t}{T},$$

$$u_2(x) = a_2 + c_4 \cos \frac{2\pi t}{T}, \quad (34)$$

$$u_3(x) = a_3 + c_5 \cos \frac{2\pi t}{T}$$

where, c_i are found:

$$c_4 = \pm 3.54491 \sqrt{\frac{a_4}{T}}, \quad c_5 = \pm 3.54491 \sqrt{\frac{a_5}{T}}.$$

Replacing a_i by b_i and c_i by d_i in (34) we obtain the following controls which stabilize the sub-system $T1$.

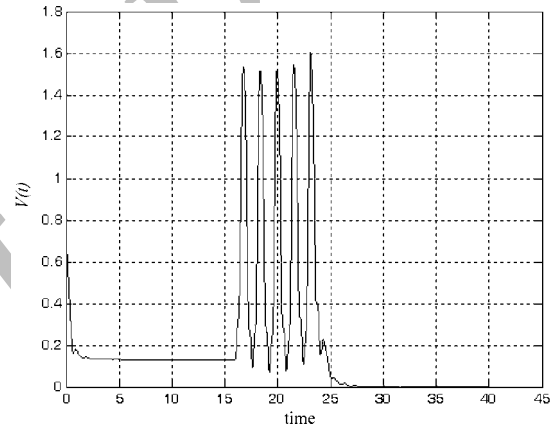


Fig. 7. Underwater vehicle Model 2: plot of the Lyapunov function $V(z(t)) = \frac{1}{2} \sum_{i=1}^5 z_i^2(t)$ versus time.

$$u_1(x) = b_1 + (d_4 + d_5) \sin \frac{2\pi t}{T},$$

$$u_2(x) = b_2 + d_4 \cos \frac{2\pi t}{T}, \quad (35)$$

$$u_3(x) = b_3 + d_5 \cos \frac{2\pi t}{T}$$

with $d_4 = \pm 3.54491 \sqrt{\frac{b_4}{T}}, d_5 = \pm 3.54491 \sqrt{\frac{b_5}{T}},$ where $b_i = -L_{X_i}W(y), i = 1, \dots, 5,$ and $W(y) = \frac{1}{2} \sum_{i=1}^5 x_i^2$.

VI. STABILIZATION ALGORITHM FOR CASE II

Repeat the following algorithm until sufficient accuracy is achieved in reaching the origin:

Algorithm:

Data: $\varepsilon > 0$

Step a: Apply the control (35) to original system (26) until its trajectories converges to $B(S_1; \varepsilon)$:

$$S_1 \stackrel{\text{def}}{=} \{x \in N : x_1 = \dots = x_5 = 0, x_6 \neq 0\}$$

where $B(S_1; \varepsilon)$ denotes the ε - neighborhood of S_1 .

Step b: To generate motion along $[Z_2, [Z_1, Z_3]]$, apply the controls

$$\begin{aligned}
u_1 &= k_1 \sin \frac{2\pi t}{T}, \\
u_2 &= k_2 \sin \frac{2\pi t}{T}, \\
u_3 &= k_3 \cos \frac{4\pi t}{T}
\end{aligned} \tag{36}$$

until the system trajectories converges to $B(S_2; \varepsilon)$, where:

$$\begin{aligned}
S_2 &\stackrel{\text{def}}{=} \{x \in N : x_6 \text{ \& } f(x) = 0\} \\
&= \{x \in N : x_6 \text{ \& } \sin x_4 \cos x_1 = 0\} \\
&= \{x \in N : x_6 = x_4 = 0\}
\end{aligned}$$

which is an invariant set for the controlled system (26)

Step c: Set $\varepsilon := \varepsilon/2$.

Remark: The outcome of Step (a) steers all state variables $x_i, i=1,2,\dots,5$ to zero except x_6 and the application of Step (b) gives $x_6 = x_4 = 0$ while the other state variables x_1, x_2, x_3 and x_5 may become nonzero. One more time application of Step (a) will make all state variables $x_i = 0, i=1,2,\dots,5,6$ and this step will not change $x_6 = x_4 = 0$ as $x = (x_1, \dots, x_6)^T \in S_2$.

The simulation results are shown in Figs. 6 and 7, where, in control (36) we have used $k_1 = 1, k_2 = -3, k_3 = 4$ and $T = 1.6$.

VII. CONCLUSION

A new approach for steering the underwater vehicle is presented by considering two different cases: (i) when all actuators are working, (ii) one actuator is not working. In first case, the model of an underwater vehicle is steered by employing a Lie bracket extension of the original system. In second case (which represents a physical example where second level Lie brackets are necessary for controllability), the original system is decomposed into two subsystems; one subsystem, which is fifth dimensional, is steered by a similar approach used in case (i) and the second subsystem, which is one dimensional is steered by using sinusoidal inputs. The mixture of both types of control is utilized to steer the actual system. The method is general and can be applied to a class of drift free systems, for which the associated controllability Lie algebra is locally nilpotent.

The approach does not necessitate conversion of the system model into a "chained form", and thus does not rely on any special transformation techniques. By introducing approximate models often permits significant simplification of the differential equations describing the evolution of the logarithmic coordinates in the open-loop problem formulation (which are usually difficult to solve analytically).

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