Digital Domain Design of Cascaded Ladder Wave Digital Filters with Tunable Parameters

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***Abstract*—A digital domain design of cascaded ladder WDFs using simple design parameters is presented. This special class of filters uses first order sections that are cascaded to realize N-order lowpass or highpass filters with tunable 3-dB cut-off frequency. To obtain the design parameters, the correspondence between the adaptor coefficient and the digital domain transfer function is shown. The transfer function is then used to derive the equations for the tunable coefficients in terms of the required cut-off frequency and filter order. The resulting structures are highly modular requiring the same type of adaptor throughout the structure. In addition, the number of coefficients required is minimal and is equal to the order of the filter.**

***Index Terms*—Cascade, ladder, tunable parameters, wave digital filters.**

I. Introduction

The simplest digital filters are first order Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) filters, individually or in cascaded. Although these filters are not as powerful as higher order filters, they are nevertheless used in many applications such as audio [1], spectroscopy [2]-[3] and biomedical [4].

Wave Digital Filters (WDF) are IIR filters with well-known good sensitivity properties [5]-[6]. WDFs are derived from classical analog filters and a classified according to the structures of their analog counterparts. Ladder WDFs are derived from resistively terminated analog filters with the passive components arranged in the ladder configuration. Ladder WDFs are known to be highly insensitive to coefficient value variations both in the passband and stopband. These WDFs have small rounding errors, high resistivity to parasitic oscillations and great dynamic range [7]. The structure of a ladder WDF usually consists of both the 3-port parallel and series adaptors for filter orders that are greater than one. These adaptors contain the adders and multipliers of the WDFs.

Ladder WDFs are conventionally designed by first synthesizing the analog filter that meets the specifications and then converting this filter to its WDF equivalent. Methods to directly synthesize WDFs purely in the digital domain have been proposed to avoid the analog filter synthesis requirement [8]-[9]. However, the complexity of these techniques is comparable to conventional WDF design as it is essentially the digital version of the reactance reduction technique used for analog network synthesis [10]-[11].

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The paper presents an alternative realization of ladder WDFs using simple design parameters. Sections of the first order ladder WDF are arranged in cascade to realize an N-order lowpass or highpass filter with tunable 3-dB cut-off frequency. A purely digital domain design is proposed. To obtain the design parameters, the correspondence between the WDF adaptor coefficient and the digital domain transfer function is shown. The transfer function is then used to derive the equation for the tunable coefficients in terms of the required cut-off frequency and filter order. The resulting structure is highly modular and has a minimum number of multipliers since only one coefficient is required per adaptor.

II. First Order IIR Filter

Designing IIR filters via the bilinear transformation requires the application of both the bilinear transformation and its inverse to arrive at the transfer function. The relationship between the digital transfer function $G(z)$ and the analog transfer function $H_A(s)$ using the bilinear transformation is

$$G(z) = H_A(s)\Big|_{\frac{1-z^{-1}}{1+z^{-1}}} \ . \tag{1}$$

The relationship between the analog frequency, Ω, on the s-plane and the digital frequency, ω, on the unit circle of the z-plane is

$$\Omega = \tan\left(\frac{\omega}{2}\right) . \tag{2}$$

A first-order lowpass transfer function with a 3-dB cutoff frequency at Ω_c is given by

$$H_{LP}(s) = \frac{\Omega_c}{s+\Omega_c} \ . \tag{3}$$

Applying the bilinear transformation, the expression for a first-order lowpass transfer function $G(z)$ is given by

$$G_{LP}(z) = \frac{\Omega_c(1+z^{-1})}{(1-z^{-1})+\Omega_c(1+z^{-1})} \ . \tag{4}$$

Rearranging terms, (4) can be written in terms of the conventional IIR transfer function form

$$G_{LP}(z) = \frac{b_0 + b_1 z^{-1}}{1+a_1 z^{-1}} \tag{5}$$

where

$$a_1 = \frac{\Omega_c - 1}{\Omega_c + 1} \tag{6}$$

or in terms of the digital cutoff frequency ω_c, using (2) and half angle identities,

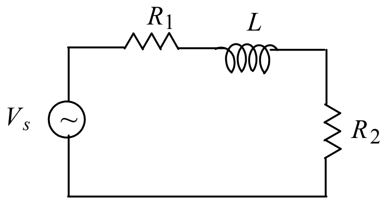

Fig. 1. First-order analog filter.

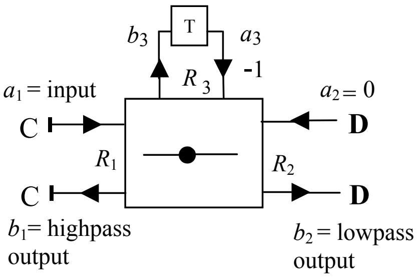

Fig. 2. First-order WDF with series adaptor.

$$a_1 = \frac{\sin\omega_c - 1}{\cos\omega_c} \tag{7}$$

while the other coefficients are

$$b_0 = \frac{1 + a_1}{2} \tag{8}$$

$$b_1 = b_0 \,. \tag{9}$$

A realization of $G_{LP}(z)$ in a Direct II IIR structure requires three non-unity multipliers [12]-[13].

III. First Order Ladder Wave Digital Filters

A. Configuration

Unlike conventional IIR filters, WDFs retain the structure of the analog filter from with it is derived since all elements and connections of the analog filter have their equivalence in the WDF structures. These equivalences are obtained using the bilinear transformation (1) and introducing wave variables. Normalized values are used for the components and frequencies. Fig. 1 shows the components of a first-order analog filter.

To obtain the WDF structure, each analog element is converted to its WDF equivalent. A 3-port series adaptor can be used to connect the WDF components as shown in Fig. 2. Ports 1 and 2 are the resistive ports with the values equal to R_1 and R_2, respectively, while port 3 is the inductive port with $R_3 = 1/L$ [5]. Due to the power complementary property, the lowpass and highpass responses are available as shown.

The configuration of the 3-port series adaptor is shown in Fig. 3.

The input and output equations are for $R_1 = R_2$ [5]

$$\begin{aligned} b_1 &= a_1 - m(a_1 + a_2 + a_3) \\ b_2 &= a_2 - m(a_1 + a_2 + a_3) \\ b_3 &= -(b_1 + b_2 + a_1 + a_2 + a_3) \end{aligned} \tag{10}$$

while the coefficient for the 3-port series adaptor is

$$m = \frac{2R_1}{R_1 + R_2 + R_3} \,. \tag{11}$$

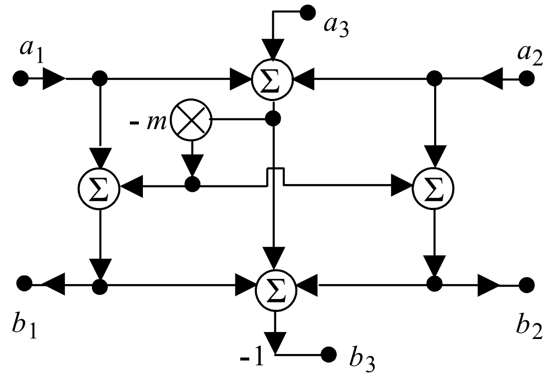

Fig. 3. Configuration of the 3-port series adaptor of Fig. 2.

B. Transfer Function

To relate the WDF coefficient to the digital domain transfer function, a Butterworth filter is used as the reference analog filter. The components of this prototype filter with reference to Fig. 1 is $R_1 = R_2 = 1$ and $L_2 = 2/\Omega_c$, where Ω_c is the 3-dB cut-off frequency of the analog filter. Substituting the port resistance values in (11) with

$$\begin{aligned} R_1 &= R_2 = 1 \\ R_3 &= \frac{2}{\Omega_c} \end{aligned} \tag{12}$$

the coefficient value is obtained as

$$m = \frac{\Omega_c}{1 + \Omega_c} \,. \tag{13}$$

By rearranging terms, (4) can be written in terms of the WDF adaptor coefficient above as

$$G_{LP}(z) = \frac{m + mz^{-1}}{1 + (2m - 1)z^{-1}} \,. \tag{14}$$

The WDF requires only one multiplier when compared to three for conventional IIR filters. By varying the WDF coefficient the 3-dB cut-off frequency of the lowpass WDF is tuned. Using (2), in terms of the digital frequency, (13) can be written as

$$m = \frac{\tan(\omega_c / 2)}{1 + \tan(\omega_c / 2)} \tag{15}$$

which has a range of zero to one.

WDFs are not only power complementary, but also allpass complementary, hence making them doubly complementary. The transfer function for the highpass response G_{HP} can be found using the allpass equation

$$G_{HP}(z) + G_{LP}(z) = 1 \tag{16}$$

yielding

$$G_{HP}(z) = \frac{(1 - m) - (1 - m)z^{-1}}{1 + (2m - 1)z^{-1}} \,. \tag{17}$$

The value of m in (15) and (17) is equal for a first order ladder WDF. The coefficient values for the lowpass and highpass responses will not be the same when the first order sections are cascaded to obtain higher order filters.

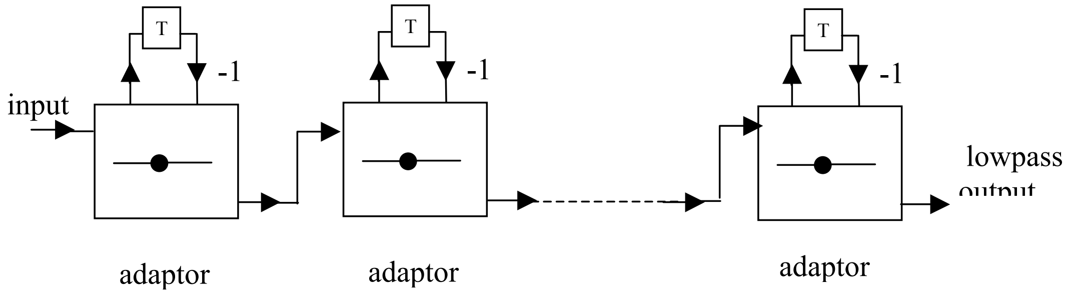

Fig. 4. An N cascade of first order WDF realizing the lowpass response.

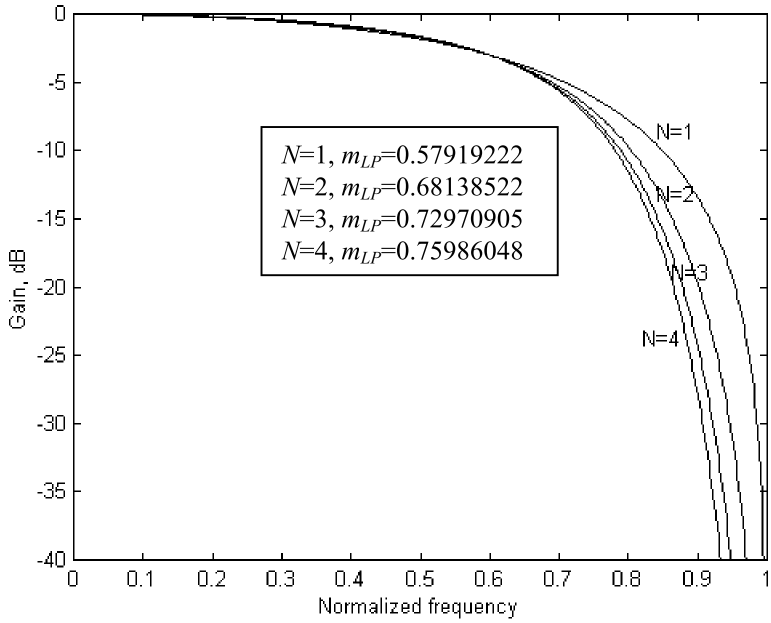

Fig. 5. Gain responses of a single and a cascade of N identical first order lowpass ladder WDFs for $\omega_{cLP} = 0.6\pi$.

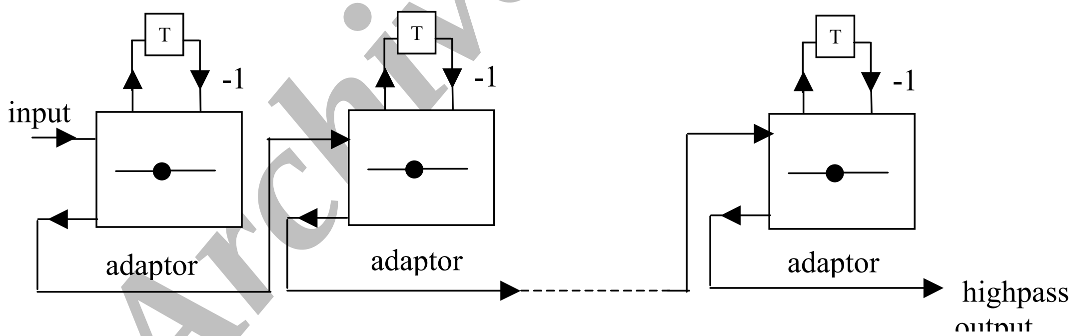

Fig. 6. An N cascade of first order WDF realizing the highpass response.

IV. Lowpass and Highpass Cascaded Ladder WDFs

An N order lowpass filter with sharper magnitude responses can be obtained by cascading the 3-port series adaptor as shown in Fig. 4. The lowpass response of each adaptor is the input of the subsequent adaptor.

For a cascade of N first order lowpass sections, with a single section transfer function given by (14), the overall structure has a transfer function given by

$$G_{LP}(z) = \left(\frac{m_{LP} + m_{LP} z^{-1}}{1 + (2m_{LP} - 1) z^{-1}} \right)^N \quad (18)$$

where m_{LP} denotes the coefficient for a lowpass response.

The corresponding squared magnitude function is

$$\left| G_{LP}(e^{j\omega}) \right|^2 = \left(\frac{m_{LP}{}^2 (2 + 2\cos\omega)}{1 + (2m_{LP} - 1)^2 + 2(2m_{LP} - 1)\cos\omega} \right)^N . \quad (19)$$

To obtain the tuning coefficient in terms of the 3-dB cut-off frequency, ω_{cLP}, requires solving the following equation for m_{LP}

$$\left(\frac{m_{LP}{}^2 (2 + 2\cos\omega_{cLP})}{1 + (2m_{LP} - 1)^2 + 2(2m_{LP} - 1)\cos\omega_{cLP}} \right)^N = \frac{1}{2}, \quad (20)$$

yielding

$$m_{LP} = \frac{Q\cos\omega_c - Q + \sqrt{Q - Q^2}\sin\omega_c}{1 + \cos\omega_c - 2Q} \quad (21)$$

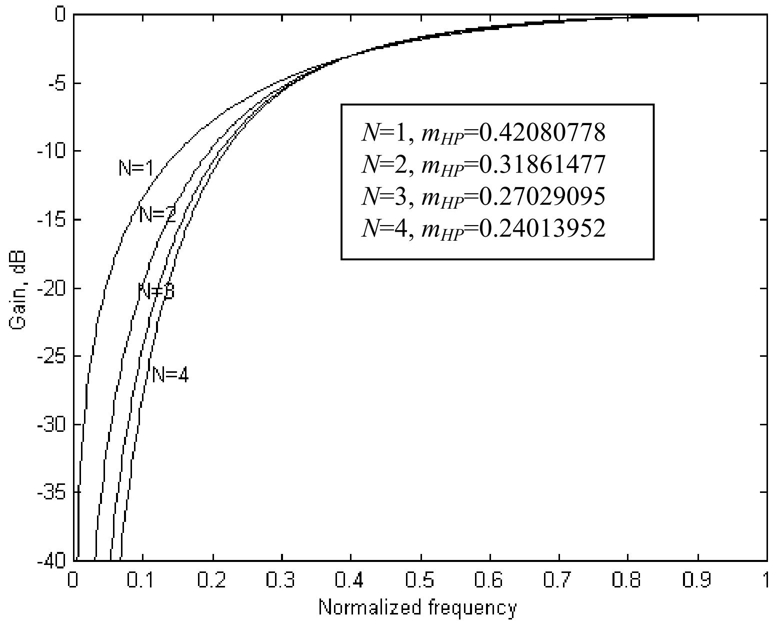

Fig. 7. Gain responses of a single and a cascade of N identical first order highpass filters for $\omega_{cHP} = 0.4\pi$.

where

$$Q = 2^{-1/N} \quad . \tag{22}$$

Cascading first order lowpass WDF sections results in a lowpass filter with a sharper roll-off in the gain response where each section increases the transition band slope by 20 dB/decade. Fig. 5 shows the gain responses of a single first order lowpass WDF and a cascade of N identical first order lowpass ladder WDFs, with the cut-off frequency fixed.

An N order highpass filter with sharper magnitude responses can be obtained by cascading the 3-port series adaptor as shown in Fig. 6. The highpass response of each adaptor is the input of the adjacent adaptor.

The same procedure that is used to derive the tuning coefficient for lowpass response can be used for the highpass response. First, the right side of (18) is replaced with the single section highpass transfer function given in (17). The corresponding squared magnitude function is used to obtain the highpass tuning coefficient m_{HP} in terms of the 3-dB cut-off frequency, ω_{cHP} , as

$$m_{HP} = \frac{Q_{HP} - Q_{HP}\cos\omega_{cHP} - \sqrt{Q - Q^2}\sin\omega_{cHP}}{1 - \cos\omega_{cHP} - 2Q} \tag{23}$$

where Q is similar to that as defined for the lowpass response (22) and

$$Q_{HP} = 1 - Q \, . \tag{24}$$

The expression for the coefficient of the cascaded ladder highpass response is different from that of the lowpass response (21), which is the case for when the cut-off frequencies for both filters are equal. The highpass and lowpass coefficient can be related to a single coefficient m where

$$m = m_{LP} \tag{25}$$

and

$$m = 1 - m_{HP} \tag{26}$$

when the corresponding cut-off frequencies are

$$\omega_c = \omega_{cLP} \tag{27}$$

and

$$\omega_c = \pi - \omega_{cHP} \, . \tag{28}$$

Cascading first order highpass sections results in a highpass filter with a sharper roll-off in the gain response where each section increases the transition band slope by 20 dB/decade. Fig. 7 shows the gain responses of a single first order highpass WDF and a cascade of N identical first order highpass WDFs with the same cut-off frequency.

V. CONCLUSION

This paper has shown an alternative realization of ladder WDFs using a cascade of first order sections. It can be used for applications where a highly modular structure is required as only the same type of adaptor is used throughout the structure. In addition, a single coefficient is required for each section thus resulting in the number of coefficients required to be equal to the order of the filter. The cascaded ladder WDFs can be used to realize an N-order lowpass or highpass filter with tunable 3-dB cut-off frequency. For both the lowpass and highpass responses, the equations for the tunable coefficients for a purely digital domain design have been derived and expressed in terms of the desired cut-off frequency and filter order.

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