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ON A LOSSY IMAGE COMPRESSION/RECONSTRUCTION METHOD BASED ON FUZZY RELATIONAL EQUATIONS

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Abstract. The pioneer work of image compression/reconstruction based on fuzzy relational equations (ICF) and the related works are introduced. The ICF regards an original image as a fuzzy relation by embedding the brightness level into [0,1]. The compression/reconstruction of ICF correspond to the composition/solving inverse problem formulated on fuzzy relational equations. Optimizations of ICF can be consequently deduced based on fuzzy relational calculus, i.e., computation time reduction/improvement of reconstructed image quality are correspond to a fast solving method/finding an approximate solution of fuzzy relational equations, respectively. Through the experiments using test images extracted from Standard Image DataBAse (SIDBA), the effectiveness of the ICF and its optimizations are shown.

1. Introduction

Fuzzy relational calculus [1] , [2] occupies a central place in fuzzy set theory as driven by the formalisms of logic and set theory. This calculus yields a new model in image processing field as well, i.e., image compression/reconstruction method based on fuzzy relational equations (ICF). The concept of ICF has been successfully proposed by Hirota and Pedrycz [3], and a number of the related works have been consequently developed [4][5]. Due to the flexibility and applicability, several extensions of ICF can be found in image database[6], watermarking[7], motion compression[9]. This paper introduces the ICF by picking up key points as well as showing recent results.

2. Image Compression Method Based on Fuzzy Relational Equations (ICF)

Image compression 2.1. The ICF regards a still grayscale image of size $M \times N$ as a fuzzy relation $R \in F(\mathbf{X} \times \mathbf{Y})$, $\mathbf{X} = \{x_1, x_2, \ldots, x_M\}$, $\mathbf{Y} = \{y_1, y_2, \ldots, y_N\}$ by embedding the intensity range of each pixel into $[0, 1]$. In the case of color image (on RGB color space), an original image can be expressed by three fuzzy relations, i.e., $\mathbf{P} = \{P^{(R)}, P^{(G)}, P^{(B)}\} \subset F(\mathbf{X} \times \mathbf{Y})$, where, $P^{(R)}$,$P^{(G)}$, and $P^{(B)}$ denote the red, green, and blue planes of the color image, respectively. For simplicity, only the case of grayscale images is considered in this paper. The study of compression/reconstruction of color image and its optimizations can be found in [8]. The

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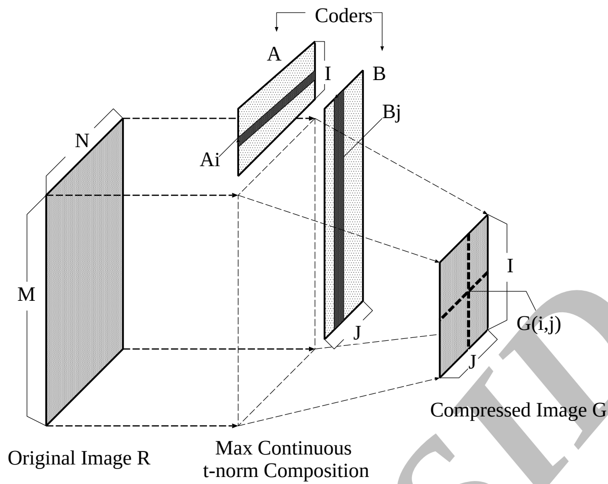

FIGURE 1. Image compression process

grayscale image $R \in F(\mathbf{X} \times \mathbf{Y})$ is compressed into $G \in F(\mathbf{I} \times \mathbf{J})$ by a max t-norm composite fuzzy relational equation [10],

$$G(i,j) = \max_{y \in \mathbf{Y}} \left\{ B_j(y) \ t \ \max_{x \in \mathbf{X}} \{A_i(x) \ t \ R(x,y)\} \right\} \qquad (\in [0,1]), \tag{1}$$

where t denotes a continuous t-norm, and $A_i \in \mathbf{A} = \{A_1, A_2, \ldots, A_I\} \subset F(\mathbf{X})$, $B_j \in \mathbf{B} = \{B_1, B_2, \ldots, B_J\} \subset F(\mathbf{Y})$, are called coders. The image compression process is shown in Fig. 1.

The composition operator of Eq. (1) can be replaced by min s-norm, adjoint of max t-norm, and adjoint of min s-norm composition. The results obtained by fuzzy relational equations with various compositions are mentioned in [5]. The coders $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A} = \{A_1, A_2, \ldots, A_I\}, \tag{2}$$

$$A_i(x_m) = \exp\left(-Sh\left(\frac{iM}{I} - m\right)^2\right), \tag{3}$$

$$(m = 1, 2, \ldots, M),$$

and

$$\mathbf{B} = \{B_1, B_2, \ldots, B_J\}, \tag{4}$$

$$B_j(y_n) = \exp\left(-Sh\left(\frac{jN}{J} - n\right)^2\right), \tag{5}$$

$$(n = 1, 2, \ldots, N),$$

where Sh denotes the sharpness of fuzzy sets A_i and B_j contained in coders $\mathbf{A}$ and $\mathbf{B}$. A preferable range of Sh depends on the compression rate, and it will be mentioned in the following image compression and reconstruction experiments. Notice that coders **A** and **B** have employed Sh. The shape of the fuzzy sets of coders are Gaussian-like that are preferable for the ICF. The compression rate of ICF can be adjusted by the number of fuzzy sets contained in coders **A** and **B**. The compression rate ρ is defined as

$$\rho = \frac{IJ}{MN}, \tag{6}$$

where IJ and MN denote the number of coefficients of compressed image and original one, respectively. In this paper, Yager's t-norm [10] is used as the continuous t-norm appearing in fuzzy relational equation (1).

$$\begin{aligned} &\text{(Yager's t-norm)} \ : \\ &at^{(p)}b = 1 - \min\left\{1, \{(1-a)^p + (1-b)^p\}^{1/p}\right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (p \geq 1). \end{aligned} \tag{7}$$

Yager's t-norm can cover all continuous t-norms by changing the parameter p, whereas Zadeh's t-norm and other major t-norms cannot cover all the continuous t-norms. Frank's t-norm can also cover all continuous t-norms, but it has high computational complexity compared with Yager's one [1].

Image reconstruction 2.2. The reconstruction of an image $R' \in F(\mathbf{X} \times \mathbf{Y})$ from the compressed image G is considered to be an inverse problem, under the condition that **A**, **B**, and G are given. That is, the image reconstruction corresponds to the estimation of the solution R' that satisfies the fuzzy relational equations (1). In the case of max t-norm composite fuzzy relational equations, the structure of the solution set $\mathbf{R} \subset F(\mathbf{X} \times \mathbf{Y})$ is shown in Fig. 2 and the greatest and minimal solutions of **R** can be obtained analytically [1].

Examples of reconstructed images (the greatest and one of the minimal solutions) are shown in Fig. 4. The reconstructed images are obtained under the conditions that compression rate $= 0.0625$ ($I \times J = 64 \times 64$) and $Sh = 0.04$. As can be seen from Fig. 4, the minimal solutions present bright pixels scattered on a black plane (background), yielding a meaningless result from the viewpoint of reconstructed image, whereas the greatest solution (Fig. 4, left) has quality comparable to the original image. The greatest solution has been presented by Hirota and Pedrycz [3], and will be referred as H-P method. In the case of H-P method, if Sh is a large value, the reconstructed image has high sharpness, but the white slit is more prominent in the image (see Fig. 5, left side image). Conversely, if Sh is a small value, then the white slit is decreased, but the reconstructed image is blurred (see Fig. 5, right side image). Therefore, a compromise value between such opposite sides is employed and the study of such the appropriate coders design problem can be found in [8].

In [3], the greatest solution $\hat{R}$ of fuzzy relational equations (1) is given by

$$\hat{R}(x,y) = \min_{(i,j)\in\mathbf{I}\times\mathbf{J}} \left\{(A_i(x)\ t\ B_j(y))\ \varphi_t\ G(i,j)\right\}, \tag{8}$$

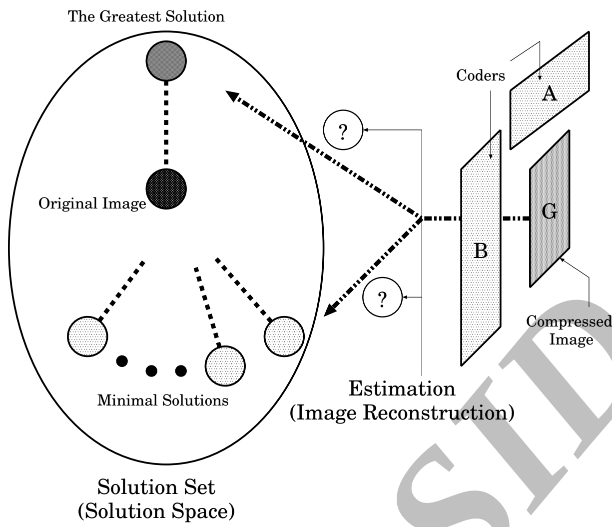

FIGURE 2. Image reconstruction

FIGURE 3. Original image (Lenna)

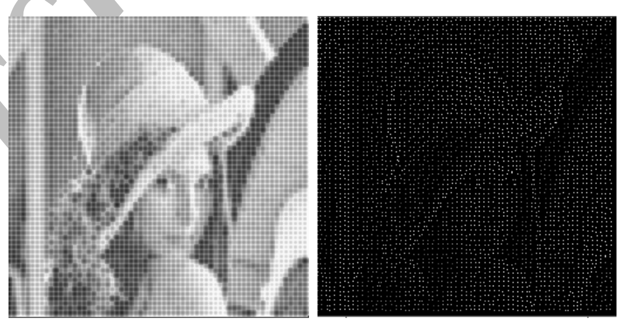

FIGURE 4. The greatest (left) and one of minimal solutions (right) (Compression Rate = 0.0625)

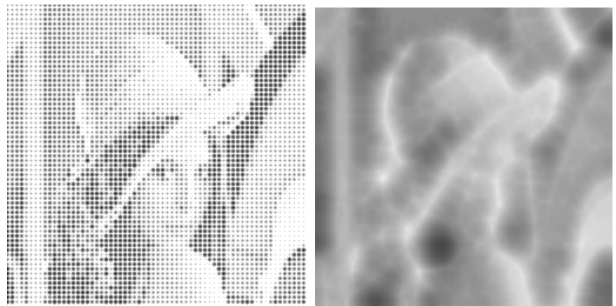

FIGURE 5. Left side image : the greatest, Sh = 0.2 , right side image : the greatest, Sh = 0.001 (Compression Rate = 0.0625)

where,

$$a \ \varphi_t \ b = \sup\{c \in [0, 1] | a \ t \ c \leq b\}. \quad (9)$$

Image compression and reconstruction experiments 2.3. In order to investigate the computation time of ICF, an image compression and reconstruction experiment using 20 images (extracted from Standard Image DataBAse, SIDBA) is performed. The original images of size $M \times N = 256 \times 256$ are compressed into images $I \times J = 32 \times 32$ and $I \times J = 64 \times 64$ pixels. In the experiment, the parameter p of Yager's t-norm is set to 1.0 and Sh of coders **A** and **B** is set to 0.01 and 0.04, respectively, under the condition that the compression rate is 0.0156 ($I \times J = 32 \times 32$) and 0.0625 ($I \times J = 64 \times 64$), respectively.

The averages of image compression and reconstruction times of ICF are shown in Table 1. One example of the original image and the reconstructed one (H-P method, the greatest solution) are shown in Figs. 3 and 4, respectively. Table 1 shows that the image reconstruction time (H-P method) is large. Therefore, a faster image reconstruction method for ICF is proposed in section 3.

	Compression Time (Average)	Reconstruction Time (Average)
$\rho = 0.0156$	0.93(s)	168.98(s)
$\rho = 0.0625$	1.43(s)	546.67(s)

TABLE 1. Image compression and reconstruction time (H-P method), computed using a Sun Ultra 10 (440MHz) workstation

3. Fast Image Reconstruction Method of ICF

Derivation of fast image reconstruction method 3.1. In order to reduce the image reconstruction time of ICF, a fast image reconstruction method has been proposed in [4]. It is based on the following theorem:

Theorem 3.2. *H-P method (Eq. (8)) is equivalent to*

$$\hat{R}(x,y) = \min_{j\in\mathbf{J}} \left\{ B_j(y)\ \varphi_t\ \left\{ \min_{i\in\mathbf{I}} \{A_i(x)\ \varphi_t\ G(i,j)\} \right\} \right\}. \tag{10}$$

Theorem 1 tells us the greatest solution obtained by the H-P method (Eq. (8)) is the same one obtained by the proposed method (Eq. (10)). The theoretical computation time of the H-P method and the proposed method correspond to

$$\text{H-P method} \quad : \quad IJMN(L+T+P), \tag{11}$$

$$\text{Proposed method} \quad : \quad JM(I+N)(L+P), \tag{12}$$

where, MN and IJ denote the size of original image R and compressed image G, respectively. The computation time of min, t-norm, and relative pseudocomplement (Eq. (9)) are denoted L, T, and P, respectively. The theoretical comparison shows that the computation time of the proposed method is shorter than that of the H-P method.

Experimental comparison of computation time 3.3. A comparison of the proposed method with the H-P method from the viewpoint of reconstruction time is presented by an image compression and reconstruction experiment using 20 images (selected from SIDBA). The original images of size $M \times N = 256 \times 256$ pixels are compressed into images of size $I \times J = 32 \times 32$ ($\rho = 0.0156$) and $I \times J = 64 \times 64$ ($\rho = 0.0625$). The image reconstruction is performed by both the H-P method and the proposed method. The result of comparison is shown in Table 2. The decrease of the computation time to 1/132.02 and 1/382.29 are obtained, respectively.

	H-P method (Average)	Proposed method (Average)
$\rho = 0.0156$	168.98(s)	1.28(s)
$\rho = 0.0625$	546.67(s)	1.62(s)

TABLE 2. Image reconstruction time comparison of H-P method with proposed method, Computed using Sun Ultra 10 (440MHz)

4. Improvement of Quality of Reconstructed Image

Solution space structure and reformulation of image reconstruction problem 4.1. As shown in 2, the quality of the reconstructed image based on the greatest solution is low due to a white slits on the image (Fig. 4, left). In order to improve the quality of the reconstructed image, an approximated solution of the fuzzy relational equations (Equation. (1) has been proposed in [5].

Approximated Solution of Fuzzy Relational Equation [5]. Let's select a t-norm t_2 such that

$$\forall a,b \in [0,1] \quad a\ t_1\ b \le a\ t_2\ b. \tag{13}$$

Then, a fuzzy relation $\tilde{R}$ is obtained by

$$\tilde{R}(x,y) = \min_{j\in\mathbf{J}} \left\{ B_j(y)\ \varphi_{t2} \Big\{ \min_{i\in\mathbf{I}} \big(A_i(x)\ \varphi_{t2}\ G(i,j) \big) \Big\} \right\}, \tag{14}$$

where φ_{t2} denotes the t-relative pseudocomplement of t-norm t_2.

The method is based on the fast solving method of fuzzy relational equations (Eq. (10)), particularly, the method is equivalent to Eq. (10) when t_1 is equal to t_2. Therefore, the computation time of the proposed method is comparable with the image reconstruction based on the greatest solution.

Image compression and reconstruction experiment 4.2. In order to evaluate the effectiveness of the proposed method, an image compression and reconstruction experiment using 20 test images of SIDBA is performed. In this experiment, Yager's t-norm (7) is used, which can adjust the order of t-norm by changing the parameter p. The greatest solution [3] [4] performs the image compression and reconstruction under the condition that the parameter $p = 1.0$. On the other hand, the proposed method compresses an image under the condition that the parameter $p = 1.0$, while the image reconstruction is performed under the condition of $p = 2.1$.

The Peak Signal to Noise Ratio (PSNR) comparison of the proposed method with the greatest solution is shown in Fig. 6. The PSNR is defined as

$$PSNR = 20\log_{10}\left(\frac{255}{RMSE}\right), \tag{15}$$

where RMSE stands for root mean square error defined by

$$RMSE = \sqrt{\frac{\sum_{(x,y)\in\mathbf{X}\times\mathbf{Y}} (R(x,y) - \hat{R}(x,y))^2}{|\mathbf{X}\times\mathbf{Y}|}}. \tag{16}$$

The results of Fig. 6 correspond to the average value of PSNR for 20 images. Furthermore, under the condition that the compression rates are 0.0625 and 0.25, the improvements for each image are presented in Table 3.

	The Greatest (PSNR, Average)	Proposed Method (PSNR, Average)
$\rho = 0.0625$	15.69	21.97
$\rho = 0.25$	20.08	25.38

Table 3. PSNR comparison

Figure 6 and Table 3 show that the quality of the reconstructed image obtained by the proposed method is better than that of the greatest solution in terms of PSNR. With respect to the aspect of the reconstructed image, the white slits on the reconstructed image of the original method can be reduced by applying the proposed method (Figs. 7 - 8).

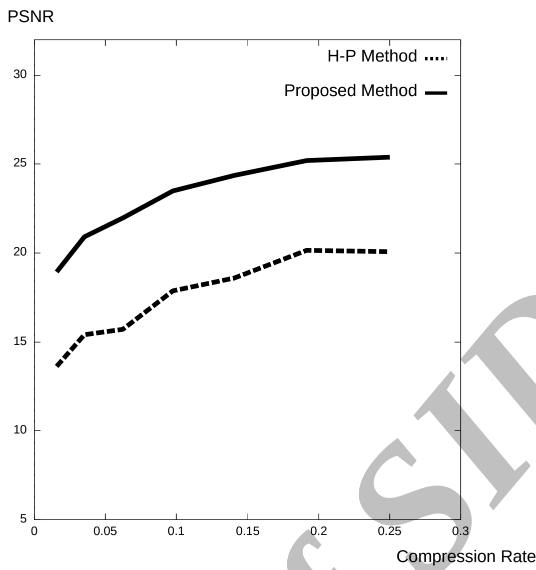

FIGURE 6. PSNR comparison

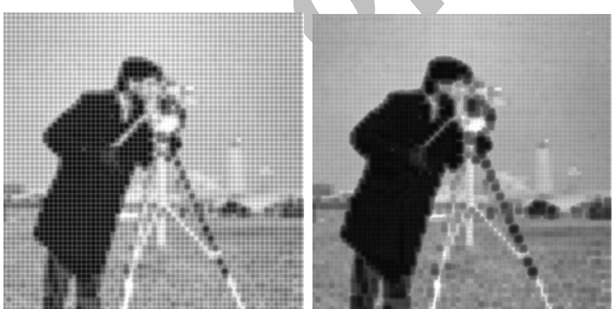

FIGURE 7. Reconstructed images (Cameraman), left : the great solution, right : proposed method

5. Conclusions

A lossy **I**mage **C**ompression and reconstruction method based on **F**uzzy relational equations (ICF) and its optimizations have been introduced.

In image compression and reconstruction experiments using 20 images (selected from SIDBA), it is confirmed that the decrease of the image reconstruction time to 1/132.02 and 1/382.29 under compression rates of 0.0156 and 0.0625, respectively. Furthermore, it is shown that the quality of the reconstructed image obtained by the proposed method is better than that of the conventional one from the viewpoint of PSNR (Peak Signal to Noise Ratio).

FIGURE 8. Reconstructed images (Barbara), left : the great solution, right : proposed method

It is still increasingly recognized that image/motion compression schemes are quite important in engineering fields, due to the development and spread of IT technology. The ICF would be a major impulse to the development of the image/motion compression to produce solutions that satisfy the demands from the IT technology.

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