MGF Approach for Diversity Analysis of QAM and PSK Signals in Fading Channels

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***Abstract*—This paper concentrates on the error rate analysis of wireless communication systems with diversity combining to derive exact closed-form expressions for QAM and PSK signals in order to provide system designers a precise and more convenient evaluation. Maximal ratio combining (MRC) with *L* independent and identical (i.i.d) diversity reception paths over slowly Nakagami-*m* fading channel is considered. The novel results obtained in this paper, via moment generation function (MGF) approach overcome the limitations of earlier works both in terms of generality and ease of simulation as they are presented in integral-free forms. At the end, several numerical results are included.**

***Index Terms*—Diversity reception, error rate analysis, Nakagami-*m* fading, phase shift keying modulation, quadrature amplitude modulation.**

I. Introduction

One of the major problems often facing a wireless system designer is to choose the best scheme that optimally matches to the desirable constraints. In this situation, making a good initial decision usually relies on exact quantitative evaluation and comparison of several options or techniques. Closed-form expressions can provide exact performance analysis that is essential to get insight into performance limits on the systems of interest while simultaneously provide useful background study for accurate system design, improvement or optimization.

Average bit or symbol error rate (SER) is an important factor that is essential to evaluate the performance of wireless communication systems. The error rate performance strongly relies on fading which is the major deteriorating factor in the system performance. An old also a well-known method to combat fading is diversity combining. The intuition behind diversity combining is to take advantage of the low probability of concurrence of deep fades in all the receiving diversity branches to reduce the probability of error [1]. Also, it has been shown that the maximal ratio combining (MRC) [2], is the optimal linear combining technique that provides maximum possible improvement in a diversity system in the presence of fading [3].

Considerable efforts have been devoted in recent years to develop statistical modeling and characterization of wireless environment in a range of models for fading channels theorizing their dependence on particular propagation environment and the underlying communication scenario. Furthermore, it is necessary to note that Nakagami- m fading statistics first introduced in [4] provide greater flexibility in various realistic experiments and involve half Gaussian fading, Rayleigh fading and pure additive white Gaussian noise (AWGN) channels as special cases.

The error rate performance of MRC diversity over Rayleigh or in general form, Nakagami- m fading, has been considered in many researches. However, the exact computable closed-form expressions of MRC diversity over Nakagami- m fading have been derived only in a few places of literature [5]-[7].

In [5] closed-form expressions for average SER for square M-QAM in Rician and Nakagami- m fading as terms of a single finite integral with an integrand composed of exponential and trigonometric functions are derived. Also, SER performance of M-QAM in Nakagami- m fading under dual MRC diversity has been reported in [6]. Most recently in [7], the probability density function (*pdf*)-based approach has been employed to derive exact closed-form expression of bit error probability for rectangular M-QAM signals with L -branch MRC diversity reception affected by a frequency non-selective slowly Nakagami- m fading while corrupted by AWGN. The proposed expression was as a function of the probabilities that the k -th bit of the in phase and the l -th bit of the quadrature components be in error in terms of signal-to-noise ratio (SNR).

It is noteworthy to mention that unlike QAM, there is no known general closed-form SER expression for coherent M-PSK; thus, *pdf*-based approach fails to get the result in these circumstances. In this paper, the *moment generation function* (MGF) is used to derive the exact SER expression for both M-QAM and M-PSK with L -branch MRC diversity reception over slowly non-selective Nakagami- m fading channel corrupted by AWGN. The superiority of this work relies on the integral-free forms of expressions superimposed to their generality and computability.

The remainder of this paper organized as follows; in Section II system and channel models are introduced. Section III deals with the derivation of average SER expressions. The proposed expressions are calculated and several simulations are presented in Section IV. Finally, the concluding remarks are given in Section V.

II. System and Channel Models

Let us consider there are L diversity channels carrying the same information bearing signal. It is also assumed that the received signals from the L different antennas are independent and identically distributed (i.i.d) random processes. This will be a true assumption if the spacing

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between any two arbitrary antennas be much larger than several wavelengths of the signal carrier. Since the slow fading is assumed, the fading stays constant in a signaling interval, so it can be represented using a complex random variable α . Thus, the instantaneous SNR per symbol, γ_S , is now a random variable given by

$$\gamma_S = |\alpha|^2 \frac{E_S}{N} \quad (1)$$

where E_S is defined as the transmit power per symbol and N is the white Gaussian noise power. The *pdf* of γ_S at the output of L-branch MRC for i.i.d Nakagami-m fading can be written as [4]

$$P(\gamma_S) = \frac{1}{\Gamma(mL)}\left(\frac{m}{\bar{\gamma}_S}\right)^{mL} \gamma_S^{mL-1} \exp\left(-\frac{m}{\bar{\gamma}_S}\gamma_S\right) \quad ;\gamma_S \geq 0 \quad (2)$$

where $\bar{\gamma}_S$ is the average SNR associated with each symbol per diversity path, $\Gamma(.)$ is the gamma function and m is the Nakagami-m fading severity parameter that assumed to be identical for all branches $(m \geq 0.5)$. The above distribution includes the half-Gaussian $(m = 0.5)$, Rayleigh $(m = 1)$ and a non-fading channel $(m \to \infty)$ as special cases.

III. Closed-Form Expressions for SER

Once the *pdf* of γ_S is known, the average SER $\bar{P}_S(e)$ in fading can be calculated using the MGF which is defined as follows

$$\phi_{\gamma_S}(s) \triangleq \int_0^\infty \exp(-s\gamma_S)\, p(\gamma_S)\, d\gamma_S \quad (3)$$

and it can be easily shown that for the distribution represented in (2)

$$\phi_{\gamma_S}(s) = \left(1 + \frac{s\bar{\gamma}_S}{m}\right)^{-mL}. \quad (4)$$

As we will see later, desired representations of the conditional symbol error rate of M-PSK and M-QAM allows obtaining the average symbol error rate in a generic fashion with the MGF-based approach.

A. Average SER for M-PSK

The conditional SER for M-PSK modulated signals is known by digital communications theory as [8]

$$P_S(e|\gamma_S) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left(-\frac{g_{MPSK}\gamma_S}{\sin^2\theta}\right) d\theta \quad (5)$$

where $g_{MPSK} = \sin^2(\pi/M)$. Employing (2)-(5) also, considering $s = g_{MPSK}/\sin^2\theta$, the average SER of the MRC with M-PSK modulation over L-diversity reception and affected by Nakagami-m fading is given by

$$\begin{aligned}\bar{P}_S(e) &= \frac{1}{\pi}\int_0^{(M-1)\pi/M} \phi_{\gamma_S}\left(\frac{g_{MPSK}}{\sin^2\theta}\right) d\theta \\ &= \frac{1}{\pi}\int_0^{(M-1)\pi/M} \left(1 + \frac{g_{MPSK}\bar{\gamma}_S}{m\sin^2\theta}\right)^{-mL} d\theta\end{aligned}. \quad (6)$$

In general, performing linear integration does not provide sufficient satisfaction to a specific term to be accepted as final desirable formula. Thus, to go further, the SER in (6) can be expressed as a closed-form formula in terms of the higher transcendental functions such as *hypergeometric functions*. We are not going to provide the detailed calculation to find a solution of (6) here. Instead, in [9], whereas the exact SER of space-time block codes in MIMO systems have been investigated a closed-form equivalence to an almost similar integration of that in (6) is calculated. The same procedure with nearly negligible changes can apply here to our problem. The final solution is given here

$$\begin{aligned}\bar{P}_S(e) = \phi_{\gamma_S}(g_{MPSK}) \Bigg\{ &\frac{1}{2\sqrt{\pi}} \frac{\Gamma(mL+1/2)}{\Gamma(mL+1)} \times \\ &{}_2F_1\left(mL, \frac{1}{2}; mL+1; \frac{m}{m+g_{MPSK}\bar{\gamma}_S}\right) \\ &+ \frac{\sqrt{1-g_{MPSK}}}{\pi} \times \\ &F_1\left(\frac{1}{2}, mL, \frac{1}{2}-mL; \frac{3}{2}; \frac{m-mg_{MPSK}}{m+g_{MPSK}\bar{\gamma}_S}, 1-g_{MPSK}\right)\Bigg\}\end{aligned} \quad (7)$$

where ${}_2F_1(a,b;c;x)$ is the hypergeometric function defined as [10]

$${}_2F_1(a,b;c;x) \triangleq \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad (8)$$

and $(a)_n \triangleq \Gamma(a+n)/\Gamma(a)$.

Additionally, $F_1(a,b,b';c;x,y)$ is the *Appell* hypergeometric function defined as [10, p. 224, Eq. 5.7 (6)]

$$F_1(a,b,b';c;x,y) \triangleq \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_{n+k}(b)_n(b')_k}{(c)_{n+k}\, n!\, k!} x^n y^k . \quad (9)$$

B. Average SER for M-QAM

Likewise, the conditional SER for M-QAM modulated signals is well known in theory [8]

$$\begin{aligned}P_S(e|\gamma_S) = &\frac{4q}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{g_{MQAM}\gamma_S}{\sin^2\theta}\right) d\theta \\ &-\frac{4q^2}{\pi}\int_0^{\frac{\pi}{4}} \exp\left(-\frac{g_{MQAM}\gamma_S}{\sin^2\theta}\right) d\theta\end{aligned} \quad (10)$$

whereas $g_{MQAM} = 3/[2(M-1)]$ and $q = 1 - 1/\sqrt{M}$.

Keeping in mind (2)-(5) also, assigning $s = g_{MQAM}/\sin^2\theta$, the average SER of MRC with M-QAM modulation over L-diversity reception paths affected by Nakagami-m fading is given by

$$\begin{aligned}\bar{P}_S(e) = &\frac{4q}{\pi}\int_0^{\frac{\pi}{2}} \phi_{\gamma_S}\left(\frac{g_{MQAM}}{\sin^2\theta}\right) d\theta \\ &-\frac{4q^2}{\pi}\int_0^{\frac{\pi}{4}} \phi_{\gamma_S}\left(\frac{g_{MQAM}}{\sin^2\theta}\right) d\theta\end{aligned} \quad (11)$$

or equivalently

$$\begin{aligned}\bar{P}_S(e) = &\frac{4q}{\pi}\int_0^{\frac{\pi}{2}} \left(1 + \frac{g_{MQAM}\bar{\gamma}_S}{m\sin^2\theta}\right)^{-mL} d\theta \\ &-\frac{4q^2}{\pi}\int_0^{\frac{\pi}{4}} \left(1 + \frac{g_{MQAM}\bar{\gamma}_S}{m\sin^2\theta}\right)^{-mL} d\theta\end{aligned}. \quad (12)$$

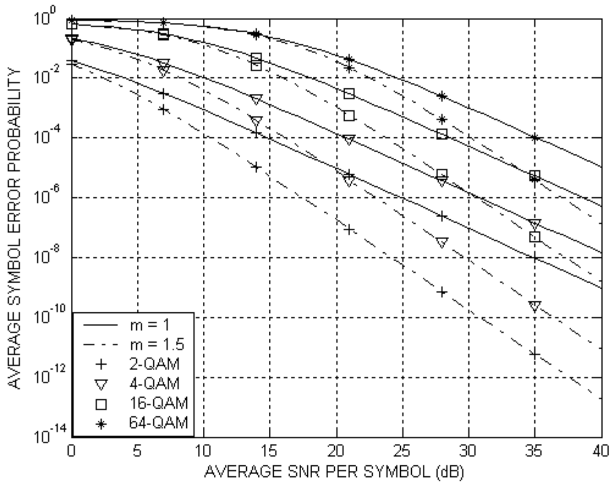

Fig. 1. Average SER for M-QAM modulation over dual-diversity reception and $m=1$ and 1.5.

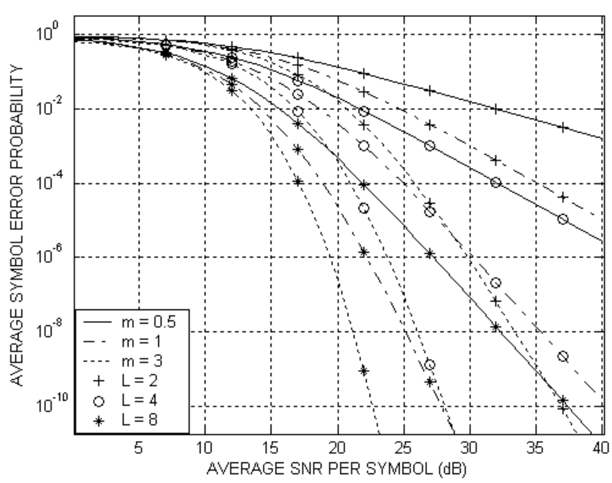

Fig. 3. Average SER for 64-QAM modulation over $L=2,4$, and 8 diversity reception and $m=0.5,1$, and 3.

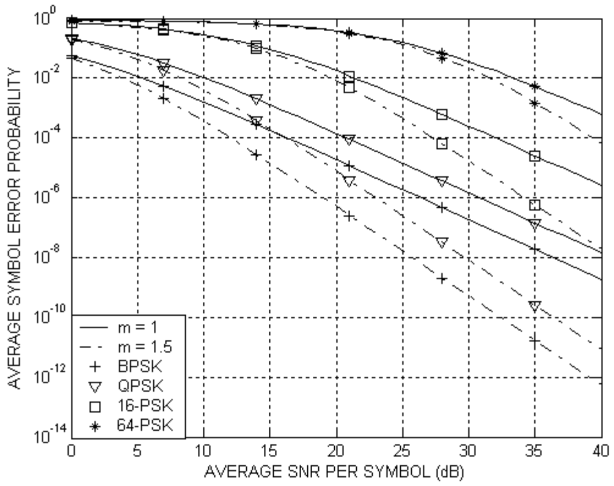

Fig. 2. Average SER for M-PSK modulation over dual-diversity reception and $m=1,1.5$.

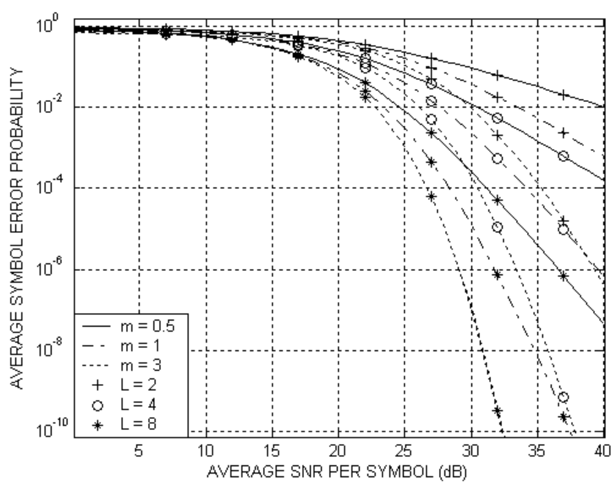

Fig. 4. Average SER for 64-PSK modulation over $L=2,4$, and 8 diversity reception and $m=0.5,1$, and 3.

Like the previous section, to our purpose the detailed calculation of the above integration may not need a major concern. Although, the research have been done in [9] have not investigated the same category; however, a detailed calculation to an almost similar integration in (12) are provided and can be applied here with little changes. Thus, the final solution will be of the form

$$\begin{aligned}\overline{P}_S(e) &= \frac{2q\phi_{\gamma_S}\left(g_{MQAM}\right)}{\sqrt{\pi}}\frac{\Gamma(mL+1/2)}{\Gamma(mL+1)} \\ &\times {}_2F_1\left(mL,\frac{1}{2};mL+1;\frac{m}{m+g_{MQAM}\overline{\gamma}_S}\right) \\ &-\frac{2q^2}{\pi}\frac{\phi_{\gamma_S}\left(2g_{MQAM}\right)}{2mL+1} \\ &\times F_1\left(1,mL,1;mL+\frac{3}{2};\frac{m+g_{MQAM}\overline{\gamma}_S}{m+2g_{MQAM}\overline{\gamma}_S},\frac{1}{2}\right)\end{aligned} \quad . \qquad (13)$$

Finally, It is worthwhile to mention that the hypergeometric functions are well developed in software like Mathematica®.

IV. Numerical Results

The final expressions for the average SER of the square M-QAM and M-PSK with L-th order MRC diversity reception over i.i.d Nakagami-m fading that obtained in the previous section are computed numerically and the results are plotted in Figs. 1-4. The results are plotted as average SER probability over average SNR per symbol per each diversity path for different values of m, L, and M.

Figs. 1 and 2 specifically represent the error rate analysis for various modulation schemes in dual-antenna reception. They are also include both a Rayleigh fading ($m=1$) and a less severe fading ($m=1.5$) environments together. The plots clearly show the effect of fading severity on the error performance of the transmission in M-QAM and M-PSK modulation techniques over a wide range of the signal strength per symbol per branch. Also, it can be concluded that the effect of fading severity parameter m is more significant in higher values of SNR.

Figs. 3 and 4 deals with the SER analysis of 64-ary PSK and QAM for different values of diversity order (L) and in fading paths with $m=0.5,1$, and 3. A close inspection of these numerical results reveals that a significant improvement in SER performance accomplishes as the order of L increases. In the meantime, the role of severity of fading is still prominent. As expected, with increasing fading parameter m the error tends to the non-fading situations.

V. CONCLUSION

In this paper, novel exact expressions for SER evaluation of *M*-ary QAM and PSK modulated signals are presented. It is assumed that the signals are received from L diversity paths and combined by MRC when have independent and identical Nakagami-*m* fading channels. The expressions obtained in this paper are capable of modeling various cases with arbitrary changing different parameters of interest. Thus, they can offer a more convenient way to precisely evaluate of arbitrary QAM and PSK with MRC diversity combining in uncorrelated slow fading environment.

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