GENERALIZED FUZZY POLYGROUPS

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ABSTRACT. small Polygroups are multi-valued systems that satisfy group-like axioms. Using the notion of "belonging $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of $(\in, \in \vee q)$-fuzzy subpolygroups is introduced. The study of $(\in, \in \vee q)$-fuzzy normal subpolygroups of a polygroup are dealt with. Characterization and some of the fundamental properties of such fuzzy subpolygroups are obtained. $(\in, \in \vee q)$-fuzzy cosets determined by $(\in, \in \vee q)$-fuzzy subpolygroups are discussed. Finally, a fuzzy subpolygroup with thresholds, which is a generalization of an ordinary fuzzy subpolygroup and an $(\in, \in, \vee q)$-fuzzy subpolygroup, is defined and relations between two fuzzy subpolygroups are discussed.

1. Introduction

In this section, we describe the motivation for our study and survey related works. The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures was first introduced by Marty [38]. Since then many researchers have studied and developed the theory of algebraic hyperstructures. A short review of this theory appears in [16,44]. A recent book [19] contains a wealth of applications. In this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. This paper deals with a certain algebraic system called a polygroup. Application of hypergroups have mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups are studied in [34] by Ioulidis and are used to study color algebra [11,13]. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced in [10], as a generalization of canonical hypergroups, introduced in [39]. Some algebraic and combinatorial properties were developed in [11-15] by Comer. Davvaz and Poursalavati in [26] introduced matrix representations of polygroups over hyperrings; they also introduced the notion of a polygroup hyperring, thus generalizing the notion of a group ring. Davvaz, using the concept of generalized permutation, defined permutation polygroups and concepts related to it [32]. The reader will find an extensive discussion of polygroup theory in [10-15,26-32,33,39,42,48-51].

A reconsideration of the concepts of classical mathematics began After the introduction of fuzzy sets by Zadeh [47]. In particular, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroup was defined by Rosenfeld [41] and its structure was investigated. This subject has been studied further in [1,3,4-10,22,46] and by many other researchers. Das characterized fuzzy subgroups by their level subgroups in [22] and since then many notions of fuzzy group theory can also be characterized equivalently with the help of notion of level subgroups. In [3], Anthony and Sherwood redefined fuzzy subgroups using statistical functions. Fuzzy quasinormality was introduced by Ajmal and Thomas [1]. A new type of fuzzy subgroup (viz, $(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [6,7] by using the combined notions of "belonging" and "quasicoincidence" of fuzzy points and fuzzy sets. In fact, a $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. This concept has been studied further in [4,5,8]. Also, a generalization of Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup is given in [46].

Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now used in the world both from the theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by Corsini, Davvaz, Leoreanu, Zahedi, Ameri, Tofan, Kehagias and others [2,17-21,23-25,29-31,33,35-37,43,48-51]. In [23,24], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined a fuzzy subhypergroup (resp. H_v-subgroup) of a hypergroup (resp. H_v-group). Zahedi, Bolurian and Hasankhani in [48] introduced the concept of a fuzzy subpolygroup of a polygroup.

Now, in this paper, using the notion of "belonging $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of an $(\in, \in \vee q)$-fuzzy subpolygroup is introduced. The $(\in, \in \vee q)$-fuzzy normal subpolygroups of a polygroup also studied and a characterization and some of the fundamental properties of such fuzzy subpolygroups are obtained. $(\in, \in \vee q)$-fuzzy cosets determined by $(\in, \in \vee q)$-fuzzy subpolygroups are discussed. Finally, a fuzzy subpolygroup with thresholds, which is a generalization of an ordinary fuzzy subpolygroup and an $(\in, \in, \vee q)$-fuzzy subpolygroup, is defined and relations between two fuzzy subpolygroups are discussed.

2. Polygroups

hypergroupoid $(P, \circ)$ is a non-empty set P with a *hyperoperation* $\circ$ defined on P, i.e. a mapping of $P \times P$ into the family of non-empty subsets of P. If $(x, y) \in P \times P$, its image under $\circ$ is denoted by $x \circ y$. If $A, B \subseteq P$ then $A \circ B$ is given by $A \circ B = \bigcup \{x \circ y \mid x \in A, \ y \in B\}$. The notation $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

Definition 2.1. A *polygroup* is a multi-valued system $\rho = < P, \circ, e, ^{-1} >$ where $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \rho^*(P)$, and the following axioms hold for all x, y, z in P.

(i) $(x \circ y) \circ z = x \circ (y \circ z)$,

(ii) $e \circ x = x \circ e = x$,

(iii) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

In the above definition, $\rho^*(P)$ is the set of all the non-empty subsets of P. The following elementary facts about polygroups follow easily from the axioms:

$$e \in x \circ x^{-1} \cap x^{-1} \circ x, \quad e^{-1} = e, \quad (x^{-1})^{-1} = x, \text{ and } (x \circ y)^{-1} = y^{-1} \circ x^{-1},$$

where $A^{-1} = \{a^{-1} \mid a \in A\}$.

Let K be a non-empty subset of P, then K is called a *subpolygroup* of P if $e \in K$ and $< K, \circ, e, ^{-1} >$ is a polygroup. The subpolygroup N of P is said to be *normal* in P if and only if

$$a^{-1} \circ N \circ a \subseteq N \quad \text{ for every } a \in P.$$

If N is a normal subpolygroup of P, the following elementary facts follows easily from the axioms:

(i) $N \circ a = a \circ N$ for all $a \in P$,

(ii) $(N \circ a) \circ (N \circ b) = N \circ a \circ b$ for all $a, b \in P$,

(iii) $N \circ a = N \circ b$ for all $b \in N \circ a$.

For a subpolygroup K of P and $x \in P$, the right coset of K is defined as usual and is denoted by $K \circ x$; P/K is the set of all right cosets of K in P. If N is a normal subpolygroup of P, then $< P/N, \odot, N, ^{-I} >$ is a polygroup, where $N \circ a \odot N \circ b = \{N \circ c | c \in N \circ a \circ b\}$ and $(N \circ a)^{-I} = N \circ a^{-1}$.

Examples of polygroups, such as double set algebras, Prenowitz algebras, conjugacy class polygroups, and character polygroups, can be found in [12,14,15,26,40,42]. These examples show how polygroups occur naturally in various context.

Rosenfeld [41] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. Since than many papers concerning various fuzzy algebraic structures have appeared in the literature. Zahedi and et. al. in [48] defined the concept of fuzzy subpolygroups of a polygroup which is a generalization of the concept of Rosenfeld's fuzzy subgroups.

Definition 2.2. [48]. Let $(P, \circ)$ be a polygroup and let A be a fuzzy subset of P. Then A is said to be a *fuzzy subpolygroup* of P if the following axioms hold:

(1) $A(x) \wedge A(y) \leq A(z)$ for all $z \in x \circ y$ and $x, y \in P$,

(2) $A(x) \leq A(x^{-1})$ for all $x \in P$.

For any fuzzy set A in P and any $t \in (0, 1]$, we define the set

$$A_t = \{x \in P \mid A(x) \geq t\}$$

which is called a *t-level cut* of A.

Theorem 2.3. [48]. *Let P be a polygroup and A a fuzzy subset of P. Then A is a fuzzy subpolygroup of P if and only if for every $t \in (0, 1]$, A_t ($\neq \emptyset$) is a subpolygroup of P.*

When A is a fuzzy subpolygroup of P, A_t is called a *level subpolygroup* of P. The concept of level subpolygroup has been used extensively to characterize various properties of fuzzy subpolygroups.

Let $K \subseteq P$. Then the characteristic function χ_K is a fuzzy subpolygroup of P if and only if K is a subpolygroup of P.

3. $(\in, \in \vee q)$-fuzzy Subpolygroups

A fuzzy subset A of P of the form

$$A(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point with support* x and *value* t and is denoted by x_t. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set A, written as $x_t \in A$ (resp. $x_t q A$) if $A(x) \geq t$ (resp. $A(x) + t > 1$). If $x_t \in A$ or $x_t q A$, then we write $x_t \in \vee q A$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Using the notion of "belonging $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of an (α, β)-fuzzy subgroup where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ is introduced in [6]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. A detailed study of $(\in, \in, \vee q)$-fuzzy subgroups has been considered in [7]. Based on [7], we can extend the concept of $(\in, \in \vee q)$-fuzzy subgroups to the concept of $(\in, \in \vee q)$-fuzzy subpolygroups in the following way:

Definition 3.1. A fuzzy subset A of a polygroup P is said to be an $(\in, \in \vee q)$-fuzzy subpolygroup of P if for all $t, r \in (0, 1]$ and $x, y \in P$,

(i) $x_t, y_r \in A$ implies $z_{t \wedge r} \in \vee q A$ for all $z \in x \circ y$,

(ii) $x_t \in A$ implies $(x^{-1})_t \in \vee q A$.

We note that if A is a fuzzy subpolygroup of P according to Definition 2.2, then A is an $(\in, \in \vee q)$-fuzzy subpolygroup of P according to Definition 3.1. However, as the following example shows, the converse is not true .

Example 3.2. Let $P = \{e, a, b\}$ be the polygroup defined by the multiplication table:

$\circ$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{e, a\}$

If $A : P \longrightarrow [0, 1]$ is defined by

$$A(e) = 0.8, \quad A(a) = 0.7, \quad A(b) = 0.6,$$

then it is easy to see that A is an $(\in, \in \vee q)$-fuzzy subpolygroup of P, but is not a fuzzy subpolygroup of P.

Proposition 3.3. *Conditions (i) and (ii) in Definition 3.1, are respectively equivalent to the following:*

(1) $A(x) \wedge A(y) \wedge 0.5 \leq \bigwedge_{z \in x \circ y} A(z)$ *for all* $x, y \in P$;

(2) $A(x) \wedge 0.5 \leq A(x^{-1})$ *for all* $x \in P$.

Proof. (i $\Longrightarrow$ 1): Suppose that $x, y \in P$. We consider the following cases:

(a) $A(x) \wedge A(y) < 0.5$,
(b) $A(x) \wedge A(y) \geq 0.5$.

Case a: Assume that there exists $z \in x \circ y$ such that $A(z) < A(x) \wedge A(y) \wedge 0.5$; this implies $A(z) < A(x) \wedge A(y)$. Choose t such that $A(z) < t < A(x) \wedge A(y)$. Then $x_t, y_t \in A$, but $z_t \overline{\in \vee q} A$ which contradicts (i).

Case b: Assume that $A(z) < 0.5$ for some $z \in x \circ y$. Then $x_{0.5}, y_{0.5} \in A$, but $z_{0.5} \overline{\in \vee q} A$, a contradiction. Hence (1) holds.

(ii $\Longrightarrow$ 2): Suppose that $x \in P$, we consider the following cases:

(a) $A(x) < 0.5$,
(b) $A(x) \geq 0.5$.

Case a: Assume that $A(x) = t < 0.5$ and $A(x^{-1}) = r < A(x)$. Choose s such that $r < s < t$ and $r + s < 1$. Then $x_s \in A$, but $(x^{-1})_s \overline{\in \vee q} A$ which contradicts (ii). So $A(x^{-1}) \geq A(x) = A(x) \wedge 0.5$.

Case b: Let $A(x) \geq 0.5$. If $A(x^{-1}) < A(x) \wedge 0.5$, then $x_{0.5} \in A$, but $(x^{-1})_{0.5} \overline{\in \vee q} A$, which contradicts (ii). So $A(x^{-1}) \geq A(x) \wedge 0.5$.

(1 $\Longrightarrow$ i): Let $x_t, y_r \in A$, then $A(x) \geq t$ and $A(y) \geq r$. For every $z \in x \circ y$ we have

$$A(z) \geq A(x) \wedge A(y) \wedge 0.5 \geq t \wedge r \wedge 0.5.$$

If $t \wedge r > 0.5$, then $A(z) \geq 0.5$ which implies $A(z) + t \wedge r > 1$.

If $t \vee r \leq 0.5$, then $A(z) \geq t \wedge r$.

Therefore $z_{t \wedge r} \in \vee q A$ for all $z \in x \circ y$.

(2 $\Longrightarrow$ ii): Let $x_t \in A$. Then $A(x) \geq t$. Now, we have $A(x^{-1}) \geq A(x) \wedge 0.5 \geq t \wedge 0.5$, which implies $A(x^{-1}) \geq t$ or $A(x^{-1}) \geq 0.5$ according as $t \leq 0.5$ or $t > 0.5$. Therefore $(x^{-1})_t \in \vee q A$. $\qquad \square$

By Definition 3.1 and Proposition 3.3, we immediately get:

Corollary 3.4. *A fuzzy subset A of a polygroup P is an $(\in, \in \vee q)$-fuzzy subpolygroup of P if and only if the conditions (1) and (2) in Proposition 3.3 hold.*

If A is an $(\in, \in \vee q)$-fuzzy subpolygroup of a polygroup P, then it is easy to see that $A(e) \geq 0.5$.

Let P be a polygroup and χ_K be the characteristic function of a subset K of P. Then it is not difficult to see that χ_K is an $(\in, \in \vee q)$-fuzzy subpolygroup if and only if K is a subpolygroup.

Now, we characterize $(\in, \in \vee q)$-fuzzy subpolygroups by their level subpolygroups.

Theorem 3.5. *Let A be an $(\in, \in \vee q)$-fuzzy subpolygroup of P. Then for all $0 < t \leq 0.5$, A_t is a non-empty set or a subpolygroup of P. Conversely, if A is a fuzzy subset of P such that A_t ($\neq \emptyset$) is a subpolygroup of P for all $0 < t \leq 0.5$, then A is an $(\in, \in \vee q)$-fuzzy subpolygroup of P.*

Proof. Let A be an $(\in, \in \vee q)$-fuzzy subpolygroup of P and $0 < t \leq 0.5$. Let $x, y \in A_t$. Then $A(x) \geq t$ and $A(y) \geq t$. Now

$$\bigwedge_{z \in x \circ y} A(z) \geq A(x) \wedge A(y) \wedge 0.5 \geq t \wedge 0.5 = t.$$

Therefore for every $z \in x \circ y$ we have $A(z) \geq t$ or $z \in A_t$, so $x \circ y \subseteq A_t$. Also, we have $A(x^{-1}) \geq A(x) \wedge 0.5 = t \wedge 0.5 = t$, and so $x^{-1} \in A_t$.

Conversely, let A be a fuzzy subset of P such that A_t $(\neq \emptyset)$ is a subpolygroup of P for all $0 < t \leq 0.5$. For every $x, y \in P$, we can write

$$A(x) \geq A(x) \wedge A(y) \wedge 0.5 = t_0,$$
$$A(y) \geq A(x) \wedge A(y) \wedge 0.5 = t_0,$$

then $x \in A_{t_0}$ and $y \in A_{t_0}$, so $x \circ y \subseteq A_{t_0}$. Therefore for every $z \in x \circ y$ we have $A(z) \geq t_0$ which implies

$$\bigwedge_{z \in x \circ y} A(z) \geq t_0,$$

and hence the condition (1) of Proposition 3.3 is verified. To verify the second condition, let $x \in P$. We can write $A(x) \geq A(x) \wedge 0.5 = t_0$, and hence $x \in A_{t_0}$, and $x^{-1} \in A_{t_0}$. Therefore $A(x^{-1}) \geq A(x) \wedge 0.5$. $\qquad\square$

Naturally, a corresponding result should be considered when A_t is a subpolygroup of P for all $t \in (0.5, 1]$.

Theorem 3.6. *Let A be a fuzzy subset of a polygroup P. Then A_t $(\neq \emptyset)$ is a subpolygroup of P for all $t \in (0.5, 1]$ if and only if*

(1) $A(x) \wedge A(y) \leq \bigwedge_{z \in x \circ y} (A(z) \vee 0.5)$ *for all $x, y \in P$;*

(2) $A(x) \leq A(x^{-1}) \vee 0.5$ *for all $x \in P$.*

Proof. $(\Longrightarrow)$: If there exist $x, y, z \in P$ with $z \in x \circ y$ such that

$$A(z) \vee 0.5 < A(x) \wedge A(y) = t,$$

then $t \in (0.5, 1]$, $A(z) < t$, $x \in A_t$, and $y \in A_t$. Since $x, y \in A_t$ and A_t is a subpolygroup, so $x \circ y \subseteq A_t$ and $A(z) \geq t$ for all $z \in x \circ y$, which contradicts $A(z) < t$. Therefore

$$A(x) \wedge A(y) \geq A(z) \vee 0.5 \text{ for all } x, y, z \in P \text{ with } z \in x \circ y,$$

which implies

$$A(x) \wedge A(y) \geq \bigwedge_{z \in x \circ y} (A(z) \vee 0.5) \text{ for all } x, y \in P.$$

Hence (1) holds.

Now, assume that for some $x \in P$, $A(x^{-1}) \vee 0.5 < A(x) = t$, then $t \in (0.5, 1]$, $A(x^{-1}) < t$ and $x \in A_t$. Since $x \in A_t$, we get $x^{-1} \in A_t$ or $A(x^{-1}) \geq t$, which is a

contradiction. Hence (2) holds.

($\Longleftarrow$): Assume that $t \in (0.5, 1]$ and $x, y \in A_t$. Then

$$0.5 < t \leq A(x) \wedge A(y) \leq \bigwedge_{z \in x \circ y} (A(z) \vee 0.5).$$

It follows that for every $z \in x \circ y$, $0.5 < t \leq A(z) \vee 0.5$ and so $t \leq A(z)$, which implies $z \in A_t$. Hence $x \circ y \subseteq A_t$.

Now, let $t \in (0.5, 1]$ and $x \in A_t$. Then using condition (2), we have

$$0.5 < t \leq A(x) \leq A(x^{-1}) \vee 0.5 < A(x^{-1}),$$

and so $x^{-1} \in A_t$. Therefore A_t is a subpolygroup of P for all $t \in (0.5, 1]$. $\square$

Let A be a fuzzy subset of a polygroup P and

$$J = \{t \mid t \in (0, 1] \text{ and } A_t \text{ is an empty} - \text{set or a subpolygroup of } P\}.$$

When $J = (0, 1]$, A is an ordinary fuzzy subpolygroup of the polygroup P (Theorem 2.3). When $J = (0, 0.5]$, A is an $(\in, \in \vee q)$-fuzzy subpolygroup of the polygroup P (Theorem 3.5).

In [46], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup as well as Bhakat and Das's fuzzy subgroup. Based on [46], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy subpolygroup with thresholds in the following way:

Definition 3.7. Let $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$. Let A be a fuzzy subset of a polygroup P. Then A is called a *fuzzy subpolygroup with thresholds of* P if for all $x, y \in P$,

(1) $A(x) \wedge A(y) \wedge \beta \leq \bigwedge_{z \in x \circ y} (A(z) \vee \alpha)$ for all $x, y \in P$;

(2) $A(x) \wedge \beta \leq A(x^{-1}) \vee \alpha$ for all $x \in P$.

If A is a fuzzy subpolygroup with thresholds of P, then we can conclude that A is an ordinary fuzzy subpolygroup when $\alpha = 0$, $\beta = 1$; and A is an $(\in, \in \vee q)$-fuzzy subpolygroup when $\alpha = 0$, $\beta = 0.5$.

Now, we characterize fuzzy subpolygroups with thresholds by their level subpolygroups.

Theorem 3.8. *A fuzzy subset A of a polygroup P is a fuzzy subpolygroup with thresholds of P if and only if A_t ($\neq \emptyset$) is a subpolygroup of P for all $t \in (\alpha, \beta]$.*

Proof. Let A be a fuzzy subpolygroup with thresholds of P and $t \in (\alpha, \beta]$. Let $x, y \in A_t$. Then $A(x) \geq t$ and $A(y) \geq t$. Now

$$\bigwedge_{z \in x \circ y} (A(z) \vee \alpha) \geq A(x) \wedge A(y) \wedge \beta \geq t \wedge \beta \geq t > \alpha.$$

So for every $z \in x \circ y$ we have $A(z) \vee \alpha \geq t > \alpha$ which implies $A(z) \geq t$ and $z \in A_t$. Hence $x \circ y \subseteq A_t$. Now let $x \in A_t$, then $A(x^{-1}) \vee \alpha \geq A(x) \wedge \beta \geq t > \alpha$. So $A(x^{-1}) \geq t$ and $x^{-1} \in A_t$. Therefore A_t is a subpolygroup of P for all $t \in (\alpha, \beta]$.

Conversely, let A be a fuzzy subset of P such that A_t $(\neq \emptyset)$ is a subpolygroup of P for all $t \in (\alpha, \beta]$. If there exist $x, y, z \in P$ with $z \in x \circ y$ such that

$$A(z) \vee \alpha < A(x) \wedge A(y) \wedge \beta = t,$$

then $t \in (\alpha, \beta]$, $A(z) < t$, $x \in A_t$ and $y \in A_t$. Since A_t is a subpolygroup of P and $x, y \in A_t$, so $x \circ y \subseteq A_t$. Hence $A(z) \geq t$ for all $z \in x \circ y$. This is a contradiction with $A(z) < t$. Therefore

$$A(x) \wedge A(y) \wedge \beta \leq A(z) \vee \alpha \ \text{ for all } x, y, z \in P \text{ with } z \in x \circ y,$$

which implies

$$A(x) \wedge A(y) \wedge \beta \leq \bigwedge_{z \in x \circ y} (A(z) \vee \alpha) \text{ for all } x, y \in P.$$

Hence condition (1) of Definition 3.7 holds.

Now, assume that there exists $x_0 \in P$ such that $A(x_0) \wedge \beta > t = A(x_0^{-1}) \vee \alpha$. Then $x \in A_t$, $t \in (\alpha, \beta]$ and $A(x_0^{-1}) < t$. Since A_t is a subpolygroup of P, so $A(x_0^{-1}) \geq t$. This contradicts $A(x_0^{-1}) < t$. Therefore $A(x) \wedge \beta \leq A(x^{-1}) \vee \alpha$ for any $x \in P$. Hence the second condition of Definition 3.7 holds. $\square$

4. $(\in, \in \vee q)$-fuzzy Normal Subgroups

In this section we first define an $(\in, \in \vee q)$-fuzzy normal subpolygroup of a polygroup and then obtain the relation between $(\in, \in \vee q)$-fuzzy normal subpolygroups. Finally $(\in, \in \vee q)$-fuzzy cosets determined by $(\in, \in \vee q)$-fuzzy subpolygroups are discussed.

Definition 4.1. [48] Let A be a fuzzy subset of a polygroup P. Then A is said to be *normal* if and only if for all $x, y \in P$,

$$A(z) = A(z'), \ \ \forall z \in x \circ y, \ \ \forall z' \in y \circ x.$$

It is obvious that if A is a fuzzy normal subpolygroup of P, then

$$A(z) = A(z'), \ \ \forall z, z' \in x \circ y, \ \ \forall x, y \in P.$$

Theorem 4.2. [48] *Let A be a fuzzy subpolygroup of a polygroup P. Then the following conditions are equivalent:*

(1) *A is a fuzzy normal subpolygroup of P,*
(2) *For all $x, y \in P$, $A(z) = A(x)$, $\forall z \in y \circ x \circ y^{-1}$,*
(3) *For all $x, y \in P$, $A(z) \geq A(x)$, $\forall z \in y \circ x \circ y^{-1}$,*
(4) *For all $x, y \in P$, $A(z) \geq A(x)$, $\forall z \in y^{-1} \circ x^{-1} \circ y \circ x.$*

Now, we give the definition of $(\in, \in \vee q)$-fuzzy normal subpolygroup of a polygroup, which is a generalization of the notion of fuzzy normal subpolygroup defined by Zahedi and et. al. [48].

Definition 4.3. An $(\in, \in \vee q)$-fuzzy subpolygroup of a polygroup P is said to be $(\in, \in \vee q)$-*fuzzy normal* if for every $x, y \in P$ and $t \in (0, 1]$,

$$x_t \in A \ \text{ implies } \ z_t \in \vee q A \ \text{ for all } z \in y \circ x \circ y^{-1}.$$

Theorem 4.4. *For an $(\in, \in \vee q)$-fuzzy subpolygroup of P, the following statements are equivalent:*

(1) *A is an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P,*

(2) $\bigwedge\limits_{z \in y \circ x \circ y^{-1}} A(z) \geq A(x) \wedge 0.5$ *for all $x, y \in P$,*

(3) $A(z) \geq A(z') \wedge 0.5, \ \forall z \in x \circ y, \quad \forall z' \in y \circ x,$

(4) $\bigwedge\limits_{z \in x^{-1} \circ y^{-1} \circ x \circ y} A(z) \geq A(x) \wedge 0.5$ *for all $x, y \in P$.*

Proof. $(1 \implies 2)$: Suppose that A is an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P and $x, y \in P$. We consider the following cases:

(a) $A(x) < 0.5$,

(b) $A(x) \geq 0.5$.

Case a: Assume that there exists $z \in y \circ y^{-1}$ such that $A(z) < A(x) \wedge 0.5$, which implies $A(z) < A(x)$. Choose t such that $A(z) < t < A(x)$. Then $x_t \in A$, but $z_t \overline{\in \vee q} A$ which contradicts (1).

Case b: Assume that $A(z) < 0.5$ for some $z \in y \circ x \circ y^{-1}$. Then we have $A(z) < 0.5 < A(x)$, which implies $x_{0.5} \in A$, but $z_{0.5} \overline{\in \vee q} A$, a contradiction. Hence (2) holds.

$(2 \implies 1)$: Suppose that $x_t \in A$ and $y \in P$, then $A(x) \geq t$. For every $z \in y \circ x \circ y^{-1}$, we have $A(z) \geq A(x) \wedge 0.5 \geq t \wedge 0.5$ which implies $A(z) \geq t$ or $A(z) \geq 0.5$ according as $t \leq 0.5$ or $t > 0.5$. Therefore $z_t \in \vee q A$.

$(2 \implies 3)$: Let $c, y \in P$, for every $z \in x \circ y$ and $z' \in y \circ x$, we get $x \in y^{-1} \circ z'$, which implies $z \in y^{-1} \circ z' \circ y$. Now, by using (2), we have $A(z) \geq A(z') \wedge 0.5$.

$(3 \implies 2)$: Suppose that there exist $x, y, z \in P$ with $z \in y \circ x \circ y^{-1}$ such that $A(z) < A(x) \wedge 0.5$. Since $z \in y \circ x \circ y^{-1}$, there exists $a \in x \circ y^{-1}$ such that $z \in y \circ a$. Since $a \in x \circ y^{-1}$, then $x \in a \circ y$. Therefore by (3), we have $A(x) \wedge 0.5 \leq A(z)$, which is a contradiction.

$(2 \implies 4)$: Let $x, y \in P$. For every $z \in x^{-1} \circ y^{-1} \circ x \circ y$, there exists $z' \in y^{-1} \circ x \circ y$ such that $z \in x^{-1} \circ z'$. Since A is an $(\in, \in \vee q)$-fuzzy subpolygroup, then $A(z) \geq A(x^{-1}) \wedge A(z') \wedge 0.5$. By (2), we have $A(z) \geq A(x^{-1}) \wedge A(x) \wedge 0.5$, and so $A(z) \geq A(x) \wedge 0.5$. Therefore

$$\bigwedge\limits_{z \in x^{-1} \circ y^{-1} \circ x \circ y} A(z) \geq A(x) \wedge 0.5.$$

$(4 \implies 2)$: Let $x, y \in P$. For every $z \in y \circ x \circ y^{-1}$, we have $z \in y \circ x \circ y^{-1} \circ x^{-1} \circ x$. Hence there exists $z' \in y \circ x \circ y^{-1} \circ x^{-1}$ such that $z \in z' \circ x$. So

$$A(z) \geq A(z') \wedge A(x) \wedge 0.5 \geq A(x) \wedge 0.5 \wedge A(x) \wedge 0.5 = A(x) \wedge 0.5.$$

Therefore $\bigwedge\limits_{z \in y \circ x \circ y^{-1}} A(z) \geq A(x) \wedge 0.5.$ $\square$

Corollary 4.5. *Let A be an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P. Then for all $x, y \in P$, we have*

$$(\bigwedge_{z \in x \circ y} A(z)) \wedge 0.5 = (\bigwedge_{z' \in y \circ x} A(z)) \wedge 0.5.$$

Proof. Let $x, y \in P$. For every $z \in x \circ y$, we have $z \in y^{-1} \circ (y \circ x) \circ y$. So there exists $z' \in y \circ x$ such that $z \in y^{-1} \circ z' \circ y$. Since A is $(\in, \in \vee q)$-fuzzy normal, we have $A(z) \geq A(z') \wedge 0.5$, which implies $A(z) \geq (\bigwedge_{z' \in y \circ x} A(z)) \wedge 0.5$, and so

$$\bigwedge_{z' \in y \circ x} A(z)) \geq (\bigwedge_{z' \in y \circ x} A(z)) \wedge 0.5. \text{ Therefore}$$

$$(\bigwedge_{z' \in y \circ x} A(z)) \wedge 0.5 \geq (\bigwedge_{z' \in y \circ x} A(z)) \wedge 0.5.$$

Similarly we obtain

$$(\bigwedge_{z' \in y \circ x} A(z)) \wedge 0.5 \leq (\bigwedge_{z' \in y \circ x} A(z)) \wedge 0.5.$$

$\square$

Theorem 4.6. *Let A be an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P. Then A_t ($\neq \emptyset$) is a normal subpolygroup of P for all $t \in (0, 0.5]$. Conversely, if A is a fuzzy subset of P such that A_t ($\neq \emptyset$) is a normal subpolygroup of P for all $t \in (0, 0.5]$, then A is $(\in, \in \vee q)$-fuzzy normal.*

Proof. First, let A be an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P. That A_t is a subpolygroup of P follows from Theorem 3.5. Now, we show that A_t is normal. Assume that $x \in A_t$ and $y \in P$. Then for every $z \in y \circ x \circ y^{-1}$, we have

$$\begin{aligned} A(z) \quad &\geq A(x) \wedge 0.5 \quad (\text{since } A \text{ is } (\in, \in \vee q) - \text{fuzzy normal}) \\ &\geq t \wedge 0.5 = t \end{aligned}$$

and so $z \in A_t$. Therefore $y \circ x \circ y^{-1} \subseteq A_t$, i.e. A_t is normal for all $t \in (0, 0.5]$.

Conversely, Let A be a fuzzy subset of P such that A_t is a normal subpolygroup of P for all $t \in (0, 0.5]$. That A is an $(\in, \in \vee q)$-fuzzy subpolygroup of P follows from Theorem 3.5. Now, we show that A is $(\in, \in \vee q)$-fuzzy normal. Assume that $x, y \in P$, we have $A(x) \geq A(x) \wedge 0.5 = t_0$, hence $x \in A_{t_0}$. Since A_{t_0} is normal, we have $y \circ x \circ y^{-1} \subseteq A_{t_0}$. So

$$z \in A_{t_0} \quad \text{for all } z \in y \circ x \circ y^{-1},$$

which implies

$$A(z) \geq t_0 \quad \text{for all } z \in y \circ x \circ y^{-1}.$$

Therefore

$$\bigwedge_{z \in y \circ x \circ y^{-1}} A(z) \geq A(x) \wedge 0.5.$$

$\square$

Definition 4.7. Let A be an $(\in, \in \vee q)$-fuzzy subpolygroup of P. For any $x \in P$, $\overline{A_x}$ (resp. $\underline{A_x}$) is define by

$$\overline{A_x}(a) = (\bigwedge_{z \in a \circ x^{-1}} A(z)) \wedge 0.5,$$

$$(\text{resp. } \underline{A_x}(a) = (\bigwedge_{z \in x^{-1} \circ a} A(z)) \wedge 0.5), \quad \forall a \in P$$

and is called $(\in, \in \vee q)$-*fuzzy left* (resp. *right*) *coset of P determined by x and A.*

The following example shows that for an $(\in, \in \vee q)$-fuzzy subpolygroup A of a polygroup P, the $(\in, \in \vee q)$-fuzzy left coset need not be equal to the corresponding $(\in, \in \vee q)$-fuzzy right coset.

Example 4.8. Let P be the polygroup defined in Example 3.2, and let $S_3 = \{e, (12), (13), (23), (123), (132)\}$ be the symmetric group of order 3. We consider the polygroup $S_3[P]$ (the extension of S_3 by P, see [13]). We define $A : S_3[P] \longrightarrow [0, 1]$ by

$$A(e) = 0.8,$$
$$A(a) = 0.3,$$
$$A(b) = 0.3,$$
$$A((12)) = 0.8,$$
$$A(x) = 0.4, \quad \forall x \in S_3 - \{e, (12)\}.$$

Then it is not difficult to see that A is an $(\in, \in \vee q)$-fuzzy subpolygroup of $S_3[P]$. We have

$$\overline{A_{(132)}}((13)) = 0.4 \quad \text{and} \quad \underline{A_{(132)}}((13)) = 0.5.$$

Hence $\overline{A_{(132)}} \neq \underline{A_{(132)}}$. This happens because A is not an $(\in, \in \vee q)$-fuzzy normal subpolygroup of $S_3[P]$. However we gave the following proposition:

Proposition 4.9. *Let A be an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P. Then*

$$\overline{A_x} = \underline{A_x} \quad \text{for all } x \in P.$$

Proof. The proof follows from Corollary 4.5. $\qquad\qquad\square$

Proposition 4.10. *Let $x, y \in P$ and for any $a \in P$, $\overline{A_x}(a) = \overline{A_y}(a)$. Then for any non-empty subset S of P, we have*

$$(\bigwedge_{a \in S \circ x^{-1}} A(a)) \wedge 0.5 = (\bigwedge_{a \in S \circ y^{-1}} A(a)) \wedge 0.5.$$

Proof. We have

$$(\bigwedge_{a \in S \circ x^{-1}} A(a)) \wedge 0.5 \leq (\bigwedge_{a \in s \circ x^{-1}} A(a)) \wedge 0.5 = (\bigwedge_{a \in s \circ y^{-1}} A(a)) \wedge 0.5, \quad \forall s \in S.$$

Therefore

$$(\bigwedge_{a \in S \circ x^{-1}} A(a)) \wedge 0.5 \leq \bigwedge_{s \in S} ((\bigwedge_{a \in s \circ y^{-1}} A(a)) \wedge 0.5) = (\bigwedge_{a \in S \circ y^{-1}} A(a)) \wedge 0.5.$$

The proof of converse inequality is similar. $\qquad\qquad\square$

The following is due to Theorem 5.3 of [48]. We shall give a proof for completeness.

Theorem 4.11. *Let A be an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P. Then*

$$\overline{A_a} = \overline{A_b} \iff A_{0.5} \circ a = A_{0.5} \circ b \ \text{ for all } a, b \in P.$$

Proof. Assume that $\overline{A_a} = \overline{A_b}$. Then

$$
\begin{aligned}
\overline{A_b}(b) \ &= (\bigwedge_{z \in b \circ b^{-1}} A(z)) \wedge 0.5 \\
&\geq A(z') \wedge 0.5, \ \ \forall z' \in b^{-1} \circ b \ \ (\text{since } A \text{ is } (\in, \in \vee q) - \text{fuzzy normal}).
\end{aligned}
$$

Since $e \in b^{-1} \circ b$ and $A(e) \geq 0.5$, we get

$$\overline{A_b}(b) \geq A(e) \wedge 0.5 \geq 0.5.$$

Hence $\overline{A_a}(b) \geq 0.5$, and so

$$
\begin{aligned}
0.5 \leq \overline{A_a}(b) \ &= (\bigwedge_{z \in b \circ a^{-1}} A(z)) \wedge 0.5 \\
&= A(z_0) \wedge 0.5, \ \ \text{for some } z_0 \in b \circ a^{-1},
\end{aligned}
$$

which implies $0.5 \leq A(z_0)$. For every $z \in a^{-1} \circ b$, we have

$$0.5 \leq A(z_0) \wedge 0.5 \leq A(z) \ \ (\text{since } A \text{ is } (\in, \in \vee q) - \text{fuzzy normal}),$$

which implies $z \in A_{0.5}$. Therefore $a^{-1} \circ b \subseteq A_{0.5}$. Suppose that $y_0 \in a^{-1} \circ b$, then $a \in b \circ y_0^{-1}$.

Now, let x be an arbitrary element of $A_{0.5} \circ a$. By Theorem 4.6, $A_{0.5}$ is a normal subpolygroup, so $x \in a \circ A_{0.5}$. Therefore $x \in a \circ c$ for some $c \in A_{0.5}$, and hence

$$x \in a \circ c \subseteq b \circ y_0^{-1} \circ c \subseteq b \circ A_{0.5} = A_{0.5} \circ b.$$

So $A_{0.5} \circ a \subseteq A_{0.5} \circ b$. Similarly, we can show that $A_{0.5} \circ b \subseteq A_{0.5} \circ a$.

Conversely, suppose that $A_{0.5} \circ a = A_{0.5} \circ b$. Then for $x \in P$, we have

$$x \circ a^{-1} \circ A_{0.5} = x \circ b^{-1} \circ A_{0.5}.$$

Also, we have

$$\overline{A_a}(x) = (\bigwedge_{z \in x \circ a^{-1}} A(z)) \wedge 0.5 = A(z_0) \wedge 0.5 \ \text{ for some } z_0 \in x \circ a^{-1},$$

and

$$
\begin{aligned}
\overline{A_b}(x) \ &= (\bigwedge_{z \in x \circ b^{-1}} A(z)) \wedge 0.5 = (\bigwedge_{z \in b^{-1} \circ x} A(z)) \wedge 0.5 \\
&= A(z_0') \wedge 0.5 \ \text{ for some } z_0' \in b^{-1} \circ x.
\end{aligned}
$$

Now, we show that

$$(A(z_0) < 0.5 \text{ and } A(z_0') < 0.5) \ \ \text{or} \ \ (A(z_0) \geq 0.5 \text{ and } A(z_0') \geq 0.5).$$

Assume that $A(z_0) < 0.5$ and $A(z_0') \geq 0.5$. Since A is an $(\in, \in \vee q)$-fuzzy normal subpolygroup, by definition for any $c \in x \circ b^{-1}$ we have $A(c) \geq A(z_0') \wedge 0.5 = 0.5$, so $x \circ b^{-1} \subseteq A_{0.5}$, which implies $x \circ b^{-1} \circ A_{0.5} = A_{0.5}$. Since $z_0 \notin A_{0.5}$, hence $x \circ a^{-1} \nsubseteq A_{0.5}$ and so $x \circ a^{-1} \circ A_{0.5} \neq A_{0.5}$. Therefore $x \circ a^{-1} \circ A_{0.5} \neq x \circ b^{-1} \circ A_{0.5}$, which is a contradiction. Similarly we can show that it is not possible to have

$A(z_0) \geq 0.5$ and $A(z_0') < 0.5$.

Now, if $A(z_0) \geq 0.5$ and $A(z_0') \geq 0.5$, then $\overline{A_a}(x) = 0.5 = \overline{A_b}(x)$ for all $x \in P$. If $A(z_0) < 0.5$ and $A(z_0') < 0.5$, then

$$\overline{A_a}(x) = A(z_0) \quad \text{for some } z_0 \in x \circ a^{-1},$$
$$\overline{A_b}(x) = A(z_0') \quad \text{for some } z_0' \in b^{-1} \circ x.$$

In this case, we show that $A(z_0) = A(z_0')$. Since $A_{0.5} \circ a = A_{0.5} \circ b$, then $b \in A_{0.5} \circ a$, and so there exists $y \in A_{0.5}$ such that $b \in y \circ a$, hence $b^{-1} \in a^{-1} \circ y^{-1}$. Since $z_0 \in x \circ a^{-1}$, then $x \in z_0 \circ a$. Therefore we get

$$z_0' \in a^{-1} \circ y^{-1} \circ z_0 \circ a.$$

Since A is an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P, we have

$$A(z_0') \quad \geq A(c) \wedge 0.5 \ \text{ for some } c \in y^{-1} \circ z_0$$
$$\geq A(y^{-1}) \wedge A(z_0) \wedge 0.5.$$

Since $A(y^{-1} \geq 0.5$ and $A(z_0) < 0.5$, we get $A(z_0') \geq A(z_0)$. Similarly, we get $A(z_0) \geq A(z_0')$. Therefore $\overline{A_a} = \overline{A_b}$. $\qquad \square$

Theorem 4.12. *Let A be an $(\in, \in \vee q)$-fuzzy normal subpolygroup of P. Let P/A be the set of all $(\in, \in \vee q)$-fuzzy left cosets of A in P. Then P/A is a polygroup if the hyperoperation is defined by*

$$\overline{A_x} \odot \overline{A_y} = \{\overline{A_z} \mid z \in A_{0.5} \circ x \circ y\}$$

and $(\overline{A_x})^{-I} = \overline{A_{x^{-1}}}$.

Proof. It is straightforward. $\qquad \square$

Naturally, a corresponding result should be considered when A_t is a normal subpolygroup of P for all $t \in (0.5, 1]$.

Theorem 4.13. *Let A be a fuzzy subset of a polygroup P. Then A_t $(\neq \emptyset)$ is a normal subpolygroup of P for all $t \in (0.5, 1]$, if and only if the conditions (1) and (2) in Theorem 3.6 and the following condition hold:*

$$A(x) \leq \bigwedge_{z \in y \circ x \circ y^{-1}} (A(z) \vee 0.5) \ \text{ for all } x, y \in P.$$

Proof. The proof is similar to the proof of Theorem 4.6. $\qquad \square$

Theorem 4.14. *Let $\alpha, \beta \in [0, 1]$, $\alpha < \beta$ and A a fuzzy subset of P. Then A_t $(\neq \emptyset)$ is a normal subpolygroup of P for all $t \in (\alpha, \beta]$ if and only if A is a fuzzy subpolygroup with thresholds of P and the following condition holds:*

$$(*) \qquad \bigwedge_{z \in y \circ x \circ y^{-1}} (A(z) \vee \alpha) \geq A(x) \wedge \beta.$$

Proof. First, let A be a fuzzy subpolygroup with thresholds of P and assume that condition $(*)$ holds. That A_t is a subpolygroup of P follows from Theorem 3.8.

Now, we show that A_t is normal. Assume that $x \in A_t$ and $y \in P$. Then for every $z \in y \circ x \circ y^{-1}$, we have

$$A(z) \vee \alpha \geq A(x) \wedge \beta \geq t \wedge \beta \geq t > \alpha.$$

So for every $z \in y \circ x \circ y^{-1}$ we have $A(z) \wedge \alpha \geq t > \alpha$, which implies $A(z) \geq t$ and $z \in A_t$. Hence $y \circ x \circ y^{-1} \subseteq A_t$, i.e. A_t is normal for all $t \in (\alpha, \beta]$.

Conversely, let A be a fuzzy subset of P such that A_t $(\neq \emptyset)$ is a normal subpolygroup of P for all $t \in (\alpha, \beta]$. That A is a fuzzy subpolygroup with thresholds of P follows from Theorem 3.8. Now, we verify the condition $(*)$. If there exist $x, y, z \in P$ with $z \in y \circ x \circ y^{-1}$ such that $A(z) \vee \alpha < A(x) \wedge \beta = t$, then $t \in (\alpha, \beta]$, $A(z) < t$ and $x \in A_t$. Since A_t is normal, so $y \circ x \circ y^{-1} \subseteq A_t$. Hence $A(z) \geq t$ for all $z \in y \circ x \circ y^{-1}$. This is a contradicts $A(z) < t$. Therefore

$$A(z) \vee \alpha \geq A(x) \wedge \beta \ \text{ for all } x, y \in P \ \text{ with } z \in y \circ x \circ y^{-1},$$

and so $(*)$ holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Implication-based Fuzzy Subpolygroup

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables (or terms of the linguistic variable truth). Some operators, like $\wedge$, $\vee$, $\neg$, $\longrightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators.

In fuzzy logic, the truth value of the fuzzy proposition P is denoted by $[P]$. In the following, we display the fuzzy logical and corresponding set-theoretical notions.

$$[x \in A] = A(x),$$
$$[x \notin A] = 1 - A(x),$$
$$[P \wedge Q] = \min\{[P], [Q]\},$$
$$[P \vee Q] = \max\{[P], [Q]\},$$
$$[P \longrightarrow Q] = \min\{1, 1 - [P] + [Q]\},$$
$$[\forall x P(x)] = \inf[P(x)],$$
$$\models P \text{ if and only if } [P] = 1 \text{ for all valuations.}$$

Of course, various implication operators have been defined. We only show a selection of them in the next table. α denotes the degree of truth (or degree of membership) of the premise, β the respective values for the consequence, and I the resulting degree of truth for the implication.

Name	Definition of Implication Operator
Lukasiewicz	$I_a\,(\alpha, \beta) = \min\{1, 1 - \alpha + \beta\}$
Standard Star (Godel)	$I_g\,(\alpha, \beta) = \begin{cases} 1 & \alpha \leq \beta \\ \beta & \text{elsewhere} \end{cases}$
Contraposition of Godel	$I_{cg}(\alpha, \beta) = \begin{cases} 1 & \alpha \leq \beta \\ 1 - \alpha & \text{elsewhere} \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha, \beta) = \begin{cases} 1 & \alpha \leq \beta \\ 0 & \text{elsewhere} \end{cases}$

The "quality" of these implication operators may be evaluated either empirically or axiomatically.

In the following definition we considered the definition of implication operator in the Lukasiewicz system of countinuous-valued logic.

Definition 5.1. If a fuzzy subset A of a polygroup P satisfies the conditions (1) and (2) below, then A is called a *fuzzifying subpolygroup of* P.:

(1) for any $x, y \in P$, $\models$ $[\ [x \in A] \wedge [y \in A] \longrightarrow [\forall z \in x \circ y, \ z \in A]\ ]$,

(2) for any $x \in P$, $\models$ $[[\ x \in A] \longrightarrow [x^{-1} \in A]]$,

Clearly, Definition 5.1 is equivalent to Definition 2.2. Therefore, a fuzzifying subpolygroup is an ordinary fuzzy subpolygroup.

In [45], the concept of t-tautology is introduced, i.e.,

$$\models_t \ P \ \text{ if and only if } \ [P] \geq t \ \text{ for all valuations.}$$

Based on [46], we can extend the concept of implication-based fuzzy subgroups to the concept of implication-based fuzzy subpolygroups in the following way:

Definition 5.2. Let A be a fuzzy subset of a polygroup P and $t \in (0, 1]$ be a fixed number. If

(1) for any $x, y \in P$, $\models_t$ $[\ [x \in A] \wedge [y \in A] \longrightarrow [\forall z \in x \circ y, \ z \in A]\ ]$,

(2) for any $x \in P$, $\models_t$ $[\ [x \in A] \longrightarrow [x^{-1} \in A]]$,

then A is called a *t-implication-based fuzzy subpolygroup of* P.

Now, let I be an implication operator. Then

Corollary 5.3. *A is a t-implication-based fuzzy subpolygroup of a polygroup P if and only if*

(i) $I(A(x) \wedge A(y), \bigwedge_{z \in x \circ y} A(z)) \geq t$ *for all* $x, y \in P$,

(ii) *for any* $x \in P$, $I(A(x) \wedge A(x^{-1})) \geq t$.

Let A be a fuzzy subset of a polygroup P. Then we have the following results:

Theorem 5.4.

(1) *Let* $I = I_{gr}$. *Then A is an 0.5-implication-based fuzzy subpolygroup of P if and only if A is a fuzzy subpolygroup with thresholds ($\alpha = 0$, $\beta = 1$) of P.*

(2) *Let* $I = I_g$. *Then A is an 0.5-implication-based fuzzy subpolygroup of P if and only if A is a fuzzy subpolygroup with thresholds ($\alpha = 0$, $\beta = 0.5$) of P.*

(3) *Let* $I = I_{cg}$. *Then A is an 0.5-implication-based fuzzy subpolygroup of P if and only if A is a fuzzy subpolygroupgroup with thresholds ($\alpha = 0.5$, $\beta = 1$) of P.*

Proof. The proof follows directly from the definitions. $\square$

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