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SOME INTUITIONISTIC FUZZY CONGRUENCES

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Abstract. First, we introduce the concept of intuitionistic fuzzy group congruence and we obtain the characterizations of intuitionistic fuzzy group congruences on an inverse semigroup and a T^*-pure semigroup, respectively. Also, we study some properties of intuitionistic fuzzy group congruence. Next, we introduce the notion of intuitionistic fuzzy semilattice congruence and we give the characterization of intuitionistic fuzzy semilattice congruence on a T^*-pure semigroup. Finally, we introduce the concept of intuitionistic fuzzy normal congruence and we prove that $(\mathrm{IFNC}(E_S), \cap, \vee)$ is a complete lattice. And we find the greatest intuitionistic fuzzy normal congruence containing an intuitionistic fuzzy congruence on E_S.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh [28] in 1965. Since then, there has been a tremendous interest in the subject due to its diverse applications ranging from engineering and computer science to social behavior studies. In particular, some researchers [9, 22-24, 26, 27] applied the notion of fuzzy sets to congruences.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1] in 1983. After that time, several researchers [3, 5-8, 10, 11, 13, 14, 16, 17, 19] applied the notion of intuitionistic fuzzy sets to relation, algebra, topology and topological group. In particular, Hur and his colleagues [15, 18] introduce the notion of intuitionistic fuzzy congruences on a lattice and a semigroup, and investigate some of their properties, respectively.

In this paper, first, we introduce the concept of intuitionistic fuzzy group congruence and we obtain the characterizations of intuitionistic fuzzy group congruences on an inverse semigroup and a T^*-pure semigroup, respectively. Next, we introduce the notion of intuitionistic fuzzy semilattice congruence and we give the characterization of intuitionistic fuzzy semilattice congruence on a T^*-pure semigroup. Finally, we introduce the concept of intuitionistic fuzzy normal congruence and we prove that $(\mathrm{IFNC}(E_S), \cap, \vee)$ is a complete lattice.

2. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections.

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For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I. And for a lattice, refer to [4]. For any ordinary relation R on a set X, we will denote the characteristic function of R as χ_R.

Definition 2.1. [2, 6] Let X be a nonempty set. A complex mapping $A = (\mu_A,\ \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 2.2. [2] Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then:
(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$,
(2) $A = B$ iff $A \subset B$ and $B \subset A$,
(3) $A^c = (\nu_A, \mu_A)$,
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$,
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$,
(6) $[\ ]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 2.3. [7] Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then:
(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$,
(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 2.4. [6] Let X be a set. Then a complex mapping $R = (\mu_R,\ \nu_R) : X \times X \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $\mu_R(x, y) + \nu_R(x, y) \leq 1$ for each $(x, y) \in X \times X$, i.e., $R \in$ IFS $(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 2.5. [6, 10] Let X be a set and let $R, Q \in$ IFR(X). Then the *composition* of R and Q, $Q \circ R$, is defined as follows : for any $x, y \in X$,
$\mu_{Q \circ R}(x, y) = \bigvee_{z \in X}[\mu_R(x, z) \wedge \mu_Q(z, y)]$
and
$\nu_{Q \circ R}(x, y) = \bigwedge_{z \in X}[\nu_R(x, z) \vee \nu_Q(z, y)]$.

Definition 2.6. [6, 10] An Intutionistic fuzzy Relation R on a set X is called an *intutionsitic fuzzy equivalence relation* (in short, IFER) on X if it satisfies the following conditions:
(i) it is *intutionsitic fuzzy reflexive*, i.e.,$R(x, x) = (1, 0)$ for each $x \in X$,

(ii) it is *intutionsitic fuzzy symmetric*, i.e.,$R(x,y) = R(y,x)$ for any $x, y \in X$,
(iii) it is *intutionsitic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \to I \times I$ as follows : for each $x \in X$

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in$ IFS(X). The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of* X *by* R and denoted by X/R.

Result 2.A. [19, Theorem 2.15] *Let R be an intuitionistic fuzzy equivalence relation on a set X. Then the followings hold :*

(1) $Ra = Rb$ if and only if $R(a, b) = (1, 0)$ for any $a, b \in X$.

(2) $R(a, b) = (0, 1)$ if and only if $Ra \cap Rb = 0_{\sim}$ for any $a, b \in X$.

(3) $\bigcup_{a \in X} Ra = 1_{\sim}$.

(4) There exists the surjection $p : X \to X/R$ defined by $p(x) = Rx$ for each $x \in X$.

Definition 2.7. [19] Let X be a set and let $R \in$ IFR(X). Then the *intuitionistic fuzzy transitive closure* of R, denoted by R^{∞}, is defined as follows :

$$R^{\infty} = \bigcup_{n \in \mathbb{N}} R^n, \text{ where } R^n = R \circ R \circ \cdots \circ R(n \text{ factors}).$$

Result 2.B. [19, Proposition 3.7] *Let X be a set and let $R, Q \in$ IFE(X). We define $R \vee Q$ as follows : $R \vee Q = (R \cup Q)^{\infty}$, i.e., $R \vee Q = \bigcup_{n \in \mathbb{N}} (R \cup Q)^n$. Then $R \vee Q \in$ IFE(X).*

Definition 2.8. [18] An IFR R on a groupoid S is said to be:
(1) *intuitionistic fuzzy left compatible* if $\mu_R(x, y) \leq \mu_R(zx, zy)$ and $\nu_R(x, y) \geq \nu_R(zx, zy)$, for any $x, y, z \in S$,
(2) *intuitionistic fuzzy right compatible* if $\mu_R(x, y) \leq \mu_R(xz, yz)$ and $\nu_R(x, y) \geq \nu_R(xz, yz)$, for any $x, y, z \in S$,
(3) *intuitionistic fuzzy compatible* if $\mu_R(x, y) \wedge \mu_R(z, t) \leq \mu_R(xz, yt)$ and $\nu_R(x, y) \vee \nu_R(z, t) \geq \nu_R(xz, yt)$, for any $x, y, z, t \in S$.

Definition 2.9. [18] An IFER R on a groupoid S is called an:
(1) *intuitionistic fuzzy left congruence* (in short, $IFLC$) if it is intuitionistic fuzzy left compatible,
(2) *intuitionistic fuzzy right congruence* (in short, $IFRC$) if it is intuitionistic fuzzy right compatible,
(3) *intuitionistic fuzzy congruence* (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLCs and IFRCs] on a groupoid S as IFC(S) [resp. IFLC(S) and IFRC(S)].

Result 2.C. [19, Theorem 2.8] *Let R be relation on a groupoid S. Then $R \in C(S)$ if and only if $(\chi_R, \chi_{R^c}) \in IFC(S)$.*

Let R be an intuitionistic fuzzy congruence on a semigroup S and let $a \in S$. The intuitionistic fuzzy set Ra in S is called an *intuitionistic fuzzy congruence class of R containing $a \in S$* and we will denote the set of all intuitionistic fuzzy congruence classes of R as S/R.

Result 2.D. [19, Proposition 2.21 and Theorem 2.22] *Let R be an intuitionistic fuzzy congruence on a semigroup S. We define the binary operation $*$ on S/R as follows:for any $a, b \in S$,*

$$Ra * Rb = Rab.$$

Then $*$ is well-defined. Moreover, $(S/R, *)$ is a semigroup.

Result 2.E. [19, Corollary 2.22-1] *Let R be an intuitionistic fuzzy congruence on an inverse semigroup S. Then $(S/R, *)$ is an inverse semigroup.*

3. Intuitionistic Fuzzy Group Congruences

A congruence R on an inverse semigroup S (or indeed on any semigroup) is called a *group congruence* [12] if S/R is a group.

Definition 3.1. An intuitionistic fuzzy congruence R on a semigroup S is called an *intuitionistic fuzzy group congruence* (in short, $IFGC$) if $(S/R, *)$ is a group.

We will denote the set of all IFGCs on S as IFGC(S).

It is clear that if S is an inverse semigroup, then S/R is an inverse semigroup by Result 1.E. Since a group is an inverse semigroup having only one idempotent , R is an IFGC if and only if Re= R_f for any $e, f \in E_S$, where E_S denotes the set of all idempotents of a semigroup S.

Result 3.A. [12, Theorem V.3.1] *If S is an inverse semigroup with semilattice of idempotents E_S, then the relation*

$$\delta = \{(a, b) \in S \times S : ea = eb \text{ for some } e \in E_S\}$$

is the least group congruence on S.

The following is the immediate result of Result 3.A and Result 2.C.

Proposition 3.2. *Let S be an inverse semigroup. Then $(\chi_\delta, \chi_{\delta^c}) \in$ IFGC(S).*

Theorem 3.3. *Let S be an inverse semigroup and let R be an intuitionistic fuzzy congruence on S. Then $R \in$ IFGC(S) if and only if $\delta \subset R^{-1}((1,0))$.*

Proof. ($\Rightarrow$) Suppose $R \in$ IFGC(S) and let $(a,b) \in H$. Then there exists an $e \in E_S$ such that $ea = eb$. By the hypothesis, (S/R, $*$) is a group. Then, we have

$$Ra = Re * Ra = Rea = Reb = Re * Rb = Rb.$$

By Result 2.A(1), $R(a,b) = (1,0)$. Thus $(a,b) \in R^{-1}((1,0))$. Hence $\delta \subset R^{-1}((1,0))$.

($\Leftarrow$) Suppose $\delta \subset R^{-1}((1,0))$. Let $e, f \in E_S$. Since S is an inverse semigroup, E_S is commutative (See [12, Theorem V.1.2]). Then $efe \in E_S$ and $(efe)e$=$(efe)f$. Thus $(e,f) \in \delta$. Since $\delta \subset R^{-1}((1,0))$, $(e,f) \in R^{-1}((1,0))$, i.e, $R_e = R_f$. So S/R is a group. Hence $R \in$ IFGC(S). This completes the proof. □

Proposition 3.4. *Let S be a regular semigroup and let $R \in$ IFC(S). If Ra is an idempotent element of S/R, then there exists an idempotent $e \in S$ such that $Re = Ra$.*

Proof. Suppose Ra is an idempotent element of S/R. Then $Ra * Ra = Ra^2 = Ra$. By Result 2.A(1), $R(a^2, a) = (1,0)$. Since S is regular and $a^2 \in S$, there exists an $x \in S$ such that $a^2 = a^2xa^2$ and $x = xa^2x$. Let $e = axa$. Then $e^2 = (axa)(axa) = (axa^2)xa = axa = e$. Thus e is an idempotent element of S. So,

$$\begin{aligned}
\mu_R(e,a^2) &= \mu_R(axa, a^2xa^2) \geq \mu_R(a,a^2) \wedge \mu_R(xa, xa^2) \\
&\geq \mu_R(a,a^2) \wedge \mu_R(x,x) \wedge \mu_R(a,a^2) \\
&= 1 \text{ (Since } \mu_R(x,x) = 1)
\end{aligned}$$

and

$$\begin{aligned}
\nu_R(e,a^2) &= \nu_R(axa, a^2xa^2) \leq \nu_R(a,a^2) \vee \nu_R(xa, xa^2) \\
&\leq \nu_R(a,a^2) \vee \nu_R(x,x) \vee \nu_R(a,a^2) = 0.
\end{aligned}$$

Thus $R(e,a^2) = (1,0)$. On the other hand,

$$\mu_R(e,a) \geq \bigvee_{z \in S} [\mu_R(e,z) \wedge \mu_R(z,a)] \geq \mu_R(e,a^2) \wedge \mu_R(a^2,a) = 1$$

and

$$\nu_R(e,a^2) \leq \bigwedge_{z \in S} [\nu_R(e,z) \vee \nu_R(z,a)] \leq \nu_R(e,a^2) \vee \nu_R(a^2,a) = 0.$$

So $R(e,a) = (1,0)$. Hence $Re = Ra$. This completes the proof. □

Let S be a semigroup. A nonempty subset A of S is called a *subsemigroup* of S if $A^2 \subset A$, and is called a *bi-ideal* of S if $ASA \subset A$. A bi-ideal A of S is said to be *T-pure* if $A \cap xSy = xAy$ for any $x, y \in S$. S is said to be *T^*-pure* if every bi-ideal of S is T-pure. It is well-known[20, Theorem 2.3] that S is T^*-pure if and only if S^3 is a semilattice of groups. Thus note that the set E_S of idempotents of a T^*-pure semigroup S is nonempty.

Result 3.B. [21, Theorem 2.6] *The set E_S of all idempotents of a T^*-pure semigroup S is a semilattice.*

Proposition 3.5. *Let R be an IFC on a T^*-pure semigroup S and let $a \in S$. If Ra is an idempotent of S/R, then there exists $e \in E_S$ such that $Ra = Re$.*

Proof. Suppose Ra is an idempotent of S/R. Then

$$Ra = Ra * Ra = Ra * Ra * Ra * Ra = Ra^4.$$

Since aSa is a bi-ideal of S and S is T^*-pure,

$$a^4 \in aSa = aSa \cap aSa = a^2Sa^2 = a^4Sa^4.$$

Thus a^4 is a regular element of S. Let x be an inverse in S of a^4. Then $a^4 = a^4xa^4$ and $x = xa^4x$. Let $e = a^2xa^2$. Then

$$e^2 = (a^2xa^2)(a^2xa^2) = a^2(xa^4x)a^2 = a^2xa^2 = e.$$

Thus $e \in E_S$. So

$$\begin{aligned} Re &= Ra^2xa^2 \\ &= Ra^2 * Rx * Ra^2 = (Ra)^2 * Rx * (Ra)^2 \\ &= (Ra)^4 * Rx * (Ra)^4 = Ra^4 * Rx * Ra^4 \\ &= Ra^4xa^4 = Ra^4 = Ra. \end{aligned}$$

This complete the proof. □

Let R be a congruence on a semigroup S. Then it is clear that $R^* = (\chi_R, \chi_{R^c})$ is an IFC on S and S/R^* is a semigroup by Result 2.D. Moreover, we can easily prove that S/R and S/R^* are isomorphic. Hence, we have the following result.

Proposition 3.6. *Let S be a T^*-pure semigroup. Then $(\chi_\delta, \chi_{\delta^c}) \in$ IFGC(S).*

Proposition 3.7. *Let S be a T^*-pure semigroup. If $R \in$ IFC(S) such that $(\chi_\delta, \chi_{\delta^c}) \subset R$, then $R \in$ IFGC(S).*

Proof. It is clear that $(S/R, *)$ is a semigroup by Result 2.D. Let $e, f \in E_S$. Then clearly $(e, f) \in \delta$. Thus

$$\mu_R(e, f) \geq \chi_\delta(e, f) = 1 \text{ and } \nu_R(e, f) \leq \chi_{\delta^c}(e, f) = 0.$$

So $R(e, f) = (1, 0)$. By Result 2.A(1), $Re = Rf$. Let $e \in E_S$ and let $a \in S$.
Then $e(ea) = (ee)a = ea$. Thus $(ea, a) \in \delta$. So

$$\mu_R(ea, a) \geq \chi_\delta(ea, e) = 1 \text{ and } \nu_R(ea, a) \leq \chi_{\delta^c}(ea, e) = 0.$$

Thus $R(ea, a) = (1, 0)$. By Result 2.A(1), $Rea = Ra$. So $Re * Ra = Ra$. Hence R_e is an identity of S/R. Since aSa is a bi-ideal of S and S is T^*-pure,

$$a^3 \in aSa = aSa \cap a^2Sa^2 = a^3Sa^3.$$

Then there exists an $x \in S$ such that $a^3 = a^3xa^3$. Since $xa^3 \in E_S$ and Re is an idempotent of S/R, by Proposition 3.5, $Rxa^3 = Re$. Thus

$$Rxa^2 * Ra = Rxa^3 = Re.$$

So Rxa^2 is an inverse of R_a in S/R. Hence S/R is a group. Therefore $R \in$ IFGC(S). This completes the proof. □

Theorem 3.8. *Let R be an IFC on a T^*-pure semigroup S. Then the following are equivalent:*
(1) $R \in$ IFGC(S),
(2) If $(a,b) \in \delta$, then $Ra = Rb$.

Proof. $(1) \Rightarrow (2)$: Suppose R is an IFGC on S. Let $a, b \in S$ such that $(a,b) \in \delta$. Then, by Result 3.A, there exists an $e \in E_S$ such that $ea = eb$. Since R is an IFGC on S, Re is an identity of S/R. Then

$$Ra = Re * Ra = Rea = Reb = Re * Rb = Rb.$$

Hence $Ra = Rb$.

$(2) \Rightarrow (1)$: Suppose the condition (2) holds. Let $a, b \in S$. Suppose there exists an $e \in E_S$ such that $ea = eb$. Then $(a,b) \in \delta$. Thus $\chi_\delta(a,b) = 1$ and $\chi_{\delta^c}(a,b) = 0$. Moreover, by the hypothess, $R_a = R_b$. Then, by Result 2.A(1), $R(a,b) = (1,0)$. Thus $(\chi_\delta,\ \chi_{\delta^c}) = R$. Suppose there exists no $e \in E_S$ such that $ea = eb$. Then $(a,b) \notin \delta$. Thus $\chi_\delta(a,b) = 0$ and $\chi_{\delta^c}(a,b) = 1$. So $\chi_\delta(a,b) \leq \mu_R(a,b)$ and $\chi_{\delta^c}(a,b) \geq \nu_R(a,b)$. Hence, in all, $(\chi_\delta,\ \chi_{\delta^c}) \subset R$. Therefore, by Proposition 3.7, $R \in$ IFGC(S). This completes the proof. □

4. Intuitionistic fuzzy semilattice congruences

A congruence R on a semigroup S is called a *semilattice congruence* if S/R is a semilattice. An IFC R on a semigroup S is called an *intuitionistic fuzzy semilattice congruence* (in short, $IFSC$) if S/R is a semilattice.

We will denote the set of all IFSCs of S as IFSC(S).

Result 4.A. [23, Lemma 9] *Let S be a T^*-pure semigroup and let $a, b \in S$. Then*
$aSb = a^2Sb^2$ *and* $abSab = baSba$.

Result 4.B. [23, Theorem 10] *Let S be a T^*-pure semigroup and let δ^* be the binary relation on S defined by*

$$\delta^* = \{(a,b) \in S \times S : a^3 \in bSb \text{ and } b^3 \in aSa\}.$$

Then δ^* is the least semilattice congruence on S.
The following is easily seen.

Proposition 4.1. *Let S be a T^*-pure Semigroup. Then $(\chi_{\delta^*},\ \chi_{\delta^{*c}}) \in$ IFSC(S).*

Proposition 4.2. *Let S be a T^*-pure semigroup. If $R \in$ IFC(S) such that $(\chi_{\delta^*},\ \chi_{\delta^{*c}}) \subset R$, then $R \in$ IFSC(S).*

Proof. It is clear that $(S/R, *)$ is a semigroup. Let $a, b \in S$. Then $a^3 \in a^2Sa^2$ and $(a^2)^3 = a^6 \in aSa$. Thus $(a, a^2) \in \delta^*$. So

$$\mu_R(a,a^2) \geq \chi_{\delta^*}(a,a^2) = 1 \text{ and } \nu_R(a,a^2) \leq \chi_{\delta^{*c}}(a,a^2) = 0.$$

Thus $R(a,a^2) = (1,0)$. By Result 2.A(1), $R_a = R_{a^2}$. Then $R_a = R_{a^2} = R_a * R_a$. Since $(ab, ba) \in \delta^*$,

$$\mu_R(ab,ba) \geq \chi_{\delta^*}(ab,ba) = 1 \text{ and } \nu_R(ab,ba) \leq \chi_{\delta^{*c}}(ab,ba) = 0.$$

Then $R(ab, ba) = (1, 0)$. By Result 1.A(1), $Rab = Rba$. Then $Ra * Rb = Rb * Ra$. So S/R is a semilattice. Hence $R \in \text{IFSC}(S)$. □

Theorem 4.3. *Let R be an IFC on a T^*-pure semigroup S. Then the following are equivalent:*
(1) $R \in IFSC(S)$,
(2) $(a, b) \in \delta^*$ implies $Ra = Rb$.

Proof. $(1) \Rightarrow (2)$: Suppose R is an IFSC on S. Let $a, b \in S$ such that $(a, b) \in \delta^*$. Then $a^3 \in bSb$ and $b^3 \in aSa$. Thus there exist $x, y \in S$ such that $a^3 = bxb$ and $b^3 = aya$. Since R is an IFSC on S,

$$\begin{aligned} Ra &= (Ra)^3 = Ra^3 = Rbxb = Rb * Rxb = (Rb)^2 * Rxb \\ &= Rb * Rbxb = Rb * Ra^3 = Rb * (Ra)^3 = Rb * Ra \\ &= (Rb)^3 * Ra = Rb^3 * Ra = Raya * Ra = Ray * (Ra)^2 \\ &= Ray * Ra = Raya = Rb^3 = (Rb)^3 = Rb. \end{aligned}$$

$(2) \Rightarrow (1)$: Suppose the condition (2) holds. Let $a, b \in S$. Then clearly $(a, a^2) \in \delta^*$ and $(ab, ba) \in \delta^*$. By the hypothesis,

$$Ra = Ra^2 \text{ and } Rab = Rba.$$

Thus $Ra = Ra * Ra$ and $Ra * Rb = Rb * Ra$. So S/R is a semilattice. Hence $R \in$ IFSC(S). This completes the proof. □

5. **Intuitionistic fuzzy normal congruences**

Let S be an inverse semigroup. It is clear that if $x \in S$ and $e \in E_S$, then xx^{-1}, $x^{-1}x$ and $x^{-1}ex \in E_S$.

Definition 5.1. Let S be an inverse semigroup. Then an intuitionistic fuzzy congruence R on E_S is called an *intuitionistic fuzzy normal congruence* (in short, $IFNC$) if for any $e, f \in E_S$ and for each $s \in S$,

$$\mu_R(s^{-1}es, s^{-1}fs) \geq \mu_R(e, f)$$

and

$$\nu_R(s^{-1}es, s^{-1}fs) \leq \nu_R(e, f).$$

We will denote the set of all IFNCs on E_S as IFNC(E_S). Then it is clear that if $P, Q \in \text{IFNC}(E_S)$, then $P \cap Q \in \text{IFNC}(E_S)$.

Definition 5.2. Let R be an IFC on an inverse semigroup S.
(1) The *intuitionistic fuzzy kernel* IFK(R) of R is an IFS in S defined as follows: for each $x \in S$,

$$\mu_{IFK(R)}(x) = \bigvee_{e \in E_S} \mu_R(x, e)$$

and

$$\nu_{IFK(R)}(x) = \bigwedge_{e \in E_S} \nu_R(x, e).$$

(2) The *intuitionistic fuzzy trace* IFT(R) of R is an intuitionistic fuzzy relation on E_S defined as follows: for any $e, f \in E_S$,

$$IFT(K)(e,f) = R(e,f).$$

It is clear that IFT(R) $\in$ IFNC(E_S).

Proposition 5.3. *Let S be an inverse semigroup. If P and Q are two IFNCs on E_S, then $P \vee Q \in$ IFNC(E_S).*

Proof. Since $P \vee Q = \bigcup_{n\in\mathbb{N}}(P\cup Q)^n$ by Result 2.B, we first show that the following holds:

$$\mu_{(P\cup Q)^n}(s^{-1}es, s^{-1}fs) \geq \mu_{(P\cup Q)^n}(e,f)$$
and $\qquad (*)$
$$\nu_{(P\cup Q)^n}(s^{-1}es, s^{-1}fs) \leq \nu_{(P\cup Q)^n}(e,f)$$
for each $n \in N$, any $e, f \in E_S$ and each $s \in S$.

Suppose $n = 1$. Then

$$\begin{aligned}\mu_{P\cup Q}(s^{-1}es, s^{-1}fs) &= \mu_P(s^{-1}es, s^{-1}fs) \vee \mu_Q(s^{-1}es, s^{-1}fs)\\ &\geq \mu_P(e,f) \vee \mu_Q(e,f) = \mu_{P\cup Q}(e,f)\end{aligned}$$

and

$$\begin{aligned}\nu_{P\cup Q}(s^{-1}es, s^{-1}fs) &= \nu_P(s^{-1}es, s^{-1}fs) \vee \nu_Q(s^{-1}es, s^{-1}fs)\\ &\leq \nu_P(e,f) \wedge \nu_Q(e,f) = \nu_{P\cup Q}(e,f).\end{aligned}$$

So $(*)$ holds for $n = 1$.

Suppose $(*)$ holds for $n = k(> 1)$. Then

$$\begin{aligned}\mu_{(P\cup Q)^{k+1}}(s^{-1}es, s^{-1}fs) &= \mu_{(P\cup Q)^k\circ(P\cup Q)}(s^{-1}es, s^{-1}fs)\\ &= \bigvee_{g\in E_S}[\mu_{(P\cup Q)^k}(s^{-1}es, g) \wedge \mu_{P\cup Q}(g, s^{-1}fs)]\\ &\geq \bigvee_{g\in E_S}[\mu_{(P\cup Q)^k}(s^{-1}es, s^{-1}hs) \wedge \mu_{P\cup Q}(s^{-1}hs, s^{-1}fs)]\\ &\geq \bigvee_{g\in E_S}[\mu_{(P\cup Q)^k}(e,h) \wedge \mu_{P\cup Q}(h,f)]\\ &= \mu_{(P\cup Q)^k\circ(P\cup Q)}(e,f) = \mu_{(P\cup Q)^{k+1}}(e,f)\end{aligned}$$

and

$$\begin{aligned}\nu_{(P\cup Q)^{k+1}}(s^{-1}es, s^{-1}fs) &= \nu_{(P\cup Q)^k\circ(P\cup Q)}(s^{-1}es, s^{-1}fs)\\ &\leq \bigwedge_{g\in E_S}[\nu_{(P\cup Q)^k}(s^{-1}es, g) \vee \nu_{P\cup Q}(g, s^{-1}fs)]\\ &\leq \bigwedge_{g\in E_S}[\nu_{(P\cup Q)^k}(s^{-1}es, s^{-1}hs) \vee \nu_{P\cup Q}(s^{-1}hs, s^{-1}fs)]\\ &\leq \bigwedge_{g\in E_S}[\nu_{(P\cup Q)^k}(e,h) \vee \nu_{P\cup Q}(h,f)]\\ &= \nu_{(P\cup Q)^k\circ(P\cup G)}(e,f) = \nu_{(P\cup Q)^{k+1}}(e,f).\end{aligned}$$

So $(*)$ holds for $n = k+1$. Hence $(*)$ holds for each $n \in \mathbb{N}$. Therefore $P \vee Q \in$ IFNC(E_S). $\square$

The following is the immediate result of Definition 5.1 and Proposition 5.3.

Theorem 5.4. *Let S be an inverse semigroup. Then (IFNC$(E_S), \cap, \vee)$ is a complete lattice.*

Proposition 5.5. *Let S be an inverse semigroup and let $R \in$ IFC(E_S). We define a complex mapping $Q = (\mu_Q, \nu_Q) : E_S \times E_S \to I \times I$ as follows : for any $e, f \in E_S$,*

$$\mu_Q(e,f) = \bigwedge_{a \in S} \mu_R(a^{-1}ea, a^{-1}fa) \text{ and } \nu_Q(e,f) = \bigvee_{a \in S} \nu_R(a^{-1}ea, a^{-1}fa)$$

Then Q is the greatest IFNC on E_S such that $Q \subset R$.

Proof. From the definition of Q, it is clear that $Q \in$ IFR(E_S). Moreover, Q is intuitionistic fuzzy reflexive and intuitionistic fuzzy symmetric. Let $a \in S$ and let $e, f, g \in E_S$. Since R is intuitionistic fuzzy transitive,

$$\mu_R(a^{-1}ea, a^{-1}fa) \geq \mu_R(a^{-1}ea, a^{-1}ga) \wedge \mu_R(a^{-1}ga, a^{-1}fa)$$

and

$$\nu_R(a^{-1}ea, a^{-1}fa) \leq \nu_R(a^{-1}ea, a^{-1}ga) \vee \nu_R(a^{-1}ga, a^{-1}fa).$$

Then

$$\begin{aligned} \mu_Q(e,f) &\geq \bigwedge_{a \in S} [\mu_R(a^{-1}ea,\ a^{-1}ga) \wedge \mu_R(a^{-1}ga,\ a^{-1}fa)] \\ &= (\bigwedge_{a \in S} \mu_R(a^{-1}ea,\ a^{-1}ga)) \wedge (\bigwedge_{a \in S} \mu_R(a^{-1}ga,\ a^{-1}fa)) \\ &= \mu_Q(e,f) \wedge \mu_Q(g,f) \end{aligned}$$

and

$$\begin{aligned} \nu_Q(e,f) &\leq \bigvee_{a \in S} [\nu_R(a^{-1}ea,\ a^{-1}ga) \vee \nu_R(a^{-1}ga,\ a^{-1}fa)] \\ &= (\bigvee_{a \in S} \nu_R(a^{-1}ea,\ a^{-1}ga)) \vee (\bigvee_{a \in S} \nu_R(a^{-1}ga,\ a^{-1}fa)) \\ &= \nu_Q(e,f) \vee \nu_Q(g,f). \end{aligned}$$

Thus

$$\mu_Q(e,f) \geq \bigvee_{g \in E_S} [(\mu_Q(e,g) \wedge \mu_Q(g,f)] = \mu_{Q \circ Q}(e,f)$$

and

$$\nu_Q(e,f) \leq \bigwedge_{g \in E_S} [(\nu_Q(e,g) \vee \nu_Q(g,f)] = \nu_{Q \circ Q}(e,f).$$

So $Q \circ Q \subset Q$. Hence $Q \in$ IFE(E_S).

Now let $a \in S$ and let $e, f \in E_S$. Then

$$\begin{aligned} \mu_Q(a^{-1}ea,\ a^{-1}fa) &= \bigwedge_{b\in S} \mu_R(b^{-1}(a^{-1}ea)b,\ b^{-1}(a^{-1}fa)b) \\ &= \bigwedge_{b\in S} \mu_R((ab^{-1})e(ab)),\ (ab^{-1})f(ab)) \\ &\geq \mu_Q(e,f) \end{aligned}$$

$$\begin{aligned} \nu_Q(a^{-1}ea,\ a^{-1}fa) &= \bigvee_{b\in S} \nu_R(b^{-1}(a^{-1}ea)b,\ b^{-1}(a^{-1}fa)b) \\ &= \bigvee_{b\in S} \nu_R((ab^{-1})e(ab)),\ (ab^{-1})f(ab)) \\ &\leq \nu_Q(e,f). \end{aligned}$$

Let $e, f, g \in E_S$. Then

$$\mu_Q(ge, gf) = \mu_Q(eg, fg) = \mu_Q(g^{-1}eg,\ g^{-1}fg) \geq \mu_Q(e,f)$$

and

$$\nu_Q(ge, gf) = \nu_Q(eg, fg) = \nu_Q(g^{-1}eg,\ g^{-1}fg) \leq \nu_Q(e,f).$$

Thus Q is intuitionistic fuzzy left and right compatible. So $Q \in \text{IFNC}(E_S)$.
Now let $e, f \in E_S$. Then

$$\mu_R(e,f) \geq \mu_R(e,ef) \wedge \mu_R(ef,f) \geq \mu_Q(e,f)$$

and

$$\nu_R(e,f) \leq \nu_R(e,ef) \vee \nu_R(ef,f) \leq \nu_Q(e,f).$$

So $Q \subset R$. Let $P \in \text{IFNC}(E_S)$ such that $P \subset R$. Then

$$\begin{aligned} \mu_P(e,f) &\leq \bigwedge_{a\in S} \mu_P(a^{-1}ea, a^{-1}fa) \text{ (Since } P \in \text{IFNC}(E_S)) \\ &\leq \mu_Q(e,f) \end{aligned}$$

and

$$\nu_P(e,f) \geq \bigvee_{a\in S} \nu_P(a^{-1}ea, a^{-1}fa) \geq \nu_Q(e,f).$$

Thus $P \subset Q$. Therefore Q is the greatest IFNC onE_S such that $Q \subset R$. This completes the proof. □

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