TRIANGULAR FUZZY MATRICES

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ABSTRACT. In this paper, some elementary operations on triangular fuzzy numbers (TFNs) are defined. We also define some operations on triangular fuzzy matrices (TFMs) such as trace and triangular fuzzy determinant (TFD). Using elementary operations, some important properties of TFMs are presented. The concept of adjoints on TFM is discussed and some of their properties are. Some special types of TFMs (e.g. pure and fuzzy triangular, symmetric, pure and fuzzy skew-symmetric, singular, semi-singular, constant) are defined and a number of properties of these TFMs are presented.

1. Introduction

After Zadeh's work [20] uncertainty can be classified into two types - probabilistic uncertainty and fuzzy uncertainty, though people were aware of fuzzy uncertainty before the mathematical formulation of fuzziness by Zadeh. Fuzziness can be represented in different ways. One of the most useful representation is membership function. Also, depending the nature or shape of membership function a fuzzy number can be classified in different ways, such as triangular fuzzy number (TFN), trapezoidal fuzzy number etc. Triangular fuzzy numbers (TFNs) are frequently used in applications. It is well known that the matrix formulation of a mathematical formula gives extra facility to handle/study the problem. Due to the presence of uncertainty in many mathematical formulations in different branches of science and technology, we introduce triangular fuzzy matrices (TFMs). To the best of our knowledge, no work is available on TFMs, though a lot of work on fuzzy matrices is available in literature . A brief review on fuzzy matrices is given below.

Fuzzy matrices were introduced for the first time by Thomason [17], who discussed the convergence of powers of fuzzy matrix. Fuzzy matrices play an important role in scientific development. Several authors have presented a number of results on the convergence of the power sequence of fuzzy matrices [2, 4, 8]. Ragab et al. [11] presented some properties on determinant and adjoint of square fuzzy matrix. Kim and Roush [5] studied the canonical form of an idempotent matrix. Hashimoto [3] studied the canonical form of a transitive matrix. Kolodiejczyk [7] presented the canonical form of a strongly transitive matrix. Xin [18, 19] studied the controllable fuzzy matrix. Ragab et al. [12] presented some properties of the min-max composition of fuzzy matrices. Kim [6] presented some important results on determinant of a square fuzzy matrices. Two new operators and some applications of

fuzzy matrices are given in [13, 14, 15, 16]. Pal [9] introduced intuitionistic fuzzy determinant. Pal et al. [10] introduced intuitionistic fuzzy matrices.

In this article, we introduce TFN, determinant, adjoint of TFMs and discuss some of their properties.

2. Triangular Fuzzy Number

Sometimes it may happen that some data or numbers cannot be specified precisely or accurately due to the error of the measuring technique or instruments etc. Suppose the height of a person is recorded as 160 cm. However, it is impossible in practice to measure the height accurately; actually this height is about 160 cm; it may be a bit more or a bit less than 160 cm. Thus the height of that person can be written more precisely as the triangular fuzzy number $(160 - \alpha, 160, 160 + \beta)$, where α and β are the left and right spreads. In general, a TFN "a" can written as $(a - \alpha, a, a + \beta)$, where α and β are the left and right spreads of a respectively. These type of numbers are alternately represented as $\langle a, \alpha, \beta \rangle$. The mathematical definition of a TFN is given below.

Definition 2.1. A triangular fuzzy number denoted by $M = \langle m, \alpha, \beta \rangle$, has the membership function

$$\mu_M(x) = \begin{cases} 0 & \text{for } x \leq m - \alpha \\ 1 - \dfrac{m - x}{\alpha} & \text{for } m - \alpha < x < m \\ 1 & \text{for } x = m \\ 1 - \dfrac{x - m}{\beta} & \text{for } m < x < m + \beta \\ 0 & \text{for } x \geq m + \beta. \end{cases}$$

The point m, with membership grade of 1, is called the mean value and α, β are the left hand and right hand spreads of M respectively.

A TFN is said to be symmetric if both its spreads are equal, i.e., if $\alpha = \beta$ and it is sometimes denoted by $M = \langle m, \alpha \rangle$.

Due to the wide field of applications of TFNs, many authors have tried to define the basic arithmetic operations on TFNs. Here we introduce the definitions of arithmetic operations due to Dubois and Prade [1].

Let $M = \langle m, \alpha, \beta \rangle$ and $N = \langle n, \gamma, \delta \rangle$ be two TFNs.
  (1) **Addition:** $M + N = \langle m + n, \alpha + \gamma, \beta + \delta \rangle$.
  (2) **Scalar multiplication:** Let λ be a scalar, $\lambda M = \langle \lambda m, \lambda \alpha, \lambda \beta \rangle$, when $\lambda \geq 0$.
      $\lambda M = \langle \lambda m, -\lambda \beta, -\lambda \alpha \rangle$, when $\lambda \leq 0$. In particular, $-M = \langle -m, \beta, \alpha \rangle$.
  (3) **Subtraction:** $M - N = \langle m, \alpha, \beta \rangle - \langle n, \gamma, \delta \rangle = \langle m - n, \alpha + \delta, \beta + \gamma \rangle$.
      For two TFNs M and N, their addition, subtraction and scalar multiplication, i.e., $M + N, M - N$ and λM are all TFMs.
  (4) **Multiplication:** It can be shown that the shape of the membership function of $M.N$ is not necessarily a triangular, but, if the spreads of M and

N are small compared to their mean values m and n then the shape of membership function is closed to a triangle. A good approximation is as follows:

(a) When $M \geq 0$ and $N \geq 0$ ($M \geq 0$, $if\ m \geq 0$)

$M.N = \langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, m\gamma + n\alpha, m\delta + n\beta \rangle.$

(b) When $M \leq 0$ and $N \geq 0$

$M.N = \langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, n\alpha - m\delta, n\beta - m\gamma \rangle.$

(c) When $M \leq 0$ and $N \leq 0$

$M.N = \langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, -n\beta - m\delta, -n\alpha - m\gamma \rangle.$

When spreads are not small compared with mean values, the following is a a better approximation:

$\langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, m\gamma + n\alpha - \alpha\gamma, m\delta + n\beta + \beta\delta \rangle$ for $M > 0, N > 0$.

Throughout the paper we use the previous definition.

Now, we define inverse of a TFN based on the definition of multiplication.

(5) **Inverse:** The inverse of a TFN $M = \langle m, \alpha, \beta \rangle$, $m > 0$ is defined as,

$M^{-1} = \langle m, \alpha, \beta \rangle^{-1} \simeq \langle m^{-1}, \beta m^{-2}, \alpha m^{-2} \rangle.$

This is also an approximate value of M^{-1} and it is valid only a neighbourhood of $1/m$.

Division of M by N is given by

$$\frac{M}{N} = M.N^{-1}.$$

Since inverse and product both are approximate, the division is also an approximate value. The formal definition of division is given below.

(6) **Division:** $\dfrac{M}{N} = M.N^{-1} = \langle m, \alpha, \beta \rangle . \langle n^{-1}, \delta n^{-2}, \gamma n^{-2} \rangle$

$$\simeq \left\langle \frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2} \right\rangle.$$

From the definition of multiplication of TFNs, the power of any TFN M is defined in the following way.

(7) **Exponentiation:** Using the definition of multiplication it can be shown that M^n is given by

$$M^n = \langle m, \alpha, \beta \rangle^n \quad \simeq \quad \langle m^n, -nm^{n-1}\beta, -nm^{n-1}\alpha \rangle, \text{ when } n \text{ is negative,}$$
$$\simeq \quad \langle m^n, nm^{n-1}\alpha, nm^{n-1}\beta \rangle, \text{ when } n \text{ is positive.}$$

Consider two TFN's with a common mean value. Then subtraction produces a TFN whose mean value is zero and the spreads are the sum of both the spreads of computed TFN. The quotient of same TFNs is a TFN having mean value one. THE Inverse of a TFN whose mean value is zero does not exist and we cannot divide by such a number. The addition and multiplication of TFNs are both commutative and associative. But the distributive law does not always hold.

For example, if $A = \langle 2, 0.5, 0.5 \rangle$, $B = \langle 3, 0.8, 0.7 \rangle$, $C = \langle 5, 1, 2 \rangle$ and $D = \langle -5, 2, 1 \rangle$, then $A.(B + C) = A.B + A.C$ holds but, $A.(C + D) \neq A.C + A.D$.

It may be remembered that

$$\langle m, \alpha, \beta \rangle . \langle 0, 0, 0 \rangle = \langle 0, 0, 0 \rangle.$$

3. **Preliminaries and Definitions**

Definition 3.1. Triangular fuzzy matrix (TFM). A triangular fuzzy matrix of order $m \times n$ is defined as $A = (a_{ij})_{m \times n}$, where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ is the ijth element of A, m_{ij} is the mean value of a_{ij} and α_{ij}, β_{ij} are the left and right spreads of a_{ij} respectively.

As for classical matrices define the following operations on TFMs. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two TFMs of same order. Then we have the following.

(i) $A + B = (a_{ij} + b_{ij})$

(ii) $A - B = (a_{ij} - b_{ij})$,

(iii) For $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times p}, A.B = (c_{ij})_{m \times p}$, where

$c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj}, \quad i = 1, 2, \cdots, m \quad \text{and} \quad j = 1, 2, \cdots, p.$

(iv) $A' = (a_{ji})$ (the transpose of A)

(v) $k.A = (ka_{ij})$, where k is a scalar.

We now define special types of TFMs corresponding to special classical matrices. However, because of fuzziness we will have more than on type of TFM corresponding to one type of classical matrix.

Definition 3.2. Pure Null TFM. A TFM is said to be a pure null TFM if all its entries are zero, i.e., all elements are $\langle 0, 0, 0 \rangle$. This matrix is denoted by O.

Definition 3.3. Fuzzy Null TFM. A TFM is said to be a fuzzy null TFM if all elements are of the form $a_{ij} = \langle 0, \epsilon_1, \epsilon_2 \rangle$, where $\epsilon_1 \cdot \epsilon_2 \neq 0$.

Definition 3.4. Pure Unit TFM. A square TFM is said to be a pure unit TFM if $a_{ii} = \langle 1, 0, 0 \rangle$ and $a_{ij} = \langle 0, 0, 0 \rangle$, $i \neq j$, for all i, j. It is denoted by I.

Definition 3.5. Fuzzy Unit TFM. A square TFM is said to be a fuzzy unit TFM if $a_{ii} = \langle 1, \epsilon_1, \epsilon_2 \rangle$ and $a_{ij} = \langle 0, \epsilon_3, \epsilon_4 \rangle$ for $i \neq j$ for all i, j, where $\epsilon_1 \cdot \epsilon_2 \neq 0$, $\epsilon_3 \cdot \epsilon_4 \neq 0$.

Definition 3.6. Pure Triangular TFM. A square TFM $A = (a_{ij})$ is said to be a pure triangular TFM if either $a_{ij} = \langle 0, 0, 0 \rangle$ for all $i > j$ or $a_{ij} = \langle 0, 0, 0 \rangle$ for all $i < j$; $i, j = 1, 2, \cdots, n$.

A pure triangular TFM $A = (a_{ij})$ is said to be pure upper triangular TFM when $a_{ij} = \langle 0, 0, 0 \rangle$ for all $i > j$ and is said to be a pure lower triangular TFM if $a_{ij} = \langle 0, 0, 0 \rangle$ for all $i < j$.

Definition 3.7. Fuzzy Triangular TFM. A square TFM $A = (a_{ij})$ is said to be a fuzzy triangular TFM if either $a_{ij} = \langle 0, \epsilon_1, \epsilon_2 \rangle$ for all $i > j$ or $a_{ij} = \langle 0, \epsilon_1, \epsilon_2 \rangle$ for all $i < j$; $i, j = 1, 2, \cdots, n$ and $\epsilon_1 \cdot \epsilon_2 \neq 0$.

Definition 3.8. Symmetric TFM. A square TFM $A = (a_{ij})$ is said to be symmetric if $A = A'$, i.e., if $a_{ij} = a_{ji}$ for all i, j.

Definition 3.9. Pure Skew-symmetric TFM. A square TFM $A = (a_{ij})$ is said to be pure skew-symmetric if $A = -A'$ and $a_{ii} = \langle 0, 0, 0 \rangle$, i.e., if $a_{ij} = -a_{ji}$ for all i, j and $a_{ii} = \langle 0, 0, 0 \rangle$.

Definition 3.10. Fuzzy Skew-symmetric TFM. A square TFM $A = (a_{ij})$ is said to be fuzzy skew-symmetric if $A = -A'$ and $a_{ii} = \langle 0, \epsilon_1, \epsilon_2 \rangle$, i.e., if $a_{ij} = -a_{ji}$ for all i, j and $a_{ii} = \langle 0, \epsilon_1, \epsilon_2 \rangle$, $\epsilon_1 \cdot \epsilon_2 \neq 0$.

4. Basic Properties

In this section some basic properties of TFMs are presented. The commutative and associative laws are valid for TFM under the operation '+' only.

Property 4.1. For any three TFMs A, B and C of order $m \times n$ we have:

(i) $A+B=B+A$,
(ii) $A+(B+C)=(A+B)+C$,
(iii) $A+A=2A$,
(iv) A-A is a fuzzy null TFM ,
(v) $A+O=A$-$O=A$.

Property 4.2. Let A and B be two TFMs of the same order and k, l be two scalars. Then:

(i) $k(lA)=(kl)A$,
(ii) $k(A+B)=kA+kB$,
(iii) $(k+l)A=kA+lA$, if $k.l¿0$,
(iv) $k(A$-$B)=kA$-kB.

Property 4.3. If A and B be two TFMs such that $A + B$ and $A.B$ are defined then:

(i) $(A')' = A$,
(ii) $(A + B)' = A' + B'$,
(iii) $(A.B)' = B'.A'$.

Corollary 4.4. *Let A and B be two TFMs and k, l be two scalars then:*

(i) $(k.A)' = k.A'$,
(ii) $(k.A + l.B)' = k.A' + l.B'$.

Property 4.5. Let A be a square TFM then

(i) $A.A'$ and $A'.A$ are both symmetric,
(ii) $A + A'$ is symmetric,
(iii) $A - A'$ is fuzzy skew-symmetric.

5. Trace of TFM

Definition 5.1. Trace of TFM. The trace of a square TFM $A = (a_{ij})$,denoted by $tr(A)$, is the sum of the principal diagonal elements.In other words,$tr(A) = \sum_{i=1}^{n} a_{ii}$.

Property 5.2. Let $A = (a_{ij})$ and $B = (b_{ij})$ be any two square TFMs of order $n \times n$ then,

(i) $tr(A) + tr(B) = tr(A + B)$

(ii) $tr(A) = tr(A^{'})$

(iii) $tr(A.B) = tr(B.A)$.

Proof. (i) Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be two TFMs where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ and $b_{ij} = \langle n_{ij}, \gamma_{ij}, \delta_{ij} \rangle$. Now, $tr(A) = \sum_{i=1}^{n} a_{ii} = \langle M, T_L, T_U \rangle$, where $M = \sum_{i=1}^{n} m_{ii}, T_L = \sum_{i=1}^{n} \alpha_{ii}$ and $T_U = \sum_{i=1}^{n} \beta_{ii}$. Similarly, $tr(B) = \langle N, P_L, P_U \rangle$ where $N = \sum_{i=1}^{n} n_{ii}, P_L = \sum_{i=1}^{n} \gamma_{ii}$ and $P_U = \sum_{i=1}^{n} \delta_{ii}$. Therefore, $tr(A) + tr(B) = \langle M, T_L, T_U \rangle + \langle N, P_L, P_U \rangle = \langle M + N, T_L + P_L, T_U + P_U \rangle$.

Again, let $A + B = D = (d_{ij})$, where $d_{ij} = \langle m_{dij}, \alpha_{dij}, \beta_{dij} \rangle$,
$m_{dij} = m_{ij} + n_{ij}$, $\alpha_{dij} = \alpha_{ij} + \gamma_{ij}$ and $\beta_{dij} = \beta_{ij} + \delta_{ij}$.
Now, $tr(D) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} \langle m_{ij} + n_{ij}, \alpha_{ij} + \gamma_{ij}, \beta_{ij} + \delta_{ij} \rangle$
$\qquad = \langle M + N, T_L + P_L, T_U + P_U \rangle$.

Hence, $tr(A) + tr(B) = tr(A + B)$.

(ii) the proof is trivial.

(iii) Let $A = (a_{ij})$ and $B = (b_{ij})$ be two TFMs of order $n \times n$, where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ and $b_{ij} = \langle n_{ij}, \gamma_{ij}, \delta_{ij} \rangle$. Also let $C = A.B = (c_{ij})$, then $c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj}$. Now $tr(A.B) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}.b_{ki}$.

Again, let $B.A = D = (d_{ij})$ where $d_{ij} = \sum_{k=1}^{n} b_{ik}.a_{kj}$.

Now $tr(B.A) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}.a_{ki}$
$\qquad = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik}.b_{ki}$ (interchanging the indices i and k).
Hence, $tr(A.B) = tr(B.A)$. $\square$

Property 5.3. The product of two pure upper triangular TFMs of order $n \times n$ is a pure upper triangular TFM.

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two upper triangular TFMs where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ and $b_{ij} = \langle n_{ij}, \gamma_{ij}, \delta_{ij} \rangle$. Since A and B are pure upper triangular TFMs then $a_{ij} = \langle 0, 0, 0 \rangle$ and $b_{ij} = \langle 0, 0, 0 \rangle$ for all $i > j; i, j = 1, 2, \cdots, n$. Let $A.B = C = (c_{ij})$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj} = \sum_{k=1}^{n} \langle m_{ik}, \alpha_{ik}, \beta_{ik} \rangle . \langle n_{kj}, \gamma_{kj}, \delta_{kj} \rangle$. We shall now show that $c_{ij} = \langle 0, 0, 0 \rangle$ if $i > j; i, j = 1, 2, \cdots, n$.

For $i > j$ we have $a_{ik} = \langle 0, 0, 0 \rangle$ for $k = 1, 2, \cdots, i - 1$ and similarly $b_{kj} = \langle 0, 0, 0 \rangle$ for $k = i, i + 1, \cdots, n$.

Therefore, $c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj} = \sum_{k=1}^{i-1} a_{ik}.b_{kj} + \sum_{k=i}^{n} a_{ik}.b_{kj} = \langle 0, 0, 0 \rangle$.

Now, $c_{ii} = \sum_{k=1}^{n} a_{ik}.b_{ki} = \sum_{k=1}^{i-1} a_{ik}.b_{ki} + a_{ii}.b_{ii} + \sum_{k=i+1}^{n} a_{ik}.b_{ki}$
$\qquad = \langle 0, 0, 0 \rangle = a_{ii}.b_{ii}$,
since $a_{ik} = \langle 0, 0, 0 \rangle$ for $k = 1, 2, \cdots, i - 1$ and $b_{ki} = \langle 0, 0, 0 \rangle$ for $k = i + 1, i + 2, \cdots, n$. Hence the result follows. $\square$

Property 5.4. The product of two pure lower triangular TFMs of order $n \times n$ is also a pure lower triangular TFM.

6. Determinant of TFM

The triangular fuzzy determinant (TFD) of a TFM, minor and cofactor are defined as in classical matrices. But, TFD has some special properties due to the sub-distributive property of TFNs.

Definition 6.1. Determinant of TFM. The triangular fuzzy determinant of a TFM A of order $n \times n$ is denoted by $|A|$ or $det(A)$ and is defined as,

$$|A| = \sum_{\sigma \in S_n} Sgn \ \sigma \ \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle$$

$$(1) \qquad = \sum_{\sigma \in S_n} Sgn \ \sigma \ \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $a_{i\sigma(i)} = \langle m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)} \rangle$ are TFNs and S_n denotes the symmetric group of all permutations of the indices $\{1, 2, ..., n\}$ and $Sgn \ \sigma = 1$ or -1 according as the permutation $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$ is even or odd respectively.

The computation of $det(A)$ involves several product of TFNs. Since the product of two or more TFNs is an approximate TFN, the value of $det(A)$ is also an approximate TFN.

Definition 6.2. Minor. Let $A = (a_{ij})$ be a square TFM of order $n \times n$. The minor of an element a_{ij} in $det(A)$ is a determinant of order $(n-1) \times (n-1)$, which is obtained by deleting the ith row and the jth column from A and is denoted by M_{ij}.

Definition 6.3. Cofactor. Let $A = (a_{ij})$ be a square TFM of order $n \times n$. The cofactor of an element a_{ij} in A is denoted by A_{ij} and is defined as, $A_{ij} = (-1)^{i+j} M_{ij}$.

Definition 6.4. Adjoint. Let $A = (a_{ij})$ be a square TFM and $B = (A_{ij})$ be a square TFM whose elements are the cofactors of the corresponding elements in $|A|$ then the transpose of B is called the adjoint or adjugate of A and it is equal to (A_{ji}). The adjoint of A is denoted by $adj(A)$.

Here $|A|$ contains $n!$ terms out of which $\frac{n}{2}!$ are positive terms and the same number of terms are negative. All these $n!$ terms contain n quantities at a time in product form, subject to the condition that from the n quantities in the product exactly one is taken from each row and exactly one from each column.

Alternatively, a TFD of a TFM $A = (a_{ij})$ may be expanded in the form $\sum_{j=1}^{n} a_{ij} \cdot A_{ij}$, $i \in \{1, 2, ..., n\}$, where A_{ij} is the cofactor of a_{ij}. Thus the TFD is the sum of the products of the elements of any row (or column) and the cofactors of the corresponding elements of the same row (or column). We refer to this method as the alternative method.

In classical mathematics, the value of a determinant is computed by any one of the aforesaid two processes and both yield same result. But, due to the failure of distributive laws of triangular fuzzy numbers, the value of a TFD, computed by

the aforesaid two processes will differ from each other. For this reason the value of a TFD should be determined according to the definition, i.e., using the following rule only

$$| A | = \sum_{\sigma \in S_n} Sgn \ \sigma \ \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \rangle \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle.$$

On the other hand the value of a TFD computed by the alternative process yields a different and less desirable result.

Property 6.5. Let $A = (a_{ij})$ be a TFM of order $n \times n$.

> (i) If all the elements of a row (column) of A are $\langle 0,0,0 \rangle$,
>   then $| A |= \langle 0,0,0 \rangle$.
>
> (ii) If a row (column) be multiplied by a scalar k,
>   then $| A |$ is multiplied by k.
>
> (iii) If A is triangular TFM,
>   then $| A |= \prod_{i=1}^{n} \langle m_{ii}, \alpha_{ii}, \beta_{ii} \rangle.$

Proof. (i) Let $A = (a_{ij})$ be a square TFM of order $n \times n$ where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$. We define the determinants $E_1, E_2, \ldots, E_n$ as follows:
$E_i(A) = \sum_{j=1}^{n} a_{ij}.A_{ij} = \sum_{j=1}^{n} \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle.A_{ij}$,
where A_{ij} is the cofactor of a_{ij} in the determinant A. Obviously, $E_1(A) = E_2(A) = \cdots = E_n(A) =| A |$. Let all the elements of rth row, $1 \leq r \leq n$, be $\langle 0,0,0 \rangle$. Then $E_r(A) = \langle 0,0,0 \rangle$ since $a_{rj} = \langle 0,0,0 \rangle$ for all $j = 1,2,\cdots,n$.
    Therefore, $| A |= E_r(A) = \langle 0,0,0 \rangle$.

(ii) If $k = 0$, then the result is obviously true since $|A| = \langle 0,0,0 \rangle$ when A has a zero row.
    Let $B = (b_{ij})_{n \times n}$ where $b_{ij} = \langle n_{ij}, \gamma_{ij}, \delta_{ij} \rangle$ be obtained from an $n \times n$ TFM $A = (a_{ij})$ by multiplying its rth row by a scalar $k \neq 0$. Obviously $\langle n_{ij}, \gamma_{ij}, \delta_{ij} \rangle = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$, for all $i \neq 0$ and $\langle n_{rj}, \gamma_{rj}, \delta_{rj} \rangle = \langle km_{rj}, k\alpha_{rj}, k\beta_{rj} \rangle$, when k is positive and $\langle n_{rj}, \gamma_{rj}, \delta_{rj} \rangle = \langle km_{rj}, k\beta_{rj}, k\alpha_{rj} \rangle$, when k is negative. Then, by definition,

$$
\begin{aligned}
| B | \quad = \quad & \sum_{\sigma \in S_n} Sgn \ \sigma \ \langle n_{1\sigma(1)}, \gamma_{1\sigma(1)}, \delta_{1\sigma(1)} \rangle \langle n_{2\sigma(2)}, \gamma_{2\sigma(2)}, \delta_{2\sigma(2)} \rangle \\
& \cdots \langle n_{r\sigma(r)}, \gamma_{r\sigma(r)}, \delta_{r\sigma(r)} \rangle \cdots \langle n_{n\sigma(n)}, \gamma_{n\sigma(n)}, \delta_{n\sigma(n)} \rangle.
\end{aligned}
$$

When k is positive scalar,

$$
\begin{aligned}
| B | \quad = \quad & \sum_{\sigma \in S_n} Sgn \ \sigma \ \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} \rangle \\
& \cdots \langle km_{r\sigma(r)}, k\alpha_{r\sigma(r)}, k\beta_{r\sigma(r)} \rangle \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle \\
= \quad & k \sum_{\sigma \in S_n} Sgn \ \sigma \ \prod_{i=1}^{n} \langle m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)} \rangle \\
= \quad & k \ | A |.
\end{aligned}
$$

When k is negative scalar,

$$
\begin{aligned}
|B| &= \sum_{\sigma \in S_n} Sgn\ \sigma\ \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} \rangle \\
&\quad \cdots \langle km_{r\sigma(r)}, k\beta_{r\sigma(r)}, k\alpha_{r\sigma(r)} \rangle \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle \\
&= k \sum_{\sigma \in S_n} Sgn\ \sigma\ \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} \rangle \\
&\quad \cdots \langle m_{r\sigma(r)}, \alpha_{r\sigma(r)}, \beta_{r\sigma(r)} \rangle \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle \\
&= k \sum_{\sigma \in S_n} Sgn\ \sigma\ \prod_{i=1}^{n} \langle m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)} \rangle \\
&= k\,|A|.
\end{aligned}
$$

Hence the result follows.

(iii) Let $A = (a_{ij})_{n \times n}$ be a square triangular TFM for $i < j$, i.e., $a_{ij} = \langle 0,0,0 \rangle$ for $i < j$. Take a term t of $|A|$,
$t = \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} \rangle \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle.$

Let $\sigma(1) \neq 1$, i.e., $1 < \sigma(1)$ and so that, $m_{1\sigma(1)} = 0, \alpha_{1\sigma(1)} = 0$ and $\beta_{1\sigma(1)} = 0$. Consequently, $t = \langle 0,0,0 \rangle$. Now, let $\sigma(1) = 1$ but $\sigma(2) \neq 2$. Then it is obvious that $\sigma(2) > 2$ therefore, $m_{2\sigma(2)} = 0, \alpha_{2\sigma(2)} = 0$ and $\beta_{2\sigma(2)} = 0$ and hence $t = \langle 0,0,0 \rangle$. This means that each term of $|A|$ is $\langle 0,0,0 \rangle$ if $\sigma(1) \neq 1$ or $\sigma(2) \neq 2$. Briefly we can say that $m_{i\sigma(i)} = 0, \alpha_{i\sigma(i)} = 0$ and $\beta_{i\sigma(i)} = 0$ for $\sigma(i) \neq i$. Therefore,
$|A| = \langle m_{11}, \alpha_{11}, \beta_{11} \rangle \langle m_{22}, \alpha_{22}, \beta_{22} \rangle \cdots \langle m_{nn}, \alpha_{nn}, \beta_{nn} \rangle$
$= \prod_{i=1}^{n} \langle m_{ii}, \alpha_{ii}, \beta_{ii} \rangle.$ $\qquad \square$

Property 6.6. If any two rows (or columns) of a square TFM A are interchanged then determinant $|A|$ of A changes the sign of $|A|$.

Proof. Let $A = (a_{ij})$ be a TFM of order $n \times n$. If $B = (b_{ij})_{n \times n}$ is obtained from A by interchanging the rth and sth row $(r < s)$ of A, then it is clear that $b_{ij} = a_{ij}, i \neq r, i \neq s$ and $b_{rj} = a_{sj}, b_{sj} = a_{rj}$.

Now,

$$
\begin{aligned}
|B| &= \sum_{\sigma \in S_n} Sgn\ \sigma\ b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{r\sigma(r)} \cdots b_{s\sigma(s)} \cdots b_{n\sigma(n)} \\
&= \sum_{\sigma \in S_n} Sgn\ \sigma\ a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{s\sigma(r)} \cdots a_{r\sigma(s)} \cdots a_{n\sigma(n)} \\
&= \sum_{\sigma \in S_n} Sgn\ \sigma\ \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \langle m_{s\sigma(r)}, \alpha_{s\sigma(r)}, \beta_{s\sigma(r)} \rangle \\
&\quad \cdots \langle m_{r\sigma(s)}, \alpha_{r\sigma(s)}, \beta_{r\sigma(s)} \rangle \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle.
\end{aligned}
$$

Let $\lambda = \begin{pmatrix} 1 & 2 & \dots & r & \dots & s & \dots & n \\ 1 & 2 & \dots & s & \dots & r & \dots & n \end{pmatrix}.$

Then λ is a transposition interchanging r and s and $Sgn\ \lambda = -1$.
Let $\sigma\lambda = \phi$. As σ runs through all permutations on $\{1, 2, \cdots, n\}$, ϕ also runs over the same permutations, because $\sigma_1\lambda = \sigma_2\lambda$ or, $\sigma_1 = \sigma_2$.

Now,

$$\phi = \sigma\lambda$$

$$= \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(r) & \ldots & \sigma(s) & \ldots & \sigma(n) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & n \\ 1 & 2 & \ldots & s & \ldots & r & \ldots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(s) & \ldots & \sigma(r) & \ldots & \sigma(n) \end{pmatrix}.$$

Therefore $\phi(i) = \sigma(i), i \neq r, i \neq s$ and $\phi(r) = \sigma(s)$, $\phi(s) = \sigma(r)$.

Since λ is odd, ϕ is even or odd according as ϕ is odd or even. Therefore, $Sgn\,\phi = -Sgn\,\sigma$.

Then,

$$\begin{aligned} |B| &= \sum_{\sigma \in S_n} Sgn\,\sigma \, \langle m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} \rangle \langle m_{2\sigma(2)}, \alpha_{2\sigma(2)}, \beta_{2\sigma(2)} \rangle \\ &\quad \cdots \langle m_{s\sigma(r)}, \alpha_{s\sigma(r)}, \beta_{s\sigma(r)} \rangle \cdots \langle m_{r\sigma(s)}, \alpha_{r\sigma(s)}, \beta_{r\sigma(s)} \rangle \\ &\quad \cdots \langle m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} \rangle \\ &= -\sum_{\phi \in S_n} Sgn\,\phi \, \langle m_{1\phi(1)}, \alpha_{1\phi(1)}, \beta_{1\phi(1)} \rangle \langle m_{2\phi(2)}, \alpha_{2\phi(2)}, \beta_{2\phi(2)} \rangle \\ &\quad \cdots \langle m_{s\phi(r)}, \alpha_{s\phi(r)}, \beta_{s\phi(r)} \rangle \cdots \langle m_{r\phi(s)}, \alpha_{r\phi(s)}, \beta_{r\phi(s)} \rangle \\ &\quad \cdots \langle m_{n\phi(n)}, \alpha_{n\phi(n)}, \beta_{n\phi(n)} \rangle \\ &= -|A|. \end{aligned}$$

$\square$

Property 6.7. If A is a square TFM then $|A| = |A'|$.

Proof. Let $A = (a_{ij})_{n \times n}$ be a square TFM and $A' = B = (b_{ij})_{n \times n}$. Then,

$$\begin{aligned} |B| &= \sum_{\sigma \in S_n} Sgn\,\sigma \, b_{1\sigma(1)}, b_{2\sigma(2)}, \cdots b_{n\sigma(n)} \\ &= \sum_{\sigma \in S_n} Sgn\,\sigma \, a_{\sigma(1)1}, a_{\sigma(2)2}, \cdots a_{\sigma(n)n}. \end{aligned}$$

Let ϕ be the permutation of $\{1, 2, \ldots, n\}$ such that $\phi\sigma = I$, the identity permutation. Then $\phi = \sigma^{-1}$. Since σ runs over the whole set of permutations, ϕ also runs over the same set of permutations. Let $\sigma(i) = j$ then $i = \sigma^{-1}(j)$ and $a_{\sigma(i)i} = a_{j\phi(j)}$ for all i, j. Therefore,

$$\begin{aligned} |B| &= \sum_{\sigma \in S_n} Sgn\,\sigma \, \langle m_{\sigma(1)1}, \alpha_{\sigma(1)1}, \beta_{\sigma(1)1} \rangle \langle m_{\sigma(2)2}, \alpha_{\sigma(2)2}, \beta_{\sigma(2)2} \rangle \cdots \\ &\quad \langle m_{\sigma(n)n}, \alpha_{\sigma(n)n}, \beta_{\sigma(n)n} \rangle. \\ &= \sum_{\phi \in S_n} Sgn\,\phi \, \langle m_{1\phi(1)}, \alpha_{1\phi(1)}, \beta_{1\phi(1)} \rangle \langle m_{2\phi(2)}, \alpha_{2\phi(2)}, \beta_{2\phi(2)} \rangle \cdots \\ &\quad \langle m_{n\phi(n)}, \alpha_{n\phi(n)}, \beta_{n\phi(n)} \rangle \\ &= |A|. \end{aligned}$$

Hence, $\mid A \mid = \mid A' \mid$ □

Property 6.8. For a square TFM A of order $n \times n$,

(i) if A is symmetric then $adj(A)$ is symmetric,
(ii) if A is a fuzzy null TFM then $adj(A)$ is a fuzzy null TFM,
(iii) if A is a pure unit TFM then $adj(A)$ is a pure unit TFM,
(iv) if A is a fuzzy unit TFM then $adj(A)$ is a fuzzy unit TFM.

Property 6.9. For a square TFM A of order $n \times n$,

(i) if A contains a zero row then $adj(A).A$ is a fuzzy null TFM.
(ii) $adj(A') = (adj(A))'$.

Proof. (i) Let $A = (a_{ij})$ be a square TFM of order $n \times n$, where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$, and $B = adj(A)$. Then by the definition of adjoint matrix, the ijth element b_{ij} of B is $|A_{ji}|$, where A_{ij} is the sub-matrix obtained from A by suppressing the ith row and jth column. That is, $|A_{ij}|$ is the cofactor of a_{ij} in A. Without loss of generality we assume that the lth row of A be the zero row. Therefore, the elements of the lth row are of the form $a_{lj} = \langle 0, \alpha_{ij}, \beta_{ij} \rangle$, for all j. Then all the elements of $adj(A)$ are of the form $|A_{ij}| = \langle 0, \alpha_{ij}^*, \beta_{ij}^* \rangle$ except $j \neq l$.

Let $C = adj(A).A$. Then the ijth element c_{ij} of C is $c_{ij} = \sum_{k=1}^{n} |A_{ik}| a_{kj} = \sum_{k \neq l} |A_{ik}| a_{kj} + |A_{il}| a_{lj}$.

Now, all $|A_{ik}|, k \neq l$ are of the form $\langle 0, \alpha_{ij}^*, \beta_{ij}^* \rangle$ and $a_{lj} = \langle 0, \alpha_{ij}, \beta_{ij} \rangle$. Hence c_{ij} is of the form $\langle 0, \gamma_{ij}, \delta_{ij} \rangle$ for all $i, j = 1, 2, \ldots, n$. Thus, C, i.e., $adj(A).A$ is a fuzzy null TFM.

(ii) The proof is straight forward. □

7. Singular and Constant TFMs

Definition 7.1. Singular TFM. A square TFM A is said to be singular if $|A| = \langle 0, 0, 0 \rangle$.

Definition 7.2. Semi-singular TFM. A square TFM A is said to be semi-singular if the value of $|A|$ is of the form $\langle 0, \alpha, \beta \rangle$.

Definition 7.3. Constant TFM. An $n \times n$ TFM $A = (a_{ij})$ is said to be constant if all its rows are equal to each other, i.e., if $\langle m_{ik}, \alpha_{ik} \beta_{ik} \rangle = \langle m_{jk}, \alpha_{jk} \beta_{jk} \rangle$ for all i, j, k.

Property 7.4. If $A = (a_{ij})$ is a constant TFM of order $n \times n$ and B is a TFM of the same order, then $A.B$ is a constant TFM.

Proof. Suppose $A = (a_{ij})$ is a constant TFM of order $n \times n$ and $B = (b_{ij})$ is a TFM of the same order, where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ and $b_{ij} = \langle n_{ij}, \gamma_{ij}, \delta_{ij} \rangle$. Then $a_{ik} = \langle m_{ik}, \alpha_{ik}, \beta_{ik} \rangle$, are the same for all $i; i, k \in \{1, 2, \ldots, n\}$. Let $A.B = \sum_{k=1}^{n} a_{ik}.b_{kj} = D = (d_{ij})$. Then
$d_{ij} = \langle p_{ij}, \mu_{ij}, \nu_{ij} \rangle$, where $p_{ij} = \sum_{k=1}^{n} m_{ik} n_{kj}$, $\mu_{ij} = \sum_{k=1}^{n} \{m_{ik} \gamma_{kj} + n_{ik} \alpha_{kj}\}$ and $\nu_{ij} = \sum_{k=1}^{n} \{m_{ik} \delta_{kj} + n_{ik} \beta_{kj}\}$. A is a constant TFM, i.e., $m_{ik} = m_{jk}$ for all i, j, k.

This implies that $p_{11} = p_{21} = \cdots = p_{n1}, \;\; p_{12} = p_{22} = \cdots = p_{n2}, \;\; \cdots \;, p_{1n} = p_{2n} = \cdots = p_{nn}$, i.e., p_{ij} is independent of $i, \;\; i \in \{1, 2, \cdots, n\}$.

Again, $\mu_{11} = (m_{11}\gamma_{11} + n_{11}\alpha_{11}) + (m_{12}\gamma_{21} + n_{21}\alpha_{12}) + \cdots + (m_{1n}\gamma_{n1} + n_{n1}\alpha_{1n}) = \mu_{21} = \cdots = \mu_{n1}, \;\; \mu_{12} = \mu_{22} = \cdots = \mu_{n2}, \;\; \cdots \;, \mu_{1n} = \mu_{2n} = \cdots = \mu_{nn}$. Similarly, $\nu_{11} = \nu_{21} = \cdots = \nu_{n1}, \;\; \nu_{12} = \nu_{22} = \cdots = \nu_{n2}, \;\; \cdots \;, \nu_{1n} = \nu_{2n} = \cdots = \nu_{nn}$. Therefore, d_{ij} is independent of $i, \;\; i \in \{1, 2, \cdots, n\}$ since $\langle m_{ik}, \alpha_{ik}, \beta_{ik} \rangle = \langle m_{jk}, \alpha_{jk}, \beta_{jk} \rangle$ for all i, j, k.

Hence, $A.B$ is constant. $\hfill\square$

Property 7.5. If A is a square constant TFM then,

(i) $A.(adjA)$ is constant,

(ii) $(adjA)^{'}$ is constant,

(iii) $A.(adjA^{'})$ is constant,

(iv) $(A^{'}.(adjA))^{'}$ is constant,

(v) $(adjA)^{'}.A$ is constant,

(vi) $|A|$ is of the form $\langle 0, \alpha, \beta \rangle$.

8. Conclusion

In this article some elementary operations on triangular fuzzy numbers are defined. Like classical matrices we also define some operations on TFMs. Using the elementary operations, some important properties of TFMs are presented. The concept of adjoint of TFM is discussed and some properties on it are also presented. The definition and some properties of determinant of TFM are presented in this article. It is well known that the determinant is a very important tool in mathematics, so an efficient method is required to evaluate a TFD. Presently, we are trying to develop an efficient method to evaluate a TFD of large size. Some special types of TFMs, i.e. pure and fuzzy triangular, symmetric, pure and fuzzy skew-symmetric, singular, semi-singular and constant TFMs are defined here. Some properties of these TFMs are also presented.

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