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SOME RESULTS ON INTUITIONISTIC FUZZY SPACES

S. B. HOSSEINI, D. O'REGAN AND R. SAADATI

Abstract. In this paper we define intuitionistic fuzzy metric and normed spaces. We first consider finite dimensional intuitionistic fuzzy normed spaces and prove several theorems about completeness, compactness and weak convergence in these spaces. In section 3 we define the intuitionistic fuzzy quotient norm and study completeness and review some fundamental theorems. Finally, we consider some properties of approximation theory in intuitionistic fuzzy metric spaces.

1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [22]. After the pioneering work of Zadeh, much interest has focused on obtaining fuzzy analogues of classical theories. We mention in particular the field of fuzzy topology [1, 10, 11, 13, 16, 20]. The concept of fuzzy topology has important applications in quantum particle physics, in particular in connection with both string and $\epsilon^{(\infty)}$ theory; see El Naschie [7, 8, 9, 21]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of an intuitionistic fuzzy metric space and an intuitionistic fuzzy normed space. These problems have been investigated by Park [17] and Saadati and Park [19] respectively; they introduced and studied a notion of an intuitionistic fuzzy metric (normed) space. In this section, using the idea of fuzzy metric (normed) spaces introduced by George and Veeramani [10, 11] and Amini and Saadati [1], we present the notion of intuitionistic fuzzy metric (normed) spaces with the help of the notion of continuous t–representable norms.

Lemma 1.1. [6] *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1$ *and* $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice .*

Definition 1.2. [2] An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_\mathcal{A}(u), \eta_\mathcal{A}(u))|u \in U\}$, where, for all $u \in U$, $\zeta_\mathcal{A}(u) \in [0, 1]$ and $\eta_\mathcal{A}(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$; we always have $\zeta_\mathcal{A}(u) + \eta_\mathcal{A}(u) \leq 1$.

Definition 1.3. For every $z_\alpha = (x_\alpha, y_\alpha) \in L^*$ we define

$$\bigvee(z_\alpha) = (\sup(x_\alpha), \inf(y_\alpha)).$$

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Since $z_\alpha \in L^*$ hence $x_\alpha + y_\alpha \le 1$ so $\sup(x_\alpha) + \inf(y_\alpha) \le \sup(x_\alpha + y_\alpha) \le 1$, i.e. $\bigvee(z_\alpha) \in L^*$. We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

Classically a triangular norm $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying $T(1,x) = 1 * x = x$, for all $x \in [0,1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \longrightarrow [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \le_{L^*})$ these definitions can be straightforwardly extended.

Definition 1.4. [4, 5] A triangular norm (t–norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

$(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$, (boundary condition)

$(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$, (commutativity)

$(\forall (x,y,z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z))$, (associativity)

$(\forall (x,x',y,y') \in (L^*)^4)(x \le_{L^*} x' \text{ and } y \le_{L^*} y' \Longrightarrow \mathcal{T}(x,y) \le_{L^*} \mathcal{T}(x',y'))$. (monotonicity)

Definition 1.5. [3] A continuous t–norm $\mathcal{T}$ on L^* is called continuous *t–representable* if and only if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

Now define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \cdots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \cdots, x^{(n)}), x^{(n+1)})$$

for $n \ge 2$ and $x^{(i)} \in L^*$.

We say the continuous t–representable norm is *natural* and write $\mathcal{T}_n$ whenever $\mathcal{T}_n(a,b) = \mathcal{T}_n(c,d)$ and $a \le_{L^*} c$ implies $b \ge_{L^*} d$.

Definition 1.6. [4, 5] A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0,1]$ defined as $N_s(x) = 1 - x$ for all $x \in [0,1]$.

Definition 1.7. Let M, N be fuzzy sets from $X^2 \times (0, +\infty)$ to $[0,1]$ such that $M(x,y,t) + N(x,y,t) \le 1$ for all $x, y \in X$ and $t > 0$. The triple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t–representable norm and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times (0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 1.2) satisfying the following conditions for every $x, y \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x,y,t) >_{L^*} 0_{L^*}$;

(b) $\mathcal{M}_{M,N}(x,y,t) = 1_{L^*}$ if and only if $x = y$;

(c) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t)$;

(d) $\mathcal{M}_{M,N}(x,y,t+s) \ge_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x,z,t), \mathcal{M}_{M,N}(z,y,s))$;

(e) $\mathcal{M}_{M,N}(x,y,\cdot) : (0,\infty) \longrightarrow L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called an *intuitionistic fuzzy metric*. Here,

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)).$$

Example 1.8. *Let (X,d) be a metric space. Define $\mathcal{T}(a,b) = (a_1b_1, \min(a_2+b_2, 1))$ for all $a = (a_1,a_2)$ and $b = (b_1,b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:*

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)) = (\frac{ht^n}{ht^n + md(x,y)}, \frac{md(x,y)}{ht^n + md(x,y)}),$$

for all $t, h, m, n \in \mathbf{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 1.9. *Let $X = \mathbf{N}$. Define $\mathcal{T}(a,b) = (\max(0, a_1+b_1-1), a_2+b_2-a_2b_2)$ for all $a = (a_1,a_2)$ and $b = (b_1,b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:*

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)) = \begin{cases} (\frac{x}{y}, \frac{y-x}{y}) & if \quad x \leq y \\ (\frac{y}{x}, \frac{x-y}{x}) & if \quad y \leq x. \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Definition 1.10. Let μ, ν be fuzzy sets from $V \times (0,+\infty)$ to $[0,1]$ such that $\mu(x,t) + \nu(x,t) \leq 1$ for all $x \in V$ and $t > 0$. The 3-tuple $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space* if V is a vector space, $\mathcal{T}$ is a continuous t–representable norm and $\mathcal{P}_{\mu,\nu}$ is a mapping $V \times (0,+\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 1.2) satisfying the following conditions for every $x, y \in V$ and $t, s > 0$:

(a) $\mathcal{P}_{\mu,\nu}(x,t) >_{L^*} 0_{L^*}$;
(b) $\mathcal{P}_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$;
(c) $\mathcal{P}_{\mu,\nu}(\alpha x,t) = \mathcal{P}_{\mu,\nu}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(d) $\mathcal{P}_{\mu,\nu}(x+y, t+s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu,\nu}(x,t), \mathcal{P}_{\mu,\nu}(y,s))$;
(e) $\mathcal{P}_{\mu,\nu}(x,\cdot) : (0,\infty) \longrightarrow L^*$ is continuous;
(f) $\lim_{t\longrightarrow\infty} \mathcal{P}_{\mu,\nu}(x,t) = 1_{L^*}$ and $\lim_{t\longrightarrow 0} \mathcal{P}_{\mu,\nu}(x,t) = 0_{L^*}$.

Then $\mathcal{P}_{\mu,\nu}$ is called an *intuitionistic fuzzy norm*. Here,

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)).$$

Example 1.11. *Let $(V, \|\cdot\|)$ be a normed space and let $\mathcal{T}(a,b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1,a_2)$ and $b = (b_1,b_2) \in L^*$. Now let μ, ν be fuzzy sets in $V \times (0,\infty)$ and define*

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = (\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}),$$

for all $t \in \mathbf{R}^+$. Then $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an intuitionistic fuzzy normed space.

Definition 1.12. A sequence $\{x_n\}$ in an intuitionistic fuzzy normed space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a *Cauchy sequence* if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon),$$

for each $n, m \geq n_0$; here N_s is the standard negator. The sequence $\{x_n\}$ is said to be *convergent* to $x \in V$. $(x_n \xrightarrow{\mathcal{P}_{\mu,\nu}} x)$ if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \longrightarrow 1_{L^*}$ whenever $n \longrightarrow \infty$ for every $t > 0$. An intuitionistic fuzzy normed space is said to be *complete* if and only if every Cauchy sequence is convergent.

Lemma 1.13. [19] *Let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space. We define*

$$\mathcal{M}_{M,N}(x, y, t) = \mathcal{P}_{\mu,\nu}(x - y, t)$$

where. Then $M(x, y, t) = \mu(x - y, t)$ and $N(x, y, t) = \nu(x - y, t)$, then $\mathcal{M}_{M,N}$ is an intuitionistic fuzzy metric on V, which is induced by the intuitionistic fuzzy norm $\mathcal{P}_{\mu,\nu}$.

Lemma 1.14. [19] *Let $\mathcal{P}_{\mu,\nu}$ be an intuitionistic fuzzy norm. Then, for any $t > 0$, the following hold:*

(1) *$\mathcal{P}_{\mu,\nu}(x, t)$ is nondecreasing with respect to t, in $(L^*, \leq_{L^*})$.*
(2) *$\mathcal{P}_{\mu,\nu}(x - y, t) = \mathcal{P}_{\mu,\nu}(y - x, t)$.*

Definition 1.15. Let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space. For $t > 0$, define the *open ball* $B(x, r, t)$ with center $x \in V$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in V : \mathcal{P}_{\mu,\nu}(x - y, t) >_{L^*} (N_s(r), r)\}.$$

A subset $A \subseteq V$ is called *open* if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{P}_{\mu,\nu}}$ denote the family of all open subsets of V. $\tau_{\mathcal{P}_{\mu,\nu}}$ is called the *topology induced by the intuitionistic fuzzy norm.*

Note that this topology is the same as the topology induced by the intuitionistic fuzzy metric which is Hausdorff (see, Remark 3.3 and Theorem 3.5 of [17]).

Definition 1.16. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. A subset A of X is said to be *IF-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $\mathcal{M}_{M,N}(x, y, t) >_{L^*} (N_s(r), r)$ for each $x, y \in A$. Also, let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space. A subset A of V is said to be *IF-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $\mathcal{P}_{\mu,\nu}(x, t) >_{L^*} (N_s(r), r)$ for each $x \in A$.

Theorem 1.17. *In an intuitionistic fuzzy normed (metric) space every compact set is closed and IF-bounded.*

Proof. By Lemma 1.13, the proof is the same as in the intuitionistic fuzzy metric space case (see, Remark 3.10 of [17]). □

Lemma 1.18. [19] *A subset A of $\mathbf{R}$ is IF-bounded in $(\mathbf{R}, \mathcal{P}_{\mu_0,\nu_0}, \mathcal{T})$ if and only if it is bounded in $\mathbf{R}$.*

Lemma 1.19. [19] *A sequence* $\{\beta_n\}$ *is convergent in the intuitionistic fuzzy normed space* $(\mathbf{R}, \mathcal{P}_{\mu_0,\nu_0}, \mathcal{T})$ *if and only if it is convergent in* $(\mathbf{R}, |\cdot|)$.

Corollary 1.20. *If the real sequence* $\{\beta_n\}$ *is IF-bounded, then it has at least one limit point.*

Definition 1.21. Let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space. Let V be a vector space, f be a real functional on V and let $(\mathbf{R}, \mathcal{P}_{\mu_0,\nu_0}, \mathcal{T})$ be an intuitionistic fuzzy normed space. We define

$$\tilde{V} = \{f : \mathcal{P}_{\mu_0,\nu_0}(f(x), t) \geq_{L^*} (\mu(cx, t), \nu(dx, t)) \ , c, d \neq 0\}$$

for every $t > 0$.

Lemma 1.22. [19] *If* $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ *is an intuitionistic fuzzy normed space, then*

(a) *The function* $(x, y) \longrightarrow x + y$ *is continuous;*
(b) *The function* $(\alpha, x) \longrightarrow \alpha x$ *is continuous.*

By the above lemma an intuitionistic fuzzy normed space is a Hausdorff TVS.

2. Intuitionistic Fuzzy Finite Dimensional Normed Spaces

Theorem 2.1. [19] *Let* $\{x_1, \cdots, x_n\}$ *be a linearly independent set of vectors in vector space* V *and* $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ *be an intuitionistic fuzzy normed space. Then there are numbers* $c, d \neq 0$ *and an intuitionistic fuzzy norm space* $(\mathbf{R}, \mathcal{P}_{\mu_0,\nu_0}, \mathcal{T})$ *such that for every choice of real scalars* $\alpha_1, \cdots, \alpha_n$ *we have*

$$\mathcal{P}_{\mu,\nu}(\alpha_1 x_1 + \cdots + \alpha_n x_n, t) \leq_{L^*} (\mu_0(c\sum_{j=1}^{n}|\alpha_j|, t), \nu_0(d\sum_{j=1}^{n}|\alpha_j|, t)). \tag{2.1}$$

Theorem 2.2. *Every finite dimensional subspace* W *of an intuitionistic fuzzy normed space* $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ *is complete. In particular, every finite dimensional intuitionistic fuzzy normed space is complete.*

Proof. Let $\{y_m\}$ be a Cauchy sequence in W such that y is its limit. We show that $y \in W$. Suppose $\dim W = n$ and let $\{x_1, ...x_n\}$ be any linearly independent subset for Y. Then each y_m has a unique representation of the form

$$y_m = \alpha_1^{(m)} x_1 + ... + \alpha_n^{(m)} x_n.$$

Since $\{y_m\}$ is Cauchy sequence, for every $\varepsilon > 0$ there is a positive integer n_0 such that,

$$(N_s(\varepsilon), \varepsilon) <_{L^*} \mathcal{P}_{\mu,\nu}(y_m - y_k, t),$$

whenever $m, k > n_0$ and for every $t > 0$. From this and the last theorem we have, for some $c, d \neq 0$ and $\mathcal{P}_{\mu_0,\nu_0}$

$$\begin{aligned}
(N_s(\varepsilon),\varepsilon) <_{L^*} \mathcal{P}_{\mu,\nu}(y_m - y_k, t) &= \mathcal{P}_{\mu,\nu}(\sum_{j=1}^{n}(\alpha_j^{(m)} - \alpha_j^{(k)})x_j, t) \\
&\leq_{L^*} (\mu_0(\sum_{j=1}^{n}|\alpha_j^{(m)} - \alpha_j^{(k)}|c, t), \nu_0(\sum_{j=1}^{n}|\alpha_j^{(m)} - \alpha_j^{(k)}|d, t)) \\
&\leq_{L^*} (\mu_0(1, \frac{t/|c|}{\sum_{j=1}^{n}|\alpha_j^{(m)} - \alpha_j^{(k)}|}), \nu_0(1, \frac{t/|d|}{\sum_{j=1}^{n}|\alpha_j^{(m)} - \alpha_j^{(k)}|})) \\
&\leq_{L^*} (\mu_0(1, \frac{t/|c|}{|\alpha_j^{(m)} - \alpha_j^{(k)}|}), \nu_0(1, \frac{t/|d|}{|\alpha_j^{(m)} - \alpha_j^{(k)}|})) \\
&= (\mu_0(\alpha_j^{(m)} - \alpha_j^{(k)}, t/|c|), \nu_0(\alpha_j^{(m)} - \alpha_j^{(k)}, t/|d|)).
\end{aligned}$$

This shows that each of the n sequences $\{\alpha_j^{(m)}\}$ where $j = 1, 2, 3, ..., n$ is Cauchy in $\mathbf{R}$. Hence it converges and let α_j denote the limit. Using these n limits $\alpha_1, ..., \alpha_n$, we define,

$$y = \alpha_1 x_1 + ... + \alpha_n x_n.$$

Clearly, $y \in W$. Furthermore

$$\begin{aligned}
\mathcal{P}_{\mu,\nu}(y_m - y, t) &= \mathcal{P}_{\mu,\nu}(\sum_{j=1}^{n}(\alpha_j^{(m)} - \alpha_j)x_j, t) \\
&\geq_{L^*} \mathcal{T}^{n-1}[\mathcal{P}_{\mu,\nu}(\alpha_1^{(m)} - \alpha_1)x_1, t/n), \cdots, \mathcal{P}_{\mu,\nu}(\alpha_n^{(m)} - \alpha_n)x_n, t/n)] \\
&\longrightarrow 1_{L^*}
\end{aligned}$$

whenever $m \longrightarrow \infty$ and every $t > 0$. This shows that an arbitrary sequence $\{y_m\}$ is convergent in W. Hence W is complete. □

Corollary 2.3. *Every finite dimensional subspace W of an intuitionistic fuzzy normed space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is closed in V.*

Theorem 2.4. *In a finite dimensional intuitionistic fuzzy normed space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$, any subset $K \subset V$ is compact if and only if K is closed and IF-bounded.*

Proof. By Theorem 1.17, compactness implies closedness and IF-boundedness. We must prove the converse. Let K be closed and IF-bounded. Let $\dim V = n$ and $\{x_1, ..., x_n\}$ be a linearly independent set of V. We consider any sequence $\{x_m\}$ in K. Each x_m has a representation,

$$x_m = \alpha_1^{(m)} x_1 + ... + \alpha_n^{(m)} x_n.$$

Since K is IF-bounded, so is $\{x_m\}$, and therefore there are $t > 0$ and $0 < r < 1$ such that $\mathcal{P}_{\mu,\nu}(x_m, t) >_{L^*} (N_s(r), r)$ for all $m \in \mathbf{N}$. On the other hand by Theorem 2.1 there are $c, d \neq 0$ and an intuitionistic fuzzy norm $\mathcal{P}_{\mu_0,\nu_0}$ such that

$$\begin{aligned}
(N_s(r), r) &<_{L^*} \mathcal{P}_{\mu,\nu}(x_m, t) \\
&= \mathcal{P}_{\mu,\nu}(\sum_{j=1}^{n} \alpha_j^{(m)} x_j, t) \\
&\leq_{L^*} (\mu_0(c\sum_{j=1}^{n} |\alpha_j^{(m)}|, t), \nu_0(d\sum_{j=1}^{n} |\alpha_j^{(m)}|, t)) \\
&\leq_{L^*} (\mu_0(1, \frac{t}{|c|\sum_{j=1}^{n} |\alpha_j^{(m)}|}), \nu_0(1, \frac{t}{|d|\sum_{j=1}^{n} |\alpha_j^{(m)}|})) \\
&\leq_{L^*} (\mu_0(1, \frac{t}{|c||\alpha_j^{(m)}|}), \nu_0(1, \frac{t}{|d||\alpha_j^{(m)}|})) \\
&= (\mu_0(\alpha_j^{(m)}, t/|c|), \nu_0(\alpha_j^{(m)}, t/|d|)).
\end{aligned}$$

Hence the sequence of $\{\alpha_j^{(m)}\}$, (j fixed), is IF-bounded and by Theorem 1.20 it has a limit point α_j, $(1 \leq j \leq n)$. Let $\{z_m\}$ be the subsequence of $\{x_m\}$ which converges to $z = \sum_{j=1}^{n} \alpha_j x_j$. Since K is closed, $z \in K$. This shows that an arbitrary sequence $\{x_m\}$ in K has a subsequence which converges in K. Hence K is compact. □

Definition 2.5. A sequence $\{x_m\}$ in an intuitionistic fuzzy normed space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be weakly convergent if there is an $x \in V$ such that for every $\tilde{V}$ and every $f \in \tilde{V}$ and $t > 0$,

$$\mathcal{P}_{\mu_0,\nu_0}(f(x_m) - f(x), t) \longrightarrow 1_{L^*}.$$

We write:

$$x_m \xrightarrow{W} x.$$

Theorem 2.6. *Let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space and $\{x_m\}$ be a sequence in V. Then:*

(i) *Convergence implies weak convergence with the same limit.*

(ii) *If $\dim V < \infty$, then weak convergence implies convergence.*

Proof. (i) Let $x_m \longrightarrow x$ then for every $t > 0$ we have

$$\mathcal{P}_{\mu,\nu}(x_m - x, t) \longrightarrow 1_{L^*}.$$

By Definition 1.21 for every $f \in \tilde{V}$ we have,

$$\begin{aligned}
\mathcal{P}_{\mu_0,\nu_0}(f(x_m) - f(x), t) &= \mathcal{P}_{\mu_0,\nu_0}(f(x_m - x), t) \\
&\geq_{L^*} (\mu(x_m - x, t/c), \nu(x_m - x, t/d))
\end{aligned}$$

for $c, d \neq 0$. Then $x_m \xrightarrow{W} x$.

(ii) Let $x_m \xrightarrow{W} x$ and $\dim V = n$. Let $\{x_1, ..., x_n\}$ be a linearly independent set of V. Then

$$x_m = \alpha_1^{(m)} x_1 + ... + \alpha_n^{(m)} x_n.$$

and,

$$x = \alpha_1 x_1 + ... + \alpha_n x_n.$$

By assumption, for every $f \in \tilde{V}$ and $t > 0$ we have

$$\mathcal{P}_{\mu_0,\nu_0}(f(x_m) - f(x), t) \longrightarrow 1_{L^*}.$$

We take in particular $f_1, ..., f_n$, defined by $f_j x_j = 1$ and $f_j x_i = 0$, $(i \neq j)$. Therefore $f_j(x_m) = \alpha_j^{(m)}$ and $f_j(x) = \alpha_j$. Hence, $f_j(x_m) \longrightarrow f_j(x)$ implies $\alpha_j^{(m)} \longrightarrow \alpha_j$. From this we obtain, for each $t > 0$

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(x_m - x, t) &= \mathcal{P}_{\mu,\nu}(\sum_{j=1}^{n}(\alpha_j^{(m)} - \alpha_j)x_j, t)\\ &\geq_{L^*} \mathcal{T}^{n-1}[\mathcal{P}_{\mu,\nu}((\alpha_1^{(m)} - \alpha_1)x_1, t/n), ..., \mathcal{P}_{\mu,\nu}((\alpha_n^{(m)} - \alpha_n)x_n, t/n)]\\ &= \mathcal{T}^{n-1}[\mathcal{P}_{\mu,\nu}(x_1, \frac{t}{n|\alpha_1^{(m)} - \alpha_1|}), ..., \mathcal{P}_{\mu,\nu}(x_n, \frac{t}{n|\alpha_n^{(m)} - \alpha_n|})]\\ &\longrightarrow 1_{L^*}\end{aligned}$$

as $m \longrightarrow \infty$. This shows that $\{x_m\}$ converges to x. □

3. Some Fundamental Theorems in Intuitionistic Fuzzy Functional Analysis

Definition 3.1. Let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space and W be a linear manifold in V. Let $Q : V \longrightarrow V/W$ be the natural map, $Qx = x + W$. We define:

$$\mathcal{P}_{\bar{\mu},\bar{\nu}}(x + W, t) = \bigvee\{\mathcal{P}_{\mu,\nu}(x + y, t) : y \in W\}, \quad t > 0.$$

Theorem 3.2. *If W is a closed subspace of the intuitionistic fuzzy normed space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ and the intuitionistic fuzzy norm $\mathcal{P}_{\bar{\mu},\bar{\nu}}$ is defined as above, then:*

(a) *$\mathcal{P}_{\bar{\mu},\bar{\nu}}$ is a fuzzy norm on V/W;*

(b) *$\mathcal{P}_{\bar{\mu},\bar{\nu}}(Qx, t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(x, t)$;*

(c) *If $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is a complete intuitionistic fuzzy normed space (intuitionistic fuzzy Banach space) and for every a, b in $[0, 1]$, $a * b \geq a \cdot b$ and $a \diamond b \leq \max(a, b)$, then so is $(V/W, \mathcal{P}_{\bar{\mu},\bar{\nu}}, \mathcal{T})$.*

Proof. By Definition 1.3, $\mathcal{P}_{\bar{\mu},\bar{\nu}} \in L^*$ and the proof follows as in [12, 18]. □

Theorem 3.3. [12, 18] *Let W be a closed subspace of an intuitionistic fuzzy normed space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$. If two of the spaces V, W, V/W are complete so is the third one.*

Theorem 3.4. (Open mapping theorem) [12, 18] *If T is a continuous linear operator from the intuitionistic fuzzy Banach space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ onto the intuitionistic fuzzy Banach space $(V', \mathcal{P}'_{\mu',\nu'}, \mathcal{T})$ and for every a, b in $[0, 1]$, $a * b \geq a \cdot b$ and $a \diamond b \leq \max(a, b)$, then T is an open mapping.*

Theorem 3.5. (Closed graph theorem) [12, 18] *Let T be a linear operator from the intuitionistic fuzzy Banach space $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ into the intuitionistic fuzzy Banach space $(V', \mathcal{P}'_{\mu',\nu'}, \mathcal{T})$. Suppose for every sequence $\{x_n\}$ in V such that $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ for some elements $x \in V$ and $y \in V'$ it follows $Tx = y$. Then T is continuous.*

Theorem 3.6. *An intuitionistic fuzzy normed space* $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ *where* $\mathcal{T}(a, b) = (\min(a_1, b_1), \max(a_2, b_2))$*, is locally convex; here* $a = (a_1, a_2)$ *and* $b = (b_1, b_2)$*.*

Proof. It suffices to consider the family of neighborhoods of the origin, $B(0, r, t)$, with $t > 0$ and $0 < r < 1$. Let $t > 0$, $0 < r < 1$, $x, y \in B(0, r, t)$ and $\alpha \in [0, 1]$. Then

$$\begin{aligned} \mathcal{P}_{\mu,\nu}(\alpha x + (1-\alpha)y, t) &\geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu,\nu}(\alpha x, \alpha t), \mathcal{P}_{\mu,\nu}((1-\alpha)y, (1-\alpha)t)) \\ &= \mathcal{T}(\mathcal{P}_{\mu,\nu}(x, t), \mathcal{P}_{\mu,\nu}(y, t)) \\ &= (\min(\mu(x, t), \mu(y, t)), \max(\nu(x, t), \nu(y, t))) \\ &>_{L^*} (N_s(r), r). \end{aligned}$$

Thus $\alpha x + (1-\alpha)y$ belongs to $B(0, r, t)$ for every $\alpha \in [0, 1]$. □

4. Approximation Theory in Intuitionistic Fuzzy Metric Spaces

Definition 4.1. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space and $A, B \subset X$. We define

$$\mathcal{M}_{M,N}(A, B, t) = \bigvee\{\mathcal{M}_{M,N}(a, b, t) : a \in A \text{ and } b \in B\}.$$

For $a \in X$, we write $\mathcal{M}_{M,N}(a, B, t)$ instead of $\mathcal{M}_{M,N}(\{a\}, B, t)$.

Definition 4.2. A sequence converges sub-sequentially if it has a convergent subsequence. In the above notation $x_n \gg x_{n'} \to x_0$ identifies the subsequence and the point to which it converges. Recall that a subset C of an intuitionistic fuzzy metric space is compact if every sequence in C converges sub-sequentially to an element of C. Also, given two sequences x_n and y_n, and a subsequence $x_{n'}$ of the first sequence, the corresponding subsequence of the second is denoted $y_{n'}$. A subset of an intuitionistic fuzzy metric space is *IF-boundedly compact* if every IF-bounded sequence in the subset is sub-sequentially convergent. In the above notation, Y is IF-boundedly compact if for any IF-bounded sequence y_n in Y, there is a point x_0 (not necessarily in Y) for which $y_n \gg y_{n'} \to x_0$.

Definition 4.3. For an intuitionistic fuzzy metric space X and nonempty subsets B and C, a sequence $b_n \in B$ is said to *converge in distance* to C if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(b_n, C, t) = \mathcal{M}_{M,N}(B, C, t).$$

The subset B is *approximately compact* relative to C if every sequence $b_n \in B$ which converges in distance to C is sub-sequentially convergent to an element of B. We call B (a subset of X) approximately compact provided that B is approximately compact relative to each of the singletons of X; B is proximinal if for every $x \in X$ some element b in B satisfies the equation $\mathcal{M}_{M,N}(x, b, t) = \mathcal{M}_{M,N}(x, B, t)$.

The following theorem says that points can be replaced by compact subsets in the definition of approximate compactness.

Theorem 4.4. *Let* B *and* C *be nonempty subsets of an intuitionistic fuzzy metric space* $(X, \mathcal{M}_{M,N}, \mathcal{T})$*. If* B *is approximately compact and* C *is compact, then* B *is approximately compact relative to* C*.*

Proof. Let $b_n \in B$ be any sequence converging in distance to C and let the sequence $c_n \in C$ satisfy

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(b_n, c_n, t) \quad = \quad \mathcal{M}_{M,N}(B, C, t). \tag{4.1}$$

Since C is compact, $c_n \gg c_{n'} \to c_0 \in C$. Hence, for every $\varepsilon > 0$ there exists n_0 such that for $n' > n_0$

$$\begin{aligned} \mathcal{M}_{M,N}(B, C, t) &\geq_{L^*} \mathcal{M}_{M,N}(b_{n'}, c_0, t) \\ &\geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(b_{n'}, c_{n'}, t-\varepsilon), \mathcal{M}_{M,N}(c_{n'}, c_0, \varepsilon)) \\ &\geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(B, C, t-\varepsilon), (N_s(\varepsilon), \varepsilon)). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, then $\lim_{n\to\infty} \mathcal{M}_{M,N}(b_{n'}, c_0, t) = \mathcal{M}_{M,N}(B, C, t)$. Therefore, $b_{n'}$ converges in distance to c_0 so, since B is approximately compact, $b_n \gg b_{n'} \to b_0 \in B$, that is, b_n converges sub-sequentially to an element of B. □

Theorem 4.5. *Let B and C be nonempty subsets of an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$. If B is approximately compact and IF-bounded, and C is IF-boundedly compact, then B is approximately compact relative to C.*

Proof. Let $b_n \in B$ be any sequence converges in distance to C and let $c_n \in C$ satisfy (2.1). As b_n is IF-bounded, so is c_n. Since C is IF-boundedly compact, $c_n \gg c_{n'} \to c_0 \in X$. Proceed as in the proof of last theorem. □

Theorem 4.6. *Let B and C be nonempty subsets of an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$. If B is closed and IF-boundedly compact and C is IF-bounded, then B is approximately compact relative to C.*

The proof is the same as the classically case (see [14]).

Lemma 4.7. [19] *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and $(Y, \mathcal{M}_{M,N}, \mathcal{T})$ be intuitionistic fuzzy metric spaces. If we define*

$$\mathbf{M}((x, y), (x', y'), t) = \mathcal{T}(\mathcal{M}_{M,N}(x, x', t), \mathcal{M}_{M,N}(y, y', t)).$$

then $(X \times Y, \mathbf{M}, \mathcal{T})$ is an intuitionistic fuzzy metric space and the topology induced on $X \times Y$ is the product topology.

Theorem 4.8. *Let S and P be nonempty subsets of intuitionistic fuzzy metric spaces $(X, \mathcal{M}_{M,N}, \mathcal{T}_n)$ and $(Y, \mathcal{M}_{M,N}, \mathcal{T}_n)$, respectively. Suppose that P is compact. If S is IF-boundedly compact or approximately compact, then so is $S \times P$.*

Proof. If S is IF-boundedly compact, we show that any sequence (s_n, p_n) in $S \times P$ which is IF-bounded has a convergent subsequence. Indeed, by definition of the product intuitionistic fuzzy metric, s_n is IF-bounded and since S is IF-boundedly compact, $s_n \gg s_{n'} \to s_0 \in X$. By compactness of P, $p_n \gg p_{n'} \to p_0 \in P$. Hence, $(s_n, p_n) \gg (s_{n''}, p_{n''}) \to (s_0, p_0) \in X \times Y$.
If S is approximately compact, let (x, y) be any element in $X \times Y$ and suppose that (s_n, p_n) is a sequence in $S \times P$ which converges in distance to (x, y), that is,

$$\lim_{n\to\infty} \mathbf{M}((s_n, p_n), (x, y), t) = \mathbf{M}(S \times P, (x, y), t).$$

By compactness of P, $p_n \gg p_{n'} \to p_0 \in P$. Hence, $\lim_{n\to\infty} \mathbf{M}((s_{n'}, p_0), (x, y), t) = \mathbf{M}(S \times P, (x, y), t)$ so

$$\lim_{n'\to\infty} \mathcal{T}_n(\mathcal{M}_{M,N}(s_{n'}, x, t), \mathcal{M}_{M,N}(p_0, y, t)) = \mathcal{T}_n(\mathcal{M}_{M,N}(S, x, t), \mathcal{M}_{M,N}(P, y, t)).$$

Since $\mathcal{M}_{M,N}(p_0, y, t) \leq_{L^*} \mathcal{M}_{M,N}(P, y, t)$ then $\lim_{n'\to\infty} \mathcal{M}_{M,N}(s_{n'}, x, t) \geq_{L^*} \mathcal{M}_{M,N}(S, x, t)$ which implies $\lim_{n'\to\infty} \mathcal{M}_{M,N}(s_{n'}, x, t) = \mathcal{M}_{M,N}(S, x, t)$. Hence, $s_{n'}$ converges in distance to x and since S is approximately compact, $s_{n'} \gg s_{n''} \to s_0 \in S$. Therefore, $(s_n, p_n) \gg (s_{n''}, p_{n''}) \to (s_0, p_0) \in S \times P$, i.e., $S \times P$ is approximately compact. □

Theorem 4.9. *Let B and C be nonempty subsets of an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$. If B is approximately compact and C is compact, then $K = \{b \in B : \exists\, c \in C,\ \mathcal{M}_{M,N}(b, c, t) = \mathcal{M}_{M,N}(B, c, t)\}$ is compact.*

Proof. Let y_n be a sequence in K and for every $n \in \mathbf{N}$ choose c_n in C so that y_n minimizes the distance from B to c_n. Since C is compact, $c_n \gg c_{n'} \to c_0 \in C$. Hence, for every $\varepsilon > 0$, there exists n_0 such that for all $n' > n_0$, $\mathcal{M}_{M,N}(c_{n'}, c_0, t) \geq_{L^*} (N_s(\varepsilon), \varepsilon)$, and therefore, for all $n' > n_0$,

$$\begin{aligned} \mathcal{M}_{M,N}(B, c_0, t) &\geq_{L^*} \mathcal{M}_{M,N}(y_{n'}, c_0, t) \\ &\geq_{L^*} \mathcal{T}^2(\mathcal{M}_{M,N}(B, c_0, t - 2\varepsilon), \mathcal{M}_{M,N}(c_{n'}, c_0, \varepsilon), \mathcal{M}_{M,N}(c_{n'}, c_0, \varepsilon)) \\ &\geq_{L^*} \mathcal{T}^2(\mathcal{M}_{M,N}(B, c_0, t - 2\varepsilon), (N_s(\varepsilon), \varepsilon), (N_s(\varepsilon), \varepsilon)). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, then $\mathcal{M}_{M,N}(B, c_0, t) = \lim_{n'\to\infty} \mathcal{M}_{M,N}(y_{n'}, c_0, t)$. Therefore, $y_{n'}$ converges in distance to c_0, so it converges sub-sequentially. □

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S. B. Hosseini, Islamic Azad University-Nour Branch, Nour, Iran

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie

Reza Saadati* Department of Mathematics, Islamic Azad University-Ayatollah Amoly Branch, Amol, Iran and Institute for Studies in Applied Mathematics 1, Fajr 4, Amol 46176-54553, Iran
E-mail address: rsaadati@eml.cc

*Corresponding author