FIXED POINT THEOREM ON INTUITIONISTIC FUZZY METRIC SPACES

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ABSTRACT. In this paper, we introduce intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces.

1. Introduction

The notion of intuitionistic fuzzy metric spaces was introduced and studied by Park in [5]. Saadati and Park in [6], further developed the theory of intuitionistic fuzzy topology (both in metric and normed) spaces. In this paper, we introduce an intuitionistic fuzzy contraction mapping and prove a fixed point theorem in intuitionistic fuzzy metric spaces. For the basic notions and concepts, we refer to [1, 3, 4, 5, 6].

2. Preliminaries

We review some basic concepts in intuitionistic fuzzy metric spaces as well as the intuitionistic fuzzy topology due to Saadati and Park [6].

Definition 2.1. [5, 6] A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions: (i) * is associative and commutative; (ii) * is continuous; (iii) a * 1 = a for all $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$. Two typical examples of continuous t-norm are

$$a * b = ab$$
,

$$a * b = min(a, b).$$

Definition 2.2. [5, 6] A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous tconorm if it satisfies the following conditions: (i) \diamond is associative and commutative; (ii) \diamond is continuous; (iii) $a \diamond 0 = a$ for all $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$. Two typical examples of t-conorm are

$$a \diamond b = min(a+b,1),$$

$$a \diamond b = max(a, b).$$

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Lemma 2.3. [5, 6] If * is a continuous t-norm and \diamond is continuous t-conorm, then: (i) For every $a,b \in [0,1]$, if a > b, there are $c, d \in [0,1]$ such that $a * c \ge b$ and $a \ge b \diamond d$.

(ii) If $a \in [0,1]$, there are $b, c \in [0,1]$ such that $b * b \ge a$ and $a \ge c \diamond c$.

The following definition is obtained from Mihet in [4].

Definition 2.4. [4] A fuzzy metric space in the sense of Kramosil and Michalek is a triple (X, M, *) where X is a nonempty set, * is a continuous t-norm and $M: X^2 \times [0, \infty) \rightarrow [0, 1]$ is a mapping which satisfies the following properties for every $x, y, z \in X$:

(FM-1)
$$M(x, y, 0) = 0;$$

- (FM-2) $M(x, y, t) = 1, \forall t > 0 \Leftrightarrow x = y;$
- (FM-3) $M(x, y, t) = M(y, x, t), \forall t > 0;$
- (FM-4) $M(x, y, \cdot) \colon [0, \infty) \to [0, 1]$ is left continuous;
- (FM-5) $\lim_{t\to\infty} M(x, y, t) = 1;$
- (FM-6) $M(x, z, t+s) \ge M(x, y, t) * M(y, z, s), \forall t, s > 0.$

In (X, M, *), the open ball $B_x(r, t)$ for t > 0 with center $x \in X$ and radius $r \in (0, 1)$ is defined as

(1)
$$B_x(r,t) = \{ y \in X \mid M(x,y,t) > 1-r \}.$$

The family $\{B_x(r,t) \mid x \in X, r \in (0,1), t > 0\}$ is a neighborhood system for a Hausdorff topology on X induced by the fuzzy metric M. In a similar fashion, the dual space of (X, M, *) is the fuzzy metric space (X, N, \diamond) defined below:

Definition 2.5. (New) A fuzzy metric space (X, N, \diamond) , where X is a nonempty set, \diamond is a continuous t-conorm and $N: X^2 \times [0, \infty) \to [0, 1]$ is a mapping assumed to satisfies the following properties for all $x, y, z \in X$:

(FM-D1) N(x, y, 0) = 1;

(FM-D2) $N(x, y, t) = 0, \forall t > 0 \Leftrightarrow x = y;$

(FM-D3) $N(x, y, t) = N(y, x, t), \forall t > 0;$

- (FM-D4) $N(x, y, \cdot) \colon [0, \infty) \to [0, 1]$ is left continuous;
- (FM-D5) $\lim_{t\to\infty} N(x, y, t) = 0;$

 $(\text{FM-D6}) \ N(x,z,t+s) \leq N(x,y,t) \diamond N(y,z,s), \forall t,s > 0.$

In (X, N, \diamond) , the open ball $D_x(r, t)$ for t > 0 with center $x \in X$ and radius $r \in (0, 1)$ is defined as

(2)
$$D_x(r,t) = \{y \in X \mid N(x,y,t) < r\}.$$

The family $\{D_x(r,t) \mid x \in X, r \in (0,1), t > 0\}$ is a neighborhood's system for a Hausdorff topology on X induced by the fuzzy metric N. The following definition is introduced and studied by Park in [5].

Definition 2.6. [5] A 5-tuple $(X, M, N, *, \diamond)$ is called a intuitionistic fuzzy metric space if X is an arbitrary nonempty set, * a continuous t-norm, \diamond a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and t, s > 0: (a) $M(x, y, t) + N(x, y, t) \leq 1$; $\begin{array}{l} (\mathbf{b})M(x,y,t) > 0;\\ (\mathbf{c})M(x,y,t) = 1 \Leftrightarrow x = y;\\ (\mathbf{d})M(x,y,t) = M(y,x,t);\\ (\mathbf{e})M(x,y,t) * M(y,z,s) \leq M(x,z,t+s);\\ (\mathbf{f})M(x,y,\cdot) \colon [0,\infty) \to [0,1] \text{ is left continuous;}\\ (\mathbf{g})N(x,y,t) = N(y,x,t);\\ (\mathbf{h})N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s);\\ (\mathbf{i})N(x,y,\cdot) \colon [0,\infty) \to [0,1] \text{ is continuous.} \end{array}$

The pair (M, N) is called an intuitionistic fuzzy metric on X. Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \diamond)$ such that t-norm * and t-conorm \diamond are associated [6], i.e $x \diamond y = 1 - [(1-x)*(1-y)]$ for any $x, y \in X$.

Let $(X, M, N, *, \diamond)$ be a intuitionistic fuzzy metric space. For t > 0, the open ball $G_x(r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ is defined by

(3)
$$G_x(r,t) = \{ y \in X \mid M(x,y,t) > 1 - r, N(x,y,t) < r \}.$$

Note that it can be easily seen that $G_x = B_x \cap D_x$ where B_x and D_x as given by (1) and (2) respectively.

Since * and \diamond are respectively a continuous t-norm and t-conorm, the family $\{G_x(r,t) \mid x \in X, r \in (0,1), t > 0\}$ generates a topology $T_{(M,N)}$, called the (M,N)-topology (see [3, 6]). We have:

 $A \in T_{(M,N)}$ if and only if $\forall x \in A, \exists t > 0, \exists r \in (0,1)$ such that $G_x(r,t) \subset A$.

We denote the (M, N)-uniformity (or the uniformity generated by M, and N) by $U_{(M,N)}$. The family $\{U_{r,t}\}_{r \in (0,1), t > 0}$, where

$$U_{r,t} = \{(x,y) \in X^2 \mid M(x,y,t) > 1 - r, N(x,y,t) < r\},\$$

is a base for this uniformity.

Definition 2.7. [5, 6] Let $(X, M, N, *, \diamond)$ be the intuitionistic fuzzy metric space endowed with (m, n)-topology and $\{x_n\}$ in X. Then

(i) $x_n \to x \Leftrightarrow M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$, for each t > 0.

(ii) $\{x_n\}$ is called a (M, N)-Cauchy sequence if for each $r \in (0, 1)$ and t > 0, there exists an integer n_0 such that $M(x_n, x_m, t) > 1 - r$ and $N(x_n, x_m, t) < r$ for each $n, m \ge n_0$.

(iii) The intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be (M, N)-complete if every (M, N)-Cauchy sequence is convergent.

3. Main Results

In the following sequel the letters \mathbb{N} and \mathbb{R}^+ denote the sets of positive integer numbers and positive real numbers, respectively.

Definition 3.1. [2] A quasi-metric on a set X is a function $d: X^2 \to \mathbb{R}^+$ satisfying the following conditions for every $x, y, z \in X$: (QM-1)d(x, y) = 0;

(QM-2)d(x, y) = d(y, x); $(QM-3)d(x, z) \le d(x, y) + d(y, z).$ M. Rafi and M. S. M. Noorani

Proposition 3.2. Let $(X, M, N, *, \diamond)$ be the intuitionistic fuzzy metric space. For any $r \in (0, 1]$, we define $d: X^2 \to \mathbb{R}^+$ as follows:

(4)
$$d_r(x,y) = \inf\{t > 0 \mid M(x,y,t) > 1 - r, N(x,y,t) < r\}$$

Then,

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(1) $(X, d_r : r \in (0, 1])$ is a generating space of a quasi-metric family.

(2) the topology $T_{(d_r)}$ on $(X, d_r : r \in (0, 1])$ coincides with the (M, N)-topology on $(X, M, N, *, \diamond)$, (i.e., d_r is a compatible symmetric for $T_{(M,N)}$).

Proof. (1)From the definition of $\{d_r : r \in (0,1]\}$, it is easy to see that $\{d_r : r \in (0,1]\}$ satisfies the condition (QM-1) and (QM-2) of Definiton 3.1. Now we prove that $\{d_r : r \in (0,1]\}$ also satisfies the condition (QM-3). Since * and \diamond are continuous, by Lemma 2.3.(ii), for any given $r \in (0,1)$, there exists $r' \in (0,r)$ such that

$$(1 - r') * (1 - r') > 1 - r$$

and

$$r' \diamond r' < r$$

. Setting $d_r(x,y) = a$ and $d_r(y,z) = b$, in equation (4), it follows that for any given t > 0,

$$M(x, y, a + t) > 1 - r', N(x, y, a + t) < r'$$

and

$$M(x, z, b+t) > 1 - r', N(y, z, b+t) < r'$$

Whence

$$M(x, z, a + b + 2t) \ge M(x, y, a + t) * M(y, z, b + t) > (1 - r') * (1 - r') > 1 - r'$$

and

$$N(x, z, a+b+2t) \leq N(x, y, a+t) \diamond N(y, z, b+t) < r' \diamond r' < r.$$

Hence, we have

$$d_r(x,z) \le a+b+2t = d_r(x,y) + d_r(y,z) + 2t$$

By the arbitrariness of t > 0, we have

$$d_r(x,z) \le d_r(x,y) + d_r(y,z).$$

(2) To prove this condition, it is only necessary to show that for any t > 0 and $r \in (0, 1)$

$$d_r(x, y) < t \Leftrightarrow M(x, y, t) > 1 - r, N(x, y, t) < r.$$

In fact, if $d_r(x,y) < t$, then by (4), we have M(x,y,t) > 1-r and N(x,y,t) < r. Conversely, if M(x,y,t) > 1-r and N(x,y,t) < r, since M and N are continuous functions, there exists an s > 0 such that M(x,y,t-s) > 1-r and N(x,y,t-s) < r and so $d_r(x,y) \le t-s < t$. This completes the proof. \Box Fixed Point Theorem on Intuitionistic Fuzzy Metric Spaces

In fuzzy metric spaces (X, M, *), the map $f: X \to X$ is said to be a fuzzy contraction if there exists $k \in (0, 1)$ such that

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(5)
$$\frac{1}{M(f(x), f(y), t)} - 1 \le k(\frac{1}{M(x, y, t) - 1}), \forall x, y \in X, \forall t > 0.$$

Several fixed point theorem has been proved by using (5) in a fuzzy metric spaces (see [1, 7]).

If the fuzzy metric space (X, N, \diamond) is a dual space of (X, M, *), the map $f: X \to X$ in (X, N, \diamond) may also enjoy a fuzzy contractive condition. Since the definition of the dual space (X, N, \diamond) is similar in the sense of metric space, we can consider the following new contractive condition for a self mapping f in (X, N, \diamond) .

Definition 3.3. (New) Let (X, N, \diamond) be a fuzzy metric space. The map $f: X \to X$ is a fuzzy contraction in (X, N, \diamond) if there exists $k \in (0, 1)$ such that

(6)
$$N(f(x), f(y), t) \le kN(x, y, t), \forall x, y \in X, \forall t > 0$$

By using the contractive conditions (5) and (6), now we are able to define an intuitionistic fuzzy contractive map f as follow:

Definition 3.4. (New) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. We say that the mapping $f: X \to X$ is intuitionistic fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \le k(\frac{1}{M(x, y, t) - 1})$$

and

$$N(f(x), f(y), t) \le kN(x, y, t),$$

for each $x, y \in X$ and t > 0.

We give a definition for a intuitionistic fixed point theorem in intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$.

Definition 3.5. (New) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and let $f: X \to X$ be an intuitionistic fuzzy contractive mapping. Then there exists $z \in X$ such that z = f(z) (We call z an intuitionistic fuzzy fixed point of f).

Now, we prove the following theorem.

Theorem 3.6. Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space. Let $f: X \to X$ be an intuitionistic fuzzy contractive mapping. Then f has a unique intuitionistic fixed point. *Proof.* We fix $x_0 \in X$. Let $x_{n+1} = f(x_n), n \in \mathbb{N}$. We have for n > m and t > 0,

$$\begin{aligned} \frac{1}{M(x_n, x_{n+m}, t)} - 1 &= \frac{1}{M(f(x_{n-1}), f(x_{n+m-1}), t)} - 1 \\ &\leq k \left(\frac{1}{M(x_{n-1}, x_{n+m-1}, t)} - 1 \right) \\ &= k \left(\frac{1}{M(f(x_{n-2}), f(x_{n+m-2}), t)} - 1 \right) \\ &\vdots \\ &\leq k^n \left(\frac{1}{M(x_0, x_m, t)} - 1 \right). \end{aligned}$$

Whence, for $n > m, \frac{1}{M(x_n, x_{n+m}, t)} - 1 \to 0$ as $n \to \infty$, that is $M(x_n, x_{n+m}, t) \to 1$ as $n \to \infty$. Also, for n > m and t > 0, by (6) we have

$$N(x_{n}, x_{n+m}, t) = N(f(x_{n-1}), f(x_{n+m-1}), t)$$

$$\leq kN(x_{n-1}, x_{n+m-1}, t)$$

$$= kN(f(x_{n-2}), f(x_{n+m-2}), t)$$

$$\vdots$$

$$\leq k^{n}N(x_{0}, x_{m}, t).$$

Whence, for n > m, $N(x_n, x_{n+m}, t) \to 0$ as $n \to \infty$. Therefore, we can conclude that $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M, N, *, \diamond)$ is complete, the sequence $\{x_n\}$ converges to some $y \in X$. We show that y is an intuitionistic fixed point of f, i.e y = f(y). By the contractive condition (5) of f, we have

$$\frac{1}{M(f(y), f(x_n), t)} - 1 \le k \left(\frac{1}{M(y, x_n, t)} - 1\right) \to 0, asn \to \infty$$

Hence, $\lim_{n\to\infty} M(f(y), f(x_n), t) = 1$ for every t > 0. And by (6), we have:

 $N(f(y), f(x_n), t) \le kN(y, x_n, t) \to 0asn \to \infty$

. Thus, $\lim_{n\to\infty}N(f(y),f(x_n),t)=0$ for every t>0. In both cases, we have $\lim_{n\to\infty}f(x_n)=f(y)$, i.e., $\lim_{n\to\infty}x_{n+1}=f(y)$, therefore y=f(y). For uniqueness, assume z=f(z) for some $z\in X$. Then for t>0, we have

$$\frac{1}{M(y,z,t)} - 1 = \frac{1}{M(f(y), f(z), t)} - 1$$

$$leq \quad k\left(\frac{1}{M(y,z,t)} - 1\right)$$

$$\vdots$$

$$leq \quad k^n\left(\frac{1}{M(y,z,t)} - 1\right) \to 0asn \to \infty.$$

whence, for every t > 0 we have M(y, z, t) = 1, it follows that z = y. This completes the proof.

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