A SHORT NOTE ON THE RELATIONSHIP BETWEEN GOAL PROGRAMMING AND FUZZY PROGRAMMING FOR VECTORMAXIMUM PROBLEMS

M. A. YAGHOOBI AND M. TAMIZ

Abstract. A theorem was recently introduced to establish a relationship between goal programming and fuzzy programming for vectormaximum problems. In this short note it is shown that the relationship does not exist under all circumstances. The necessary correction is proposed.

1. Introduction

The vectormaximum problem, known today as multi-objective programming, was first mentioned by H.W. Kuhn and A.W. Tucker in [6] and is applied extensively in practice [3, 10]. However, there exist many imprecise factors in real world problems. R. Bellman and L.A. Zadeh [1] first proposed the concept of decision making in a fuzzy environment involving several objectives and H.J. Zimmerman in [13] applied their approach to a vectormaximum problem. He transformed the fuzzy multi-objective linear programming (FMLP) problem to a classic single objective linear programming (LP). Extensive research in FMLP has been carried out after this pioneering work [7]. Another major methodology for dealing with multi-objective programming is goal programming (GP). GP was first introduced by A. Charnes and W.W. Cooper in [2] and it is widely used for solving real life problems [4, 5, 11, 12]. Recently, a relationship between fuzzy programming (FP) and GP for solving an FMLP is introduced in [8]. In this paper it is shown that the relationship is not correct in general, and a correction is suggested.

2. Two Methods for FMLP

Consider the following FMLP model.

$$\begin{aligned} \max \ &: \ Z = CX \\ &\text{s.t.} \\ &AX \leq b, \end{aligned} \tag{1}$$

where $Z = (z_1, ..., z_k)^T$ is the vector of objectives, C is a $k \times n$ matrix of constants, X is an $n \times 1$ vector of decision variables, A is an $m \times n$ matrix of constant and b

Received: July 2004; Accepted: July 2005
Key words and phrases: Fuzzy programming, Goal programming, Fuzzy multi-objective programming .

is an $m \times 1$ vector of constants.
An adapted fuzzy model of (1) due to Zimmerman [13] is:

$$(2) \qquad \begin{array}{rcl} CX & \gtrsim & \bar{Z} \\ \text{s.t.} & & \\ AX & \lesssim & b, \end{array}$$

where $\bar{Z} = (\bar{z_1}, ..., \bar{z_k})^T$ is the vector of aspiration levels, $\gtrsim$ and $\lesssim$ are the fuzzification of $\geq$ and $\leq$ respectively. $\gtrsim$ ($\lesssim$) means essentially greater (lower) than.
For measurement of solutions X in model (2) Zimmerman suggested the simplest kind of membership functions as follows.

$$(3)\, \mu_i(c^iX) = \begin{cases} 1 & & c^iX \geq \bar{z}_i \\ 1 - \frac{\bar{z}_i - c^iX}{d_{1i}} & \bar{z}_i - d_{1i} \leq & c^iX \leq \bar{z}_i \\ 0 & & c^iX \leq \bar{z}_i - d_{1i} \end{cases} \qquad i = 1, ..., k$$

$$(4)\, \mu_i(a^iX) = \begin{cases} 1 & & a^iX \leq b_i \\ 1 - \frac{a^iX - b_i}{d_{2i}} & b_i \leq & a^iX \leq b_i + d_{2i} \\ 0 & & a^iX \geq b_i + d_{2i} \end{cases} \qquad i = 1, ..., m,$$

where c^i $(i = 1, ..., k)$ and a^i $(i = 1, ..., m)$ are the ith row of matrices C and A respectively. d_{1i} $(i = 1, ..., k)$ and d_{2i} $(i = 1, ..., m)$ are subjectively chosen constants of admissible violations from aspiration levels of objective and constraints respectively.

In [8], FP and GP are introduced as two main approaches for solving (2). The min-operator is used to transform (2) with membership functions (3) and (4) to the following LP by FP approach. This approach was first also suggested by Zimmerman [13].

$$(5) \qquad \begin{array}{lll} \max & y & \\ \text{s.t.} & & \\ y & \leq & 1 - (\bar{z}_i - C^iX)/d_{1i} \qquad i = 1, ..., k, \\ y & \leq & 1 - (a^iX - b_i)/d_{2i} \qquad i = 1, ..., m, \\ y & \geq & 0, \ X \geq 0, \end{array}$$

Another approach based on GP for solving (2) is proposed by Mohamed [8]: "It uses the fact that the maximum value of any membership function is 1, hence maximizing any of them is equivalent to making them as close as possible to 1 by minimizing its negative deviational variable from 1." Thus (2) is converted to a GP problem that can be solved by any variants of GP [4, 12]. The following model based on a MinMax variant of GP is suggested by Mohamed [8]:

$$
\begin{array}{rcll}
\min\ \phi & & & \\
\text{s.t.} & & & \\
1-(\bar{z}_i - C^iX)/d_{1i} + n_{1i} - p_{1i} & = & 1 & i=1,...,k \\
1-(a^iX - b_i)/d_{2i} + n_{2i} - p_{2i} & = & 1 & i=1,...,m \\
\phi & \geq & n_{1k} & i=1,...,k \\
\phi & \geq & n_{2i} & i=1,...,m \\
X\geq 0,\ n_{1k}\geq 0,\ p_{1k}\geq 0,\ n_{2i}\geq 0,\ p_{2i} & \geq & 0, & \\
n_{1k}.p_{1k}=0,\ n_{2i}.p_{2i} & = & 0, & \\
k=1,...,K,\ i=1,...,M. & & &
\end{array} \tag{6}
$$

Model (6) can be rewritten in a simple form as follows [8]:

$$
\begin{array}{rcll}
\min\ u & & & \\
\text{s.t.} & & & \\
C^iX + n_{1i} - p_{1i} & = & \bar{z}_i & i=1,...,k \\
a^iX + n_{2i} - p_{2i} & = & b_i & i=1,...,m \\
u & \geq & n_{1k}/d_{1i} & i=1,...,k \\
u & \geq & n_{2i}/d_{2i} & i=1,...,m \\
X\geq 0,\ n_{1k}\geq 0,\ p_{1k}\geq 0,\ n_{2i}\geq 0,\ p_{2i} & \geq & 0, & \\
n_{1k}.p_{1k}=0,\ n_{2i}.p_{2i} & = & 0, & \\
k=1,...,K,\ i=1,...,M. & & &
\end{array} \tag{7}
$$

In a theorem in [8] an equivalence between (5) and (7) is established. In [8], it is stated that every fuzzy linear program has an equivalent weighted linear goal programming where the weights are the reciprocals of the admissible violation constants. For proving the relationship, it is asserted that model(5s a fuzzy linear program and model (7) as a linear goal program are equivalent. However, this paper shows that the equivalence is not correct in general by using the following example.

Example 2.1. This example is taken from Mohamed's paper [8, PP. 221-222].

$$
\begin{array}{rcl}
\max\quad z_1 & = & -x_1 + 2x_2 \\
\max\quad z_2 & = & 2x_1 + x_2 \\
\text{s.t.} & &
\end{array}
$$

$$-x_1 + 3x_2 \leq 21 \tag{8}$$

$$x_1 + 3x_2 \leq 27 \tag{9}$$

$$4x_1 + 3x_2 \leq 45 \tag{10}$$

$$3x_1 + x_2 \leq 30 \tag{11}$$

$$x_1,\ x_2 \geq 0. \tag{12}$$

Mohamed considered $\bar{z_1} = 14$ and $\bar{z_2} = 21$ as the aspiration levels for the two objective functions, where 14 and 21 are the optimum values of solving z_1 and z_2 subject to constraints (8)-(12) respectively. Also, the admissible violations for the two objective functions d_{11} and d_{12} are arbitrarily set as 17 and 14 respectively by Mohamed.

Models (5) and (7) are applied to this problem and since the same results are yielded, the proposed approach (model (7)) and their equivalence are considered to be valid [8]. However, in this paper, the values of d_{11} and d_{12} are changed from 17 and 14 to 4 and 2, respectively. These changes are permissible as d_{11} and d_{12} are subjectively chosen constants of admissible violations. By applying these changes to the proposed GP approach by Mohamed, model (7), the following model is obtained:

$$\begin{array}{rrcl} \min & u & & \\ & \text{s.t.} & & \\ & -x_1 + 2x_2 + n_1 - p_1 & = & 14 \\ (13) & 2x_1 + x_2 + n_2 - p_2 & = & 21 \\ & u & \geq & n_1/4 \\ & u & \geq & n_2/2 \\ & \text{Plus constraints (8)} & - & (12) \\ & n_1,\ n_2,\ p_1,\ p_2 & \geq & 0, \end{array}$$

with the optimal solution of $x_1^\star = 5.6,\ x_2^\star = 7.13,\ n_1^\star = 5.33,\ n_2^\star = 2.66,\ p_1^\star = p_2^\star = 0$ and $u^\star = 1.33$.

By applying the FP approach, model (5), to this example with $d_{11} = 4$ and $d_{12} = 2$ the following LP is obtained.

$$\begin{array}{rrcl} \max & y & & \\ & \text{s.t.} & & \\ (14) & y & \leq & -0.25x_1 + 0.5x_2 - 2.5, \\ & y & \leq & x_1 + 0.5x_2 - 9.5, \\ & y & \geq & 0, \\ & & & \text{Plus constraints } (8) - (12). \end{array}$$

Model (14) has no feasible solution. □

The results obtained from models (13) and (14) show the stated equivalence in the theorem of [8] is incorrect. This is due to the fact that in (6) and (7) deviational variables n_{1k} and n_{2i} are deviations from a fuzzy membership function. Since fuzzy membership functions have always a value between zero and one, therefore, n_{1k} and n_{2i} cannot take values greater than one. Since ϕ and u are upper bounds for n_{1k} and n_{2i}, it is suggested that $\phi \leq 1$ and $u \leq 1$ should be added to the constraints of (6) and (7) respectively. The conditions $\phi \leq 1$ and $u \leq 1$ are used implicitly by Mohamed [8] and it may make a confusion for reader. Some authors have realized this point such as B.B. Pal *et al.* [9] on page 398, line -3. But it may happen that someone does not realize it. Here in this paper, we made it clear.

3. Concluding Remarks

In this note the equivalence between two methods for solving fuzzy multi-objective programming are considered and it is shown, by an example, the equivalence in its reported form does not hold for all cases. The problem is rectified by addition of a simple bound constraint to the model.

Acknowledgements. The authors wish to thank the Scholarship and Overseas Student Office, Student Deputy-Ministry of Science, Research and Technology in Iran for supporting Mr Yaghoobi's visit to Portsmouth to carry out the research. The authors are also grateful to Dr Dylan Jones in Portsmouth for his valuable discussions and contributions.

References

[1] R. Bellman and L. A. Zadeh, *Decision making in a fuzzy environment*, Management Sciences, **17(4)** (1970) B141-B164.
[2] A. Charnes and W. W. Cooper, *Management models and industrial applications of linear programming*, John Wiley and Sons, New York, 1961.
[3] M. Ehrgott, *Multicriteria optimization*, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 2000.
[4] J. P. Ignizio, *Goal programming and extentions*, Lexington Books, London, 1976.
[5] D. F. Jones and M. Tamiz, *Goal programming in the period 1990-2000*, In: M. Ehrgott, X. Gandibleux, (Eds.), Multicriteria Optimization: State of the Art Annotated Bibliographic Survey, Kluwer Academic Publisher, Boston, 2002, Chapter 3.
[6] H. W. Kuhn and A. W. Tucker, *Nonlinear programming*, In: J. Neyman, (Ed.), Proceedings of 2nd Berkeley Symposium on Mathematical Statistics and Probabilities, 1951.
[7] Y. -J. Lai and C. L. Hwang, *Fuzzy Multiple Objective Decision Making: Methods and Applications*, Lecture Notes in Economics and Mathematical Systems, Vol. 404, Springer, New York, 1994.
[8] R. H. Mohamed, *The relationship between goal programming and fuzzy programming*, Fuzzy Sets and Systems, **89** (1997) 215-222.
[9] B. B. Pal, B. N. Morita and U. Maulik, *A goal programming procedure for fuzzy multiobjective linear programming problem*, Fuzzy Sets and Systems, **139** (2003) 395-405.
[10] R. E. Steuer, *Multiple Criteria Optimization: Theory, Computation and Application*, John Wiley, New York, 1986.
[11] M. Tamiz, D. F. Jones and E. El-Darzi, *A review of goal programming and its applications*, Annals of Operations Research, **58** (1993) 39-53.
[12] M. Tamiz, D. Jones and C. Romero, *Goal programming for decision making: An overview of the current state-of-the-art*, European Journal of Operational Research, **111** (1998) 569-581.
[13] H. J. Zimmerman, *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets and Systems, **1** (1978) 45-55.

M. A. Yaghoobi*, Faculty of Mathematics and Computer Sciences, University of Kerman, Kerman, Iran
E-mail address: yaghoobi@mail.uk.ac.ir

M. Tamiz, Department of Mathematics, University of Portsmouth, Buckingham Building, Lion Terrace, Portsmouth, PO1 3HE, UK
E-mail address: mehrdad.tamiz@port.ac.uk

∗Corresponding author