# INTUITIONISTIC FUZZY HYPER $BCK$-IDEALS OF HYPER $BCK$-ALGEBRAS

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ABSTRACT. The intuitionistic fuzzification of (strong, weak, $s$-weak) hyper $BCK$-ideals is introduced, and related properties are investigated. Characterizations of an intuitionistic fuzzy hyper $BCK$-ideal are established. Using a collection of hyper $BCK$-ideals with some conditions, an intuitionistic fuzzy hyper $BCK$-ideal is built.

## 1. Introduction

The hyper structure theory (called also multialgebras) was introduced in 1934 by Marty [8] at the 8th congress of Scandinavian Mathematiciens. In [7], Jun et al. applied the hyper structures to $BCK$-algberas, and introduced the concept of a hyper $BCK$-algebra which is a generalization of a $BCK$-algebra. They also introduced the notion of a (weak, $s$-weak, strong) hyper $BCK$-ideal, and gave relations among them. After the introduction of the concept of fuzzy sets by Zadeh [9], several researchers were conducted on the generalization of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. In this paper, using the Atanassov's idea, we establish the intuitionistic fuzzification of the notion of (strong, weak, $s$-weak) hyper $BCK$-ideals in hyper $BCK$-algebras, and investigate some of their properties. We give characterizations of intuitionistic fuzzy hyper $BCK$-ideals. Using a collection of hyper $BCK$-ideals with some conditions, we build an intuitionistic fuzzy hyper $BCK$-ideal.



## 2. Preliminaries

We include some elementary aspects of hyper $BCK$-algebras that are necessary for this paper, and for more details we refer to [5], [6], and [7].

Let $H$ be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets $A$ and $B$ of $H$, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper $BCK$-algebra* we mean a nonempty set $H$ endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

Received: June 2003; Accepted: November 2003
*Key words and Phrases:* Hyper $BCK$-algebra, **inf-sup** property, Intuitionistic Fuzzy (Weak, $s$-weak, Strong) Hyper $BCK$-ideal.

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(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in $H$.

Note that the condition (HK3) is equivalent to the condition:

(p1) $x \circ y \ll \{x\}$ for all $x, y \in H$.

In any hyper $BCK$-algebra $H$, the following hold.

(p2) $x \circ 0 \ll \{x\}$, $0 \circ x \ll \{0\}$ and $0 \circ 0 \ll \{0\}$.
(p3) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ and $0 \circ A \ll \{0\}$.
(p4) $0 \circ 0 = \{0\}$.
(p5) $0 \ll x$.
(p6) $x \ll x$.
(p7) $A \ll A$.
(p8) $A \subseteq B$ implies $A \ll B$.
(p9) $0 \circ x = \{0\}$
(p10) $0 \circ A = \{0\}$.
(p11) $A \ll \{0\}$ implies $A = \{0\}$.
(p12) $A \circ B \ll A$.
(p13) $x \in x \circ 0$.
(p14) $x \circ 0 \ll \{y\}$ implies $x \ll y$.
(p15) $y \ll z$ implies $x \circ z \ll x \circ y$.
(p16) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$.
(p17) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$.

for all $x, y, z \in H$ and for all nonempty subsets $A$, $B$ and $C$ of $H$.

A nonempty subset $I$ of a hyper $BCK$-algebra $H$ is said to be a *hyper $BCK$-ideal* of $H$ if it satisfies

(I1) $0 \in I$,
(I2) $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

A nonempty subset $I$ of a hyper $BCK$-algebra $H$ is called a *strong hyper $BCK$-ideal* of $H$ if it satisfies (I1) and

(I3) $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Note that every strong hyper $BCK$-ideal of a hyper $BCK$-algebra is a hyper $BCK$-ideal.

A nonempty subset $I$ of a hyper $BCK$-algebra $H$ is called a *weak hyper $BCK$-ideal* of $H$ if it satisfies (I1) and

(I4) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

We now review some fuzzy logic concepts. A *fuzzy set* in a set $H$ is a function $\mu : X \to [0, 1]$, and the complement of $\mu$, denoted by $\bar{\mu}$, is the fuzzy set in $H$ given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in H$. For any $t \in [0, 1]$ and a fuzzy set $\mu$ in a nonempty set $H$, the set

$$U(\mu;t) = \{x \in H \mid \mu(x) \geq t\} \ \text{(resp. } L(\mu;t) = \{x \in H \mid \mu(x) \leq t\})$$

is called an *upper* (resp. *lower*) *level set* of $\mu$.

A fuzzy set $\mu$ in a hyper $BCK$-algebra $H$ is called a *fuzzy hyper $BCK$-ideal* of $H$ if it satisfies

- $x \ll y$ implies $\mu(y) \leq \mu(x)$,
- $\mu(x) \geq \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\}$,

for all $x, y \in H$.

An *intuitionistic fuzzy set* (briefly, IFS) $A$ in a nonempty set $X$ is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$. An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ in $X$ can be identified with an ordered pair $(\mu_A, \gamma_A)$ in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

## 3. Intuitionistic Fuzzy Hyper $BCK$-Ideals

In what follows let $H$ denote a hyper $BCK$-algebra unless otherwise specified.

**Definition 3.1.** An IFS $A = (\mu_A, \gamma_A)$ in $H$ is called an *intuitionistic fuzzy hyper $BCK$-ideal* of $H$ if it satisfies

(k1) $x \ll y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$,
(k2) $\mu_A(x) \geq \min\{\inf_{a \in x \circ y} \mu_A(a),\ \mu_A(y)\}$,
(k3) $\gamma_A(x) \leq \max\{\sup_{b \in x \circ y} \gamma_A(b),\ \gamma_A(y)\}$

for all $x, y \in H$.

**Example 3.2.** Let $H = \{0, a, b\}$ be a hyper $BCK$-algebra with the following Cayley table:

| $\circ$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0, a\}$ | $\{0, a\}$ |
| $b$ | $\{b\}$ | $\{a, b\}$ | $\{0, a, b\}$ |

Define an IFS $A = (\mu_A, \gamma_A)$ in $H$ by $\mu_A(0) = 0.7$, $\mu_A(a) = 0.4$, $\mu_A(b) = 0.2$, $\gamma_A(0) = 0.07$, $\gamma_A(a) = 0.5$, and $\gamma_A(b) = 0.6$. It is easily verified that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$.

**Definition 3.3.** An IFS $A = (\mu_A, \gamma_A)$ in $H$ is called an *intuitionistic fuzzy strong hyper $BCK$-ideal* of $H$ if it satisfies

$$\inf_{a \in x \circ x} \mu_A(a) \geq \mu_A(x) \geq \min\{\sup_{b \in x \circ y} \mu_A(b),\ \mu_A(y)\}$$

and

$$\sup_{c\in x\circ x}\gamma_A(c)\le\gamma_A(x)\le\max\{\inf_{d\in\, x\circ y}\gamma_A(d),\,\gamma_A(y)\}$$

for all $x, y \in H$.

**Example 3.4.** Let $H = \{0, a, b\}$ be a hyper $BCK$-algebra with the following Cayley table:

| $\circ$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0\}$ | $\{a\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0, b\}$ |

Define an IFS $A = (\mu_A, \gamma_A)$ in $H$ by $\mu_A(0) = 0.9$, $\mu_A(a) = 0.6$, $\mu_A(b) = 0.3$, $\gamma_A(0) = 0.09$, $\gamma_A(a) = 0.16$, and $\gamma_A(b) = 0.23$. It is routine to check that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy strong hyper $BCK$-ideal of $H$.

**Definition 3.5.** An IFS $A = (\mu_A, \gamma_A)$ in $H$ is called an *intuitionistic fuzzy s-weak hyper BCK-ideal* of $H$ if it satisfies

(s1) $\mu_A(0) \ge \mu_A(x)$ and $\gamma_A(0) \le \gamma_A(x)$ for all $x \in H$,
(s2) for every $x, y \in H$ there exist $a, b \in x \circ y$ such that

$$\mu_A(x)\ge\min\{\mu_A(a),\mu_A(y)\}\ \text{ and }\ \gamma_A(x)\le\max\{\gamma_A(b),\gamma_A(y)\}.$$

**Definition 3.6.** An IFS $A = (\mu_A, \gamma_A)$ in $H$ is called an *intuitionistic fuzzy weak hyper BCK-ideal* of $H$ if it satisfies

$$\mu_A(0)\ge\mu_A(x)\ge\min\{\inf_{a\in x\circ y}\mu_A(a),\,\mu_A(y)\}$$

and

$$\gamma_A(0)\le\gamma_A(x)\le\max\{\sup_{b\in x\circ y}\gamma_A(b),\,\gamma_A(y)\}$$

for all $x, y \in H$.

Let an IFS $A = (\mu_A, \gamma_A)$ in $H$ be an intuitionistic fuzzy $s$-weak hyper $BCK$-ideal of $H$ and let $x, y \in H$. Then there exist $a, b \in x \circ y$ such that

$$\mu_A(x)\ge\min\{\mu_A(a),\mu_A(y)\}\ \text{ and }\ \gamma_A(x)\le\max\{\gamma_A(b),\gamma_A(y)\}.$$

Since $\mu_A(a) \ge \inf_{c\in x\circ y} \mu_A(c)$ and $\gamma_A(b) \le \sup_{d\in x\circ y} \gamma_A(d)$, it follows that

$$\mu_A(x)\ge\min\{\inf_{c\in x\circ y}\mu_A(c),\,\mu_A(y)\}$$

and

$$\gamma_A(x)\le\max\{\sup_{d\in x\circ y}\gamma_A(d),\,\gamma_A(y)\}.$$

Hence every intuitionistic fuzzy $s$-weak hyper $BCK$-ideal is an intuitionistic fuzzy weak hyper $BCK$-ideal.

An IFS $A = (\mu_A, \gamma_A)$ in $H$ is said to satisfy the **inf-sup** *property* if for any subset $T$ of $H$ there exist $x_0, y_0 \in T$ such that $\mu_A(x_0) = \inf_{x\in T} \mu_A(x)$ and $\gamma_A(y_0) = \sup_{y\in T} \gamma_A(y)$.

It is not easy to find an example of an intuitionistic fuzzy weak hyper $BCK$-ideal which is not an intuitionistic fuzzy $s$-weak hyper $BCK$-ideal. But we have the following proposition.

**Proposition 3.7.** *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy weak hyper $BCK$-ideal of $H$. If $\mu$ satisfies the* **inf-sup** *property, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $s$-weak hyper $BCK$-ideal of $H$.*

*Proof.* Since $A = (\mu_A, \gamma_A)$ satisfies the **inf-sup** property, there exist $a_0, b_0 \in x \circ y$ such that $\mu_A(a_0) = \inf_{a \in x \circ y} \mu_A(a)$ and $\gamma_A(b_0) = \sup_{b \in x \circ y} \gamma_A(b)$. It follows that

$$\mu_A(x) \geq \min\{\inf_{a \in x \circ y} \mu_A(a),\, \mu_A(y)\} = \min\{\mu_A(a_0), \mu_A(y)\}$$

and

$$\gamma_A(x) \leq \max\{\sup_{b \in x \circ y} \gamma_A(b),\, \gamma_A(y)\} = \max\{\gamma_A(b_0), \gamma_A(y)\}.$$

This completes the proof. □

Note that, in a finite hyper $BCK$-algebra, every fuzzy set satisfies the **inf-sup** property. Hence the concept of intuitionistic fuzzy weak hyper $BCK$-ideals and intuitionistic fuzzy $s$-weak hyper $BCK$-ideals coincide in a finite hyper $BCK$-algebra.

**Proposition 3.8.** *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy strong hyper $BCK$-ideal of $H$ and let $x, y \in H$. Then*

(i) *$\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$.*
(ii) *$x \ll y$ implies $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$.*
(iii) *$\mu_A(x) \geq \min\{\mu_A(a), \mu_A(y)\}$ and $\gamma_A(x) \leq \max\{\gamma_A(b), \gamma_A(y)\}$ for all $a, b \in x \circ y$.*

*Proof.* (i) Since $0 \in x \circ x$ for all $x \in H$, we have $\mu_A(0) \geq \inf_{a \in x \circ x} \mu_A(a) \geq \mu_A(x)$ and $\gamma_A(0) \leq \sup_{b \in x \circ x} \gamma_A(b) \leq \gamma_A(x)$, which proves (i).

(ii) Let $x, y \in H$ be such that $x \ll y$. Then $0 \in x \circ y$ and so $\sup_{c \in x \circ y} \mu_A(c) \geq \mu_A(0)$ and $\inf_{d \in x \circ y} \gamma_A(d) \leq \gamma_A(0)$. It follows from (i) that

$$\mu_A(x) \geq \min\{\sup_{c \in x \circ y} \mu_A(c),\, \mu_A(y)\} \geq \min\{\mu_A(0),\, \mu_A(y)\} = \mu_A(y)$$

and

$$\gamma_A(x) \leq \max\{\inf_{d \in x \circ y} \gamma_A(d),\, \gamma_A(y)\} \leq \max\{\gamma_A(0),\, \gamma_A(y)\} = \gamma_A(y).$$

(iii) Let $x, y \in H$. Since

$$\mu_A(x) \geq \min\{\sup_{c \in x \circ y} \mu_A(c),\, \mu_A(y)\} \geq \min\{\mu_A(a),\, \mu_A(y)\}$$

and

$$\gamma_A(x) \leq \max\{\inf_{d \in x \circ y} \gamma_A(d),\, \gamma_A(y)\} \leq \max\{\gamma_A(b),\, \gamma_A(y)\}$$

for all $a, b \in x \circ y$, we have the desired result. □

The following corollaries are straightforward.

**Corollary 3.9.** *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy strong hyper $BCK$-ideal of $H$, then*

$$\mu_A(x) \geq \min\{\inf_{a \in x \circ y} \mu_A(a),\, \mu_A(y)\} \text{ and } \gamma_A(x) \leq \max\{\sup_{b \in x \circ y} \gamma_A(b),\, \gamma_A(y)\}$$

*for all $x, y \in H$.*

**Corollary 3.10.** *Every intuitionistic fuzzy strong hyper $BCK$-ideal is both an intuitionistic fuzzy s-weak hyper $BCK$-ideal (and hence an intuitionistic fuzzy weak hyper $BCK$-ideal) and an intuitionistic fuzzy hyper $BCK$-ideal.*

**Proposition 3.11.** *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy hyper $BCK$-ideal of $H$ and let $x, y \in H$. Then*

(i) $\mu_A(0) \geq \mu_A(x)$ *and* $\gamma_A(0) \leq \gamma_A(x)$.

(ii) *If $A = (\mu_A, \gamma_A)$ satisfies the* **inf-sup** *property, then*

$$\mu_A(x) \geq \min\{\mu_A(a),\, \mu_A(y)\} \text{ and } \gamma_A(x) \leq \max\{\gamma_A(b),\, \gamma_A(y)\}$$

*for some $a, b \in x \circ y$.*

*Proof.* (i) Since $0 \ll x$ for all $x \in H$, it follows from (k1) that $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$.

(ii) Since $A = (\mu_A, \gamma_A)$ satisfies the **inf-sup** property, there exist $a_0, b_0 \in x \circ y$ such that $\mu_A(a_0) = \inf_{a \in x \circ y} \mu_A(a)$ and $\gamma_A(b_0) = \sup_{b \in x \circ y} \gamma_A(b)$. Hence

$$\mu_A(x) \geq \min\{\inf_{a \in x \circ y} \mu_A(a),\, \mu_A(y)\} = \min\{\mu_A(a_0),\, \mu_A(y)\}$$

and

$$\gamma_A(x) \leq \max\{\sup_{b \in x \circ y} \gamma_A(b),\, \gamma_A(y)\} = \max\{\gamma_A(b_0),\, \gamma_A(y)\}.$$

This completes the proof. □

**Corollary 3.12.** (i) *Every intuitionistic fuzzy hyper $BCK$-ideal is an intuitionistic fuzzy weak hyper $BCK$-ideal.*

(ii) *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$ satisfying the* **inf-sup** *property, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy s-weak hyper $BCK$-ideal of $H$.*

*Proof.* Straightforward. □

The following example shows that the converse of Corollary 3.10 and Corollary 3.12(i) may not be true.

**Example 3.13.** (1) Consider the hyper $BCK$-algebra $H$ in Example 3.2. Define an IFS $A = (\mu_A, \gamma_A)$ in $H$ by

$$\mu_A(x) := \begin{cases} 0.8 & \text{if } x = 0, \\ 0.5 & \text{if } x = a, \\ 0.3 & \text{if } x = b, \end{cases} \qquad \gamma_A(x) := \begin{cases} 0.08 & \text{if } x = 0, \\ 0.15 & \text{if } x = a, \\ 0.23 & \text{if } x = b, \end{cases}$$

Then we can see that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$ and hence it is also an intuitionistic fuzzy weak hyper $BCK$-ideal of $H$. But it is not an intuitionistic fuzzy strong hyper $BCK$-ideal of $H$ since

$$\min\{\sup_{w\in b\circ a} \mu_A(w),\, \mu_A(a)\} = \min\{\mu_A(a),\, \mu_A(a)\} = 0.5 \nleqq \mu_A(b)$$

also

$$\max\{\inf_{w\in b\circ a} \gamma_A(w),\, \gamma_A(a)\} = \max\{\gamma_A(a),\, \gamma_A(a)\} = 0.15 \ngeqq \gamma_A(b).$$

(2) Consider the hyper $BCK$-algebra $H$ in Example 3.2. Define an IFS $A = (\mu_A, \gamma_A)$ in $H$ by

$$\mu_A(x) := \begin{cases} 0.8 & \text{if } x = 0, \\ 0.5 & \text{if } x = b, \\ 0.3 & \text{if } x = a, \end{cases} \qquad \gamma_A(x) := \begin{cases} 0.08 & \text{if } x = 0, \\ 0.15 & \text{if } x = b, \\ 0.23 & \text{if } x = a, \end{cases}$$

Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy weak hyper $BCK$-ideal of $H$ but it is not an intuitionistic fuzzy hyper $BCK$-ideal of $H$ since $a \ll b$ but $\mu_A(a) \ngeqq \mu_A(b)$, also $\gamma_A(a) \nleqq \gamma_A(b)$.

**Theorem 3.14.** *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy strong hyper $BCK$-ideal of $H$, then for every $s, t \in [0, 1]$, the nonempty level sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are strong hyper $BCK$-ideals of $H$.*

*Proof.* Assume that $U(\mu_A; t) \neq \emptyset \neq L(\gamma_A; s)$ for $s, t \in [0, 1]$. Then there exist $a \in U(\mu_A; t)$ and $b \in L(\gamma_A; s)$, and so $\mu_A(a) \geq t$ and $\gamma_A(b) \leq s$. Using Proposition 3.11(i), we get $\mu_A(0) \geq \mu_A(a) \geq t$ and $\gamma_A(0) \leq \gamma_A(b) \leq s$, so $0 \in U(\mu_A; t) \cap L(\gamma_A; s)$. Let $x, y \in H$ be such that $(x \circ y) \cap U(\mu_A; t) \neq \emptyset$ and $y \in U(\mu_A; t)$. Then there exists $a_0 \in (x \circ y) \cap U(\mu_A; t)$ and hence $\mu_A(a_0) \geq t$. It follows that

$$\mu_A(x) \geq \min\{\sup_{a\in x\circ y} \mu_A(a),\, \mu_A(y)\} \geq \min\{\mu_A(a_0),\, \mu_A(y)\} \geq t$$

so that $x \in U(\mu_A; t)$. Now let $u, v \in H$ be such that $(u \circ v) \cap L(\gamma_A; s) \neq \emptyset$ and $v \in L(\gamma_A; s)$. Then we can take $b_0 \in (u \circ v) \cap L(\gamma_A; s)$ and so $\gamma_A(b_0) \leq s$. Hence

$$\gamma_A(u) \leq \max\{\inf_{b\in u\circ v} \gamma_A(b),\, \gamma_A(v)\} \leq \max\{\gamma_A(b_0),\, \gamma_A(v)\} \leq s,$$

which implies $u \in L(\gamma_A; s)$. Consequently, $U(\mu_A; t)$ and $L(\gamma_A; s)$ are strong hyper $BCK$-ideals of $H$. □

We now consider the converse of Theorem 3.14. We need the following notion. An IFS $A = (\mu_A, \gamma_A)$ in $H$ is said to satisfy the **sup-inf** *property* if for any subset $T$ of $H$ there exist $a_0, b_0 \in T$ such that $\mu_A(a_0) = \sup_{a\in T} \mu_A(a)$ and $\gamma_A(b_0) = \inf_{b\in T} \gamma_A(b)$.

**Lemma 3.15.** *Let $A$ be a subset of $H$. If $I$ is a hyper $BCK$-ideal of $H$ such that $A \ll I$, then $A$ is contained in $I$.*

**Theorem 3.16.** *Let $A = (\mu_A, \gamma_A)$ be an IFS in $H$ satisfying the* **sup-inf** *property. If for every $t, s \in [0, 1]$, the nonempty level sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are strong hyper $BCK$-ideals of $H$, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy strong hyper $BCK$-ideal of $H$.*

*Proof.* Suppose that for each $t, s \in [0,1]$ the nonempty level sets $U(\mu_A;t)$ and $L(\gamma_A;s)$ are strong hyper $BCK$-ideals of $H$. For every $x, y \in H$, let $\mu_A(x) = t$ and $\gamma_A(y) = s$. Then $x \in U(\mu_A;t)$ and $y \in L(\gamma_A;s)$. Since $x \circ x \ll x$ and $y \circ y \ll y$ by (p1), we have $x \circ x \ll U(\mu_A;t)$ and $y \circ y \ll L(\gamma_A;s)$. It follows from Lemma 3.15 that $x \circ x \subseteq U(\mu_A;t)$ and $y \circ y \subseteq L(\gamma_A;s)$. Thus, for each $a \in x \circ x$ and $b \in y \circ y$, we have $a \in U(\mu_A;t)$ and $b \in L(\gamma_A;s)$, and so $\mu_A(a) \geq t$ and $\gamma_A(b) \leq s$. Therefore $\inf\limits_{a \in x \circ x} \mu_A(a) \geq t = \mu_A(x)$ and $\sup\limits_{b \in y \circ y} \gamma_A(b) \leq s = \gamma_A(y)$. Now let

$$k := \min\{\sup_{a \in x \circ y} \mu_A(a),\ \mu_A(y)\} \ \text{ and } \ h := \max\{\inf_{b \in x \circ y} \gamma_A(b),\ \gamma_A(y)\}.$$

Since $A = (\mu_A, \gamma_A)$ satisfies the **sup-inf** property, we have

$$\mu_A(a_0) = \sup_{a \in x \circ y} \mu_A(a) \geq \min\{\sup_{a \in x \circ y} \mu_A(a), \mu_A(y)\} = k$$

and

$$\gamma_A(b_0) = \inf_{b \in x \circ y} \gamma_A(b) \leq \max\{\inf_{b \in x \circ y} \gamma_A(b), \gamma_A(y)\} = h$$

for some $a_0, b_0 \in x \circ y$. It follows that $a_0 \in U(\mu_A;k)$ and $b_0 \in L(\gamma_A;h)$ so that $(x \circ y) \cap U(\mu_A;k) \neq \emptyset$ and $(x \circ y) \cap L(\gamma_A;h) \neq \emptyset$. Since $y \in U(\mu_A;k) \cap L(\gamma_A;h)$, we have $x \in U(\mu_A;k) \cap L(\gamma_A;h)$ by (I3). Hence

$$\mu_A(x) \geq k = \min\{\sup_{a \in x \circ y} \mu_A(a),\ \mu_A(y)\}$$

and

$$\gamma_A(x) \leq h = \max\{\inf_{b \in x \circ y} \gamma_A(b),\ \gamma_A(y)\}.$$

Consequently, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy strong hyper $BCK$-ideal of $H$. □

**Theorem 3.17.** *Let $A = (\mu_A, \gamma_A)$ be an IFS in $H$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$ if and only if for every $s, t \in [0,1]$, the nonempty level sets $U(\mu_A;t)$ and $L(\gamma_A;s)$ are hyper $BCK$-ideals of $H$.*

*Proof.* Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$ and $U(\mu_A;t) \neq \emptyset \neq L(\gamma_A;s)$ for any $s, t \in [0,1]$. It is clear that $0 \in L(\mu_A;t) \cap L(\gamma_A;s)$ by Proposition 3.11(i). Let $x, y \in H$ be such that $x \circ y \ll U(\mu_A;t)$ and $y \in U(\mu_A;t)$. Then for any $a \in x \circ y$, there exists $a_0 \in U(\mu_A;t)$ such that $a \ll a_0$. It follows from (k1) that $\mu_A(a) \geq \mu_A(a_0) \geq t$ for all $a \in x \circ y$ so that $\inf\limits_{a \in x \circ y} \mu_A(a) \geq t$. Thus

$$\mu_A(x) \geq \min\{\inf_{a \in x \circ y} \mu_A(a), \mu_A(y)\} \geq t,$$

and so $x \in U(\mu_A;t)$. Therefore $U(\mu_A;t)$ is a hyper $BCK$-ideal of $H$. Now let $x, y \in H$ be such that $x \circ y \ll L(\gamma_A;s)$ and $y \in L(\gamma_A;s)$. Then $x \circ y \ll L(\gamma_A;s)$ implies that for every $b \in x \circ y$ there is $b_0 \in L(\gamma_A;s)$ such that $b \ll b_0$, so $\gamma_A(b) \leq \gamma_A(b_0)$ by (k1). It follows that $\gamma_A(b) \leq \gamma_A(b_0) \leq s$ for all $b \in x \circ y$ so that $\sup\limits_{b \in x \circ y} \gamma_A(b) \leq s$. Then

$$\gamma_A(x) \leq \max\{\sup_{b \in x \circ y} \gamma_A(b), \gamma_A(y)\} \leq s,$$

which implies that $x \in L(\gamma_A; s)$. Consequently, $L(\gamma_A; s)$ is a hyper $BCK$-ideal of $H$.

Conversely, suppose that for each $t, s \in [0,1]$ the nonempty level sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are hyper $BCK$-ideals of $H$. Let $x, y \in H$ be such that $x \ll y$, $\mu_A(y) = t$ and $\gamma_A(y) = s$. Then $y \in U(\mu_A; t) \cap L(\gamma_A; s)$, and so $x \ll U(\mu_A; t)$ and $x \ll L(\gamma_A; s)$. It follows from Lemma 3.15 that $x \in U(\mu_A; t)$ and $x \in L(\gamma_A; s)$ so that $\mu_A(x) \geq t = \mu_A(y)$ and $\gamma_A(x) \leq s = \gamma_A(y)$. Now for any $x, y \in H$ let $t = \min\{\inf_{c \in x \circ y} \mu_A(c),\, \mu_A(y)\}$ and $s = \max\{\sup_{d \in x \circ y} \gamma_A(d),\, \gamma_A(y)\}$. Then $y \in U(\mu_A; t) \cap L(\gamma_A; s)$, and for each $a, b \in x \circ y$ we have

$$\mu_A(a) \geq \inf_{c \in x \circ y} \mu_A(c) \geq \min\{\inf_{c \in x \circ y} \mu_A(c),\, \mu_A(y)\} = t$$

and

$$\gamma_A(b) \leq \sup_{d \in x \circ y} \gamma_A(d) \leq \max\{\sup_{d \in x \circ y} \gamma_A(d),\, \gamma_A(y)\} = s.$$

Hence $a \in U(\mu_A; t)$ and $b \in L(\gamma_A; s)$, which implies that $x \circ y \subseteq U(\mu_A; t)$ and $x \circ y \subseteq L(\gamma_A; s)$. Using (p8), we get $x \circ y \ll U(\mu_A; t)$ and $x \circ y \ll L(\gamma_A; s)$. Combining $y \in U(\mu_A; t) \cap L(\gamma_A; s)$ and $U(\mu_A; t)$ and $L(\gamma_A; s)$ being hyper $BCK$-ideals of $H$, we conclude that $x \in U(\mu_A; t) \cap L(\gamma_A; s)$, and so

$$\mu_A(x) \geq t = \min\{\inf_{c \in x \circ y} \mu_A(c),\, \mu_A(y)\}$$

and

$$\gamma_A(x) \leq s = \max\{\sup_{d \in x \circ y} \gamma_A(d),\, \gamma_A(y)\}.$$

This completes the proof. □

**Lemma 3.18.** *An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$ if and only if the fuzzy sets $\mu_A$ and $\bar{\gamma}_A$ are fuzzy hyper $BCK$-ideals of $H$.*

*Proof.* Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$. Obviously $\mu_A$ is a fuzzy hyper $BCK$-ideal of $H$. Let $x, y \in H$. If $x \ll y$, then $\gamma_A(x) \leq \gamma_A(y)$, and so $\bar{\gamma}_A(x) = 1 - \gamma_A(x) \geq 1 - \gamma_A(y) = \bar{\gamma}_A(y)$. Now

$$\begin{aligned} \bar{\gamma}_A(x) = 1 - \gamma_A(x) &\geq 1 - \max\{\sup_{b \in x \circ y} \gamma_A(b),\, \gamma_A(y)\} \\ &= \min\{\inf_{b \in x \circ y} (1 - \gamma_A(b)),\, 1 - \gamma_A(y)\} \\ &= \min\{\inf_{b \in x \circ y} \bar{\gamma}_A(b),\, \bar{\gamma}_A(y)\}. \end{aligned}$$

Hence $\bar{\gamma}_A$ is a fuzzy hyper $BCK$-ideal of $H$. Conversely, assume that $\mu_A$ and $\bar{\gamma}_A$ are fuzzy hyper $BCK$-ideals of $H$. Let $x, y \in H$. If $x \ll y$, then $1 - \gamma_A(x) =$

$\bar{\gamma}_A(x) \geq \bar{\gamma}_A(y) = 1 - \gamma_A(y)$, and so $\gamma_A(x) \leq \gamma_A(y)$. Moreover,

$$\begin{aligned} 1 - \gamma_A(x) = \bar{\gamma}_A(x) &\geq \min\{\inf_{c \in x \circ y} \bar{\gamma}_A(c), \bar{\gamma}_A(y)\} \\ &= \min\{\inf_{c \in x \circ y} (1 - \gamma_A(c)), 1 - \gamma_A(y)\} \\ &= \min\{1 - \sup_{c \in x \circ y} \gamma_A(c), 1 - \gamma_A(y)\} \\ &= 1 - \max\{\sup_{c \in x \circ y} \gamma_A(c), \gamma_A(y)\}, \end{aligned}$$

and so $\gamma_A(x) \leq \max\{\sup_{c \in x \circ y} \gamma_A(c), \gamma_A(y)\}$. Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$. □

The following two theorems are immediately from Lemma 3.18 and Theorem 3.17, respectively.

**Theorem 3.19.** *Let $A = (\mu_A, \gamma_A)$ be an IFS in $H$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$ if and only if $\Box A := (\mu_A, \bar{\mu}_A)$ and $\Diamond A := (\bar{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy hyper $BCK$-ideals of $H$.*

**Theorem 3.20.** *For any subset $I$ of $H$, let $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ be an IFS in $H$ defined by*

$$\mu_{A(I)}(x) := \begin{cases} t_1 & \text{if } x \in I, \\ t_2 & \text{otherwise,} \end{cases} \qquad \gamma_{A(I)}(x) := \begin{cases} s_1 & \text{if } x \in I, \\ s_2 & \text{otherwise,} \end{cases}$$

*for all $x \in H$, where $t_1, t_2, s_1, s_2 \in [0,1]$ with $t_1 > t_2$, $s_1 < s_2$, $t_i + s_i \leq 1$ for $i = 1, 2$. Then $I$ is a hyper $BCK$-ideal of $H$ if and only if $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$.*

**Theorem 3.21.** *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$, then the set*

$$I := \{x \in H \mid \mu_A(x) = \mu_A(0) \text{ and } \gamma_A(x) = \gamma_A(0)\}$$

*is a hyper $BCK$-ideal of $H$.*

*Proof.* Obviously $0 \in I$. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $\mu_A(x) = \mu_A(0)$, $\gamma_A(x) = \gamma_A(0)$, and for each $a \in x \circ y$ there exists $z \in I$ such that $a \ll z$. Thus $\mu_A(a) \geq \mu_A(z) = \mu_A(0)$ and $\gamma_A(a) \leq \gamma_A(z) = \gamma_A(0)$ by (k1). It follows from (k2) and (k3) that

$$\mu_A(x) \geq \min\{\inf_{a \in x \circ y} \mu_A(a), \mu_A(y)\} = \mu_A(0)$$

and

$$\gamma_A(x) \leq \max\{\sup_{a \in x \circ y} \gamma_A(a), \gamma_A(y)\} = \gamma_A(0)$$

so that $\mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(0)$. Hence $x \in I$, which shows that $I$ is a hyper $BCK$-ideal of $H$. □

The following example shows that arbitrary union of hyper $BCK$-ideals may not be a hyper $BCK$-ideal.

**Example 3.22.** Let $H = \mathbb{N} \cup \{0\} \cup \{\alpha\}$, where $\alpha(\neq 0) \notin \mathbb{N}$. Define a hyperoperation "$\circ$" on $H$ as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } x \leq y, \text{or} x \neq \alpha,\ y = \alpha, \\ \{x\} & \text{if } x > y, \\ \{\alpha\} & \text{if } x = \alpha, y = 0, \\ \mathbb{N} & \text{if } x = \alpha, y \in \mathbb{N}, \\ \mathbb{N} \cup \{0\} & \text{if } x = y = \alpha, \end{cases}$$

for all $x, y \in H$. Then $(H, \circ)$ is a hyper $BCK$-algebra. Now, we prove that for each $k \in \mathbb{N}$, $\mathbb{N}_k = \{0, 1, 2, \cdots, k\}$ is a hyper $BCK$-ideal of $H$. Let $x, y \in H$ be such that $x \circ y \ll \mathbb{N}_k$ and $y \in \mathbb{N}_k$. Assume that $x \notin \mathbb{N}_k$. Then either $x = \alpha$ or $x \in \mathbb{N}$, $x \geq k+1$. If $x \in \mathbb{N}$ and $x \geq k+1$, then $\{x\} = x \circ y \ll \mathbb{N}_k$ since $y \leq k$. Hence there is $0 \leq i \leq k$ such that $x \ll i$, and so $0 \in x \circ i$. It follows from $x > i$ that $x \circ i = \{x\}$ so that $x = 0$, which is a contradiction. Now let $x = \alpha$. If $y = 0$, then $\{\alpha\} = \alpha \circ 0 = x \circ y \ll \mathbb{N}_k$. Hence there is $0 \leq i \leq k$ such that $\alpha \ll i$ and so $0 \in \alpha \circ i$. If $i = 0$, then $0 \in \alpha \circ i = \{\alpha\}$ which is a contradiction. If $i \neq 0$, then $0 \in \alpha \circ i = \mathbb{N}$, which is a contradiction. If $0 < y \leq k$, then $\mathbb{N} = \alpha \circ y = x \circ y \ll \mathbb{N}_k$. Since $k + 1 \in \mathbb{N}$, there exists $0 \leq i \leq k$ such that $k + 1 \ll i$ and thus $0 \in (k + 1) \circ i = \{k + 1\}$ which is a contradiction. Therefore $\mathbb{N}_k$ is a hyper $BCK$-ideal of $H$. Consequently, $\{\mathbb{N}_k \mid k \in \mathbb{N}\}$ is a collection of hyper $BCK$-ideals of $H$. We note that $\bigcup_{k \in \mathbb{N}} \mathbb{N}_i = \mathbb{N}_0$ is not a hyper $BCK$-ideal of $H$.

**Lemma 3.23.** *Let for all $x, y \in H, |x \circ y| < \infty$ and $\{I_p \mid p \in \Lambda \subseteq [0, 1]\}$ be a collection of hyper $BCK$-ideals of $H$ such that $s > t$ if and only if $I_s \subset I_t$ for all $s, t \in \Lambda$. If $q \in [0, 1]$, then $\bigcup_{p \in \Lambda, p \geq q} I_p$ is also a hyper $BCK$-ideal of $H$.*

*Proof.* Obviously, $0 \in \bigcup_{p \in \Lambda, p \geq q} I_p$. Let $x, y \in H$ be such that $x \circ y \ll \bigcup_{p \in \Lambda, p \geq q} I_p$, $y \in \bigcup_{p \in \Lambda, p \geq q} I_p$ and $x \circ y = \{a_1, a_2, ..., a_n\}$. Then $y \in I_r$ for some $r \in \Lambda$ with $r \geq q$, and for every $a_i \in x \circ y$ there exists $b_i \in \bigcup_{p \in \Lambda, p \geq q} I_p$, and so $b_i \in I_{t_i}$ for some $t_i \in \Lambda$ with $t_i \geq q$, such that $a_i \ll b_i$. Let $t = \min\{t_i : 1 \leq i \leq n\}$. Then, for all $1 \leq i \leq n$, $I_{t_i} \subset I_t$ and so $x \circ y \ll I_t$ with $t \geq q$. Without loss of generality we may assume that $r > t$, so $I_r \subset I_t$. It follows from (I2) that $x \in I_t \subset \bigcup_{p \in \Lambda, p \geq q} I_p$. Hence $\bigcup_{p \in \Lambda, p \geq q} I_p$ is a hyper $BCK$-ideal of $H$. □

**Theorem 3.24.** *Let for all $x, y \in H, |x \circ y| < \infty$ and $\{I_t \mid t \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a collection of hyper $BCK$-ideals of $H$ such that $H = \bigcup_{t \in \Lambda} I_t$, and $s > t$ if and only if $I_s \subset I_t$ for all $s, t \in \Lambda$. Then an IFS $A = (\mu_A, \gamma_A)$ in $H$ defined by*

$$\mu_A(x) := \sup\{t \in \Lambda \mid x \in I_t\},\ \gamma_A(x) := \inf\{t \in \Lambda \mid x \in I_t\}$$

*for all $x \in H$ is an intuitionistic fuzzy hyper $BCK$-ideal of $H$.*

*Proof.* According to Theorem 3.17, it is sufficient to show that nonempty level sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are hyper $BCK$-ideals of $H$ for every $s, t \in [0, 1]$. In order to prove that $U(\mu_A;t)$ $(\neq \emptyset)$ is a hyper $BCK$-ideal of $H$, we consider into the following two cases:

$$\text{(i) } t = \sup\{q \in \Lambda \mid q < t\}, \text{ (ii) } t \neq \sup\{q \in \Lambda \mid q < t\}.$$

Case (i) implies that

$$x \in U(\mu_A;t) \Leftrightarrow x \in I_q \, \forall q < t \Leftrightarrow x \in \bigcap_{q<t} I_q,$$

so that $U(\mu_A;t) = \bigcap_{q<t} I_q$, which is a hyper $BCK$-ideal of $H$. In case (ii), we claim that $U(\mu_A;t) = \bigcup_{q\geq t} I_q$. If $x \in \bigcup_{q\geq t} I_q$, then $x \in I_q$ for some $q \geq t$. It follows that $\mu_A(x) \geq q \geq t$, so that $x \in U(\mu_A;t)$. This shows that $\bigcup_{q\geq t} I_q \subseteq U(\mu_A;t)$. Now assume that $x \notin \bigcup_{q\geq t} I_q$. Then $x \notin I_q$ for all $q \geq t$. Since $t \neq \sup\{q \in \Lambda \mid q < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Lambda = \emptyset$. Hence $x \notin I_q$ for all $q > t-\varepsilon$, which means that if $x \in I_q$, then $q \leq t-\varepsilon$. Thus $\mu_A(x) \leq t-\varepsilon < t$, and so $x \notin U(\mu_A;t)$. Therefore $U(\mu_A;t) \subseteq \bigcup_{q\geq t} I_q$, and thus $U(\mu_A;t) = \bigcup_{q\geq t} I_q$ which is a hyper $BCK$-ideal of $H$, by Lemma 3.23. Next we prove that $L(\gamma_A;s)$ is a hyper $BCK$-ideal of $H$. We consider the following two cases:

$$\text{(iii) } s = \inf\{r \in \Lambda \mid s < r\}, \text{ (iv) } s \neq \inf\{r \in \Lambda \mid s < r\}.$$

For case (iii) we have

$$x \in L(\gamma_A;s) \Leftrightarrow x \in I_r \, \forall s < r \Leftrightarrow x \in \bigcap_{s<r} I_r,$$

and hence $L(\gamma_A;s) = \bigcap_{s<r} I_r$ which is a hyper $BCK$-ideal of $H$. For case (iv), there exists $\varepsilon > 0$ such that $(s, s+\varepsilon) \cap \Lambda = \emptyset$. We will show that $L(\gamma_A;s) = \bigcup_{s\geq r} I_r$. If $x \in \bigcup_{s\geq r} I_r$, then $x \in I_r$ for some $r \leq s$. It follows that $\gamma_A(x) \leq r \leq s$ so that $x \in L(\gamma_A;s)$. Hence $\bigcup_{s\geq r} I_r \subseteq L(\gamma_A;s)$. Conversely if $x \notin \bigcup_{s\geq r} I_r$, then $x \notin I_r$ for all $r \leq s$, which implies that $x \notin I_r$ for all $r < s+\varepsilon$, i.e., if $x \in I_r$, then $r \geq s+\varepsilon$. Thus $\gamma_A(x) \geq s+\varepsilon > s$, i.e., $x \notin L(\gamma_A;s)$. Therefore $L(\gamma_A;s) \subseteq \bigcup_{s\geq r} I_r$ and consequently $L(\gamma_A;s) = \bigcup_{s\geq r} I_r$ which is a hyper $BCK$-ideal of $H$, by Lemma 3.23. This completes the proof. □

**Open Problem 3.25.** Does the Lemma 3.23 (and so Theorem 3.24), hold if we omit the condition " $|x \circ y| < \infty$, for all $x, y \in H$" in this lemma?

**Acknowledgements.** Authors would like to express their sincere thanks to the referees for their valuable suggestions and comments.

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