

## OPTIMIZATION OF LINEAR OBJECTIVE FUNCTION SUBJECT TO FUZZY RELATION INEQUALITIES CONSTRAINTS WITH MAX-AVERAGE COMPOSITION

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**ABSTRACT.** In this paper, the finitely many constraints of a fuzzy relation inequalities problem are studied and the linear objective function on the region defined by a fuzzy max-average operator is optimized. A new simplification technique which accelerates the resolution of the problem by removing the components having no effect on the solution process is given together with an algorithm and a numerical example to illustrate the steps of the problem resolution process.

### 1. Introduction

Fuzzy set theory was first introduced in 1965 by Zadeh [30]. The operations proposed by him are specified as the membership function of intersection and union of two fuzzy sets, and that of complement of the normalized fuzzy set. Since the resolution of fuzzy relation equations was proposed by Sanchez [25], fuzzy relation equations (FRE), fuzzy relation inequalities (FRI) and the problems related to them have been studied by many researchers (for instance, see Han et al [14], Hu [16], Loetamonphong and Fang [19, 20], Pedrycz [23], Wang [27, 28], Zhang et al [31] and Guo, Xia [11], and also [3, 6, 9, 10, 12, 15, 18, 22, 24, 29]). The max-min composition is commonly used when a system requires conservative solutions in the sense that the goodness of one value can not compensate for the badness of another [20]. Recent results in the literature, however, show that the min operator is not always the best choice for the intersection operation, but, in fact, the max-product composition provides results better or equivalent to the max-min composition in some applications [27].

The fundamental result for fuzzy relation equations with max-product composition goes back to Pedrycz [23] and recent study in this regard can be found in Bourk and Fisher [2], who extended the study of an inverse solution of a system of fuzzy relation equations with max-product composition and provided theoretical results for determining the complete solution sets as well as the conditions for the existence of resolutions. Their results show that such

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complete solution sets can be characterized by one maximum solution and a number of minimal solutions. The monograph by Di Nola, Sessa, Pedrycz and Sanchez [7] contains a thorough discussion of this class of equations. The optimization problem with max-product and max-min compositions was first considered by Loetamonphong and Fang [19] and Fang and Li [8].

The optimization problem of max-min and max-product can be separated into two sub problems by separating the nonnegative and negative coefficients in the objective function. The sub- problem formed by the negative coefficients can be solved easily by obtaining the maximum solution of the feasible solutions set. On the other hand, the sub-problem formed by the nonnegative coefficients can be converted into a 0-1 integer programming problem. The associated 0-1 integer programming problem was solved by Fang and Li [8] using the branch and bound technique. Lu and Fang [21] proposed a genetic algorithm to solve nonlinear optimization problem with max-min composition and Guu and Wu [13] improved Fang and Li's method by providing an upper bound for the branch-and-bound procedure. For the optimization problem with max-product composition, Loetamonphong and Fang solved the corresponding 0-1 integer problem by reducing its size and by employing the branch and bound method. On other hand, Loetamonphong, Fang, and Young [20] extended the study of the problem of max-min composition to the solution with multi-objective functions. The application of (FRE) and (FRI) can be seen in many areas, for instance, fuzzy control, fuzzy decision making, fuzzy symptom, diagnosis, and especially fuzzy medical diagnosis [1,4,26].

In this paper, we generalize the linear optimization problem of the FRE with max-average operator [17] by considering the FRI constraints. This problem can be formulated as follows:

$$\begin{aligned}
 & \min c^t x \\
 \text{s.t. } & A \bullet x \geq d^1 \\
 & B \bullet x \leq d^2 \\
 & x \in [0,1]^n
 \end{aligned} \tag{1}$$

Where  $A = (a_{ij})_{m \times n}$ ,  $a_{ij} \in [0,1]$ ,  $B = (b_{ij})_{l \times n}$ ,  $b_{ij} \in [0,1]$ , are fuzzy matrices,  $d^1 = (d_i^1)_{m \times 1} \in [0,1]^m$ ,  $d^2 = (d_i^2)_{l \times 1} \in [0,1]^l$ ,  $x = (x_j)_{n \times 1} \in [0,1]^n$  are fuzzy vectors and,  $c = (c_j)_{n \times 1} \in R^n$ , and “ $\bullet$ ” denotes the fuzzy max-average operator. Problem (1) can be rewritten as follows:

$$\begin{aligned}
 & \min c^t x \\
 \text{s.t. } & a_i \bullet x \geq d_i^1, \quad i \in I^1 = \{1, 2, \dots, m\} \\
 & b_i \bullet x \leq d_i^2 \quad i \in I^2 = \{m+1, m+2, \dots, m+1\} \\
 & 0 \leq x_j \leq 1, \quad j \in J = \{1, 2, \dots, n\}
 \end{aligned} \tag{2}$$

Where  $a_i$  and  $b_i$  are  $i$ 'th row of the matrices  $A$  and  $B$ , respectively and the constraints are expressed by the max-average operator definition as;

$$\begin{aligned}
 a_i \bullet x &= \max_{j \in J} \left\{ \frac{a_{ij} + x_j}{2} \right\} \geq d_i^1 \quad \forall i \in I^1 \\
 b_i \bullet x &= \max_{j \in J} \left\{ \frac{b_{ij} + x_j}{2} \right\} \leq d_i^2 \quad \forall i \in I^2
 \end{aligned} \tag{3}$$

In section 2, the feasible solutions set of the problem (2) and its properties are studied. Moreover, necessary and sufficient conditions are given for the feasibility of the problem (2). In section 3, some simplification operations are presented to accelerate the resolution process. Finally, in section 4 an algorithm is introduced to solve the problem and a numerical example is given to illustrate the algorithm. Section 5 provides a conclusion.

## 2. The Characteristics of the Feasible Solution Set

**Definition 2.1.** Let

$$\begin{aligned}
 {}^i S(A, d^1) &= \{x \in [0, 1]^n : a_i \bullet x \geq d_i^1\} \text{ for each } i \in I^1, \\
 {}^i S(B, d^2) &= \{x \in [0, 1]^n : b_i \bullet x \leq d_i^2\} \text{ for each } i \in I^2, \\
 S(A, d^1) &= \bigcap_{i \in I^1} {}^i S(A, d^1) = \{x \in [0, 1]^n : A \bullet x \geq d^1\}, \\
 S(B, d^2) &= \bigcap_{i \in I^2} {}^i S(B, d^2) = \{x \in [0, 1]^n : B \bullet x \leq d^2\} \text{ and} \\
 S(A, B, d^1, d^2) &= S(A, d^1) \cap S(B, d^2) = \{x \in [0, 1]^n : A \bullet x \geq d^1, B \bullet x \leq d^2\}.
 \end{aligned}$$

**Corollary 2.2.** According to (3), we have  $x \in {}^i S(A, d^1)$  for each  $i \in I^1$  iff there exists  $j \in J$  such that  $x_j \geq 2d_i^1 - a_{ij}$ . Similarly,  $x \in {}^i S(B, d^2)$  for each  $i \in I^2$  iff  $x_j \leq 2d_i^2 - b_{ij}$ ,  $\forall j \in J$ .

**Lemma 2.3.**

(a)  $S(A, d^1) \neq \emptyset$  iff for each  $i \in I^1$  there exists  $j \in J$  such that  $2d_i^1 - a_{ij} \leq 1$ .

(b) If  $S(A, d^1) \neq \emptyset$  then  $\bar{1} = [1, 1, \dots, 1]_{1 \times n}^t$  is the single maximum solution of  $S(A, d^1)$ .

*Proof.*

(a) Suppose  $x \in S(A, d^1)$ . Hence by Definition 2.1,  $\forall i \in I^1$ ,  $x \in {}^i S(A, d^1)$  and thus, by Corollary 2.2, for each  $i \in I^1$ , and some  $j_i \in J$ , we have  $x_{j_i} \geq 2d_i^1 - a_{ij_i}$ . Moreover, since  $x \in [0, 1]^n$ ,  $x \in S(A, d^1)$ , we have  $2d_i^1 - a_{ij} \leq 1 \forall i \in I^1$ . Conversely, suppose there exists  $j_i \in J$  such that  $2d_i^1 - a_{ij} \leq 1, \forall i \in I^1$ .

Set  $x = \bar{1} = [1, 1, \dots, 1]_{1 \times n}^t$ . Since  $x \in [0, 1]^n$  and  $x_{j_i} = 1 \geq 2d_i^1 - a_{ij_i}$ ,  $\forall i \in I^1$ , hence, by Corollary 2.2,  $x \in {}^i S(A, d^1)$ ,  $\forall i \in I^1$ , and as a result,  $x \in S(A, d^1)$ .

(b) Proof of this part follows easily from part (a) and Corollary 2.2.

**Lemma 2.4.**

(a)  $S(B, d^2) \neq \emptyset$  iff  $2d_i^2 - b_{ij} \geq 0$ ,  $\forall i \in I^2$  and  $\forall j \in J$ .

(b) If  $S(B, d^2) \neq \emptyset$  then  $\bar{0} = [0, 0, \dots, 0]_{1 \times n}^t$  is the single minimum solution of  $S(B, d^2)$ .

*Proof.* It is similar to the proof of the Lemma 2.3 by using the Corollary 2.2.

**Theorem 2.5.** (necessary condition) if  $S(A, B, d^1, d^2) \neq \emptyset$  then

(a)  $\forall i \in I^1 \exists j \in J$  Such that  $2d_i^1 - a_{ij} \leq 1$ .

(b)  $2d_i^2 - b_{ij} \geq 0$ ,  $\forall i \in I^2$  and  $\forall j \in J$ .

*Proof.* This theorem follows from Lemmas 2.3, 2.4 and Definition 2.1.

**Definition 2.6.** Set  $\bar{x} = (\bar{x}_j)_{n \times 1}$ , where  $\bar{x}_j = \max\{\min\{1, \min_{1 \leq i \leq l} \{2d_i^2 - b_{ij}\}\}, 0\}$ .

**Lemma 2.7.** If  $S(B, d^2) \neq \emptyset$  then  $\bar{x}$  is the single maximum solution of  $S(B, d^2)$ .

*Proof.* See the proof of the Theorem 2 in [17].

**Corollary 2.8.** If  $S(B, d^2) \neq \emptyset$  then  $S(B, d^2) = [\bar{0}, \bar{x}]$ .

*Proof.* We recall that, by Lemmas 2.3 and 2.4,  $\bar{0}$  and  $\bar{x}$  are, respectively, the single minimum and the maximum solutions. Now, let  $x \in [\bar{0}, \bar{x}]$ . Then  $x \in [0, 1]^n$  and  $x \leq \bar{x}$ . Thus,  $b_i \bullet x \leq b_i \bullet \bar{x} \leq d_i^2$ ,  $\forall i \in I^2$  that implies  $x \in S(B, d^2)$ . Conversely, let  $x \in S(B, d^2)$ . By Lemma 2.4 and Definition 2.1,  $\bar{0} \leq x$  and  $x \in {}^i S(B, d^2)$ ,  $\forall i \in I^2$ . Next, by Corollary 2.2 we have  $x_j \leq 2d_i^2 - b_{ij}$ ,  $\forall i \in I^2$  and  $\forall j \in J$ . Hence,  $x_j \leq \bar{x}_j$ ,  $\forall j \in J$  i.e.  $x \leq \bar{x}$ . It follows that  $x \in [\bar{0}, \bar{x}]$ .

**Definition 2.9.** Let  $J_i = \{j \in J : 2d_i^1 - a_{ij} \leq 1\}$ ,  $\forall i \in I^1$ . For each  $j \in J_i$ , we define  $i_{x(j)} = (i_{x(j)_k})_{n \times 1}$  such that

$$i_{x(j)k} = \begin{cases} \max\{2d_i^1 - a_{ij}, 0\} & k = j \\ 0 & k \neq j \end{cases}$$

**Lemma 2.10.** Assume  $i \in I^1$  is a fixed number.

(a) For each  $j \in J_i$ , the vectors  $i_{x(j)}$  are the minimal solutions of  ${}^i S(A, d^1)$ .

(b) If there exists  $j \in J_i$  such that  $2d_i^1 - a_{ij} \leq 0$  then  $\bar{0}$  is the single minimum solution of  ${}^i S(A, d^1)$ .

*Proof.*

(a) Suppose  $j \in J_i$  and  $i \in I^1$ , since  $i_{x(j)} = \max\{2d_i^1 - a_{ij}, 0\}$  then, by Corollary 2.2,  $i_{x(j)} \in {}^i S(A, d^1)$ ,  $\forall i \in I^1$ .

Now suppose that there exists  $x \in {}^i S(A, d^1)$  such that  $x < i_{x(j)}$ . Then  $x_j < 2d_i^1 - a_{ij}$  and, for  $k \in J - \{j\}$ ,  $x_k = 0$ . Hence  $x_j < 2d_i^1 - a_{ij}$ ,  $\forall j \in J$  and so, by Corollary 2.2,  $x \notin {}^i S(A, d^1)$ , which is a contradiction.

(b) The proof of this part is similar to (a).

**Corollary 2.11.** If  ${}^i S(A, d^1) \neq \emptyset$  then  ${}^i S(A, d^1) = \bigcup_{j \in J_i} [i_{x(j)}, \bar{1}]$ , where  $i \in I^1$ .

*Proof.* We recall that  ${}^i S(A, d^1) \neq \emptyset$  means that the vector  $\bar{1}$  is the maximum solution and the vectors  $i_{x(j)}$ ,  $\forall j \in J_i$  are the minimal solutions in  ${}^i S(A, d^1)$  as a result of Lemmas 2.3, 2.4, 2.7 and 2.10, respectively. Now, let  $x \in \bigcup_{j \in J_i} [i_{x(j)}, \bar{1}]$ .

Hence, for some  $j \in J_i$ ,  $x \in [i_{x(j)}, \bar{1}]$ .

Also,  $x \in [0, 1]^n$  and, by Definition 2.9,  $x_j \geq i_{x(j)} = 2d_i^1 - a_{ij}$ . Hence, by Corollary 2.2,  $x \in {}^i S(A, d^1)$ .

Conversely, let  $x \in {}^i S(A, d^1)$ . Then, by Corollary 2.2, there exists  $j' \in J$  such that  $x_{j'} \geq 2d_i^1 - a_{ij'}$ . Since  $x \in [0, 1]^n$ , so  $2d_i^1 - a_{ij'} \leq 1$  and hence  $j' \in J_i$ . Therefore  $i_{x(j')} \leq x \leq \bar{1}$ , which implies that  $x \in \bigcup_{j \in J_i} [i_{x(j)}, \bar{1}]$ .

**Definition 2.12.** Suppose that  $e = (e(1), e(2), \dots, e(m)) \in J_1 \times J_2 \times \dots \times J_m$  such that  $e(i) = j \in J_i$ .  $x(e)_j = \max\{\max_{i \in I_j^e} \{2d_i^1 - a_{ij}\}, 0\}$  if  $I_j^e \neq \emptyset$  and  $x(e)_j = 0$  if  $I_j^e = \emptyset$ , where  $I_j^e = \{i \in I^1 : e(i) = j\}$ .

**Corollary 2.13.**

- (a) If  $2d_i^1 - a_{ij} \leq 0$  for some  $j \in J_i$ , then we can remove the  $i$  th row of the matrix  $A$ .
- (b) If  $j \notin J_i, \forall i \in I^1$  then we can omit the  $j$  th column of the matrix  $A$  for the purpose of finding  $x(e)$ .

We recall that in part (a), by Definition 2.12 and part (b) of Lemma 2.10, the  $i$  th row of the matrix  $A$  has no effect in the calculation of the vectors  $x(e)$  for any  $e \in J_I = J_1 \times J_2 \times \dots \times J_m$ . Also, in part (b), before calculating the vectors  $x(e), \forall e \in I_J$ , we can remove the  $j$  th column of the matrix  $A$  using Definition 2.12.

**Lemma 2.14.** Let  $S(A, d^1) \neq \emptyset$ , then  $S(A, d^1) = \bigcup_{X(e)} [x(e), \bar{1}]$ , where  $X(e) = \{x(e) : e \in I_J\}$ .

*Proof.* If  $S(A, d^1) \neq \emptyset$  then,  ${}^i S(A, d^1) \neq \emptyset, \forall i \in I^1$ . Hence, by Corollary 2.11 and Definitions 2.1, 2.6, 2.9 and 2.12, we have

$$S(A, d^1) = \bigcap_{i \in I^1} {}^i S(A, d^1) = \bigcap_{i \in I^1} \left[ \bigcup_{j \in J_i} [i_{x(j)}, \bar{1}] \right] = \bigcap_{i \in I^1} \left[ \bigcup_{e(i) \in J_i} [i_{x(e(i))}, \bar{1}] \right] = \bigcup_{e \in J_I} \left[ \bigcap_{i \in I^1} [i_{x(e(i))}, \bar{1}] \right] = \bigcup_{e \in J_I} [x(e), \bar{1}] = \bigcup_{X(e)} [x(e), \bar{1}]$$

From Lemma 2.14 it is obvious that

$$S(A, d^1) = \bigcup_{X_0(e)} [x(e), \bar{1}] \text{ and } X_0(e) = S_0(A, d^1),$$

where  $X_0(e)$  and  $S_0(A, d^1)$  are the set of minimal solutions of  $X(e)$  and  $S(A, d^1)$ , respectively.

**Theorem 2.15.** If  $S(A, B, d^1, d^2) \neq \emptyset$  then  $S(A, B, d^1, d^2) = \bigcup_{X_0(e)} [x(e), \bar{x}]$ .

*Proof.* Similar to the proof of the Lemma 2.14.

**Corollary 2.16.** (Necessary and Sufficient Conditions) Assume that  $S(B, d^2) \neq \emptyset$ . Then  $S(A, B, d^1, d^2) \neq \emptyset$  iff  $\bar{x} \in S(A, d^1)$  or, equivalently,  $S(A, B, d^1, d^2) \neq \emptyset$  iff there exists  $e \in J_I$  such that  $x(e) \leq \bar{x}$ .

*Proof.* the proof follows easily from Definition 2.1, Theorem 2.15 and Lemma 2.7.

### 3. Simplification Operations and the Resolution Algorithm

In order to solve the problem (2), it is initially converted into the two following sub-problems

$$\begin{array}{ll}
 \min c^{+t} x & \min c^{-t} x \\
 \text{s.t. } A \bullet x \geq d^1 & \text{s.t. } A \bullet x \geq d^1 \\
 B \bullet x \leq d^2 & B \bullet x \leq d^2 \\
 x \in [0,1]^n & x \in [0,1]^n
 \end{array} \quad (4a) \qquad (4b)$$

Where,  $c_j^+ = \max(0, c_j)$  and  $c_j^- = \min(0, c_j)$ .

It is clear that  $\bar{x}$  is an optimal solution of (4b) and (4a) achieves its optimal points at some  $x(e) \in X_0(e)$ . Once  $x(e_0)$  optimizes (4a), we set  $x^* = (x_j^*)_{n \times 1}$  such that

$$x_j^* = \begin{cases} \bar{x}_j & , c_j \leq 0 \\ x(e_0)_j & , c_j > 0 \end{cases}$$

Now the following lemma gives us an optimal point of the problem (2).

**Lemma 3.1.**  $x^*$  is an optimal solution of the problem (2).

*Proof.* See Theorem 2.1 in [11].

In order to calculate  $x^*$ , it is enough to find  $\bar{x}$  and  $x(e_0)$ . Although  $\bar{x}$  is easily obtained by Definition 2.6,  $x(e_0)$  is not so, because,  $X_0(e)$  is obtained by the pairwise comparison of  $X(e)$  members. Hence, having a complete set of  $X_0(e)$  is time-consuming, especially when  $X(e)$  has several members. Simplification operations can hasten the resolution of the problem (4a). With the intention of simplification the vectors  $e \in J_i$  are removed when  $x(e)$  is not optimal for (4a). One such operation is given in Corollary 2.13. Other operations are obtained by the following theorems.

**Definition 3.2.** Let  $\bar{J}_i = \{j \in J_i : 2d_i^1 - a_{ij} \leq \bar{x}_j\}$ ,  $\forall i \in I^1$  where  $\bar{x}$  is as in Definition 2.6.

**Theorem 3.3.** Let  $S(B, d^2) \neq \emptyset$ . Then  $S(A, B, d^1, d^2) \neq \emptyset$  iff  $\bar{J}_i \neq \emptyset, \forall i \in I^1$ .

*Proof.* Suppose  $S(A, B, d^1, d^2) \neq \emptyset$ . Therefore, by Theorem 2.15,  $\bar{x} \in S(A, B, d^1, d^2)$  and so we have  $\bar{x} \in S(A, d^1)$ ,  $\forall i \in I^1$ . Thus, by Corollary 2.2, for each  $i \in I^1$ , there exists  $j \in J$  such that  $\bar{x}_j \geq 2d_i^1 - a_{ij}$ . Consequently,  $\bar{J}_i \neq \emptyset, \forall i \in I^1$ .

Conversely, suppose that  $\forall i \in I^1, \bar{J}_i \neq \emptyset$ . Then  $\forall i \in I^1$  there exists  $j \in J$  such that  $\bar{x}_j \geq 2d_i^1 - a_{ij}$  and, by Corollary 2.2, it follows that  $\bar{x} \in S(A, d^1), \forall i \in I^1$ . As a result,  $\bar{x} \in S(A, d^1)$ . This fact, together with Lemma 2.7, implies  $\bar{x} \in S(A, B, d^1, d^2)$  and therefore,  $S(A, B, d^1, d^2) \neq \emptyset$ .

**Theorem 3.4.** *Let  $S(A, B, d^1, d^2) \neq \emptyset$ , then  $S(A, B, d^1, d^2) = \bigcup_{\bar{X}(e)} [x(e), \bar{x}]$ , where  $\bar{X}(e) = \{x(e) : e \in \bar{J}_I = \bar{J}_1 \times \bar{J}_2 \times \dots \times \bar{J}_m\}$ .*

*Proof.* By Theorem 2.15, it is sufficient to show that  $x(e) \notin S(A, B, d^1, d^2)$  when  $e \notin \bar{J}_I$ . Suppose  $e \notin \bar{J}_I$ . Then there exist  $i' \in I^1$  and  $j' \in J_{i'}$  such that  $e(i') = j'$  and  $2d_{i'}^1 - a_{i'j'} > \bar{x}_{j'}$ . Hence  $i' \in I_{j'}^c$  and, by Definition 2.12,  $x(e)_{j'} = \max_{i \in I_{j'}^c} \{\max\{2d_i^1 - a_{ij'}\}, 0\} \geq 2d_{i'}^1 - a_{i'j'} > \bar{x}_{j'}$ . Therefore the inequality  $x(e) \leq \bar{x}$  does not hold and, as a consequence of Theorem 2.15, we obtain  $x(e) \notin S(A, B, d^1, d^2)$ .

We remark that as a result of Definition 3.2,  $\bar{J}_i \subseteq J_i, \forall i \in I^1$ . In other words,  $\bar{X}(e) \subseteq X(e)$ . Also, by Theorem 3.4,  $S_0(A, B, d^1, d^2) \subseteq \bar{X}(e)$ . Thus, the region of search can be reduced to the smaller set  $S_0(A, B, d^1, d^2)$ .

**Definition 3.5.** Let  $J_i^* = \{j \in \bar{J}_i : c_j^- \neq 0\}, \forall i \in I^1$ .

**Theorem 3.6.** *Suppose  $x(e_0)$  is an optimal solution of (4a) and  $J_{i'}^* \neq \emptyset$  for some  $i' \in I^1$ , then there exists  $x(e')$  such that  $e'(i') \in J_{i'}^*$  and  $x(e')$  is the optimal solution of (4a).*

*Proof.* Suppose  $J_{i'}^* \neq \emptyset$  for some  $i' \in I^1$  and  $e_0(i') = j'$ . Define  $e' \in \bar{J}_I$  such that  $e'(i') = k \in J_{i'}^*$  and  $e'(i) = e_0(i)$  for each  $i \in I^1$  and  $i \neq i'$ . By means of Definition 2.12, we have

$x(e_0)_{j'} = \max_{i \in I_{j'}^{e_0}} \{\max\{2d_i^1 - a_{ij'}\}, 0\} \geq \max_{\substack{i \in I_{j'}^{e_0} \\ i \neq i'}} \{\max\{2d_i^1 - a_{ij'}\}, 0\} = x(e')_{j'}$  and  $x(e_0)_j = x(e')_j$  for each  $j \in J$  and  $j \neq j', k$ . Therefore, since  $c_k^+ = 0$ ,

$$c^{+t} x(e_0) = c_{j'}^+ x(e_0)_{j'} + \sum_{\substack{j \in J \\ j \neq j'}} c_j^+ x(e_0)_j \geq c_{j'}^+ x(e')_{j'} + \sum_{\substack{j \in J \\ j \neq j'}} c_j^+ x(e')_j = c^{+t} x(e')$$

Hence  $x(e')$  is an optimal solution of (4a) and the proof is complete.



**Corollary 3.7.** *If  $J_i^* \neq \emptyset$  for some  $i \in I^1$  then by omitting the  $i$  th row we reach a reduced problem for which each optimal solution is an optimal solution for the previous (main) problem.*

*Proof.* Results from Theorem 3.6 and the fact that  $c_j^+ = 0$  for each  $j \in J_i^*$ .

**Definition 3.8.** Let  $j_1, j_2 \in J$ ,  $c_{j_1} > 0$  and  $c_{j_2} > 0$ . We say that  $j_2$  dominates  $j_1$  iff

- (a)  $j_1 \in \bar{J}_i$  implies  $j_2 \in \bar{J}_i, \forall i \in I^1$ .
- (b) For each  $i \in I^1$  such that  $j_1 \in \bar{J}_i$ ,  $c_{j_1}(2d_i^1 - a_{ij_1}) \geq c_{j_2}(2d_i^1 - a_{ij_2})$ .

**Theorem 3.9.** *Suppose  $x(e_0)$  is optimal in (4a) and, for  $j_1, j_2 \in J$ ,  $j_2$  dominates  $j_1$ . Then there exists  $x(e')$  such that  $I_{j_1}^{e'} = \emptyset$  and  $x(e')$  is an optimal solution of (4a).*

*Proof.* Define  $e' = (e'(i))_{m \times 1}$  such that

$$e'(i) = \begin{cases} e_0(i) & i \notin I_{j_1}^{e_0} \\ j_2 & i \in I_{j_1}^{e_0} \end{cases}$$

It is obvious that  $I_{j_1}^{e'} = \emptyset$  and so  $x(e')_{j_1} = 0$ . Also,  $x(e_0)_j = x(e')_j$  for each  $j \in J$  and  $j \neq j_1, j_2$ ,  $x(e')_{j_2} = 2d_{i_0}^1 - a_{i_0 j_2}$ .

If  $i_0 \notin I_{j_1}^{e_0}$  then:

$$x(e_0)_{j_2} = x(e')_{j_2} = 2d_{i_0}^1 - a_{i_0 j_2},$$

So we have:

$$c^{+t} x(e_0) = c_{j_1}^+ x(e_0)_{j_1} + \sum_{\substack{j \in J \\ j \neq j_1}} c_j^+ x(e_0)_j \geq \sum_{\substack{j \in J \\ j \neq j_1}} c_j^+ x(e')_j = c^{+t} x(e')$$

and the result follows.

Now suppose  $i_0 \in I_{j_1}^{e_0}$ . We show that  $c^{+t} x(e_0) \geq c^{+t} x(e')$ .

Let  $x(e_0)_{j_2} = 2d_{i_0}^1 - a_{i_0 j_2}$ . Then, by part (a) of the Corollary 2.13 and Definition 3.8,  $c_{j_2}^+ x(e_0)_{j_2} \geq 0$  and since

$$c^{+t} x(e_0) = c_{j_1}^+ x(e_0)_{j_1} + c_{j_2}^+ x(e_0)_{j_2} + \sum_{j \neq j_1, j_2} c_j^+ x(e_0)_j$$

and

$$c^{+t} x(e') = c_{j_2}^+ x(e')_{j_2} + \sum_{j \neq j_1, j_2} c_j^+ x(e')_j$$

it is sufficient to show that  $c_{j_1}^+ x(e_0)_{j_1} \geq c_{j_2}^+ x(e')_{j_2}$ . By Definition 2.12, set  $x(e_0)_{j_1} = 2d_{i'}^1 - a_{i'j_1}$ .

Since  $j_2$  dominates  $j_1$ , we have  $c_{j_1}^+ (2d_{i'}^1 - a_{i'j_1}) \geq c_{j_2}^+ (2d_{i_0}^1 - a_{i_0j_2})$ . i.e.  $c_{j_1}^+ x(e_0)_{j_1} \geq c_{j_2}^+ x(e')_{j_2}$  once  $i_0 = i'$ . Otherwise, suppose  $i_0 \neq i'$ . Since  $i_0 \in I_{j_1}^{e_0}$  and  $j_2$  dominates  $j_1$ , hence

$$c_{j_1}^+ (2d_{i_0}^1 - a_{i_0j_1}) \geq c_{j_2}^+ (2d_{i_0}^1 - a_{i_0j_2})$$

Also, by Definition 2.12 we have  $x(e_0)_{j_1} = \max_{i \in I_{j_1}^{e_0}} \{\max\{2d_i^1 - a_{ij_1}, 0\}\} = 2d_{i'}^1 - a_{i'j_1}$  which implies that

$$2d_{i'}^1 - a_{i'j_1} \geq 2d_{i_0}^1 - a_{i_0j_1}, \quad \forall i \in I_{j_1}^{e_0}$$

Therefore

$$c_{j_1}^+ (2d_{i'}^1 - a_{i'j_1}) \geq c_{j_1}^+ (2d_{i_0}^1 - a_{i_0j_1}) \geq c_{j_2}^+ (2d_{i_0}^1 - a_{i_0j_2})$$

It follows that  $c_{j_1}^+ x(e_0)_{j_1} \geq c_{j_2}^+ x(e')_{j_2}$ . Hence  $c^{+t} x(e_0) \geq c^{+t} x(e')$  and the proof is complete.

**Corollary 3.10.** *If  $j_2$  dominates  $j_1$  for each  $j_1, j_2 \in J$ , then, by omitting the  $j_1$ th column we reach a reduced problem for which each optimal solution is an optimal solution for the previous (main) problem.*

#### 4. An Algorithm for Finding an Optimal Solution and Example

**Definition 4.1.** Consider problem (1). We call  $\bar{A} = (\bar{a}_{ij})_{m \times n}$  the characteristic matrix of the matrix  $A$ , where  $\bar{a}_{ij} = 2d_i^1 - a_{ij}$  for each  $i \in I^1$  and  $j \in J$ . The characteristic matrix  $\bar{B} = (\bar{b}_{ij})_{l \times n}$  is similarly defined.

**Algorithm:** Given problem (2),

- 1- Find the matrices  $\bar{A}$  and  $\bar{B}$  by Definition 4.1.
- 2- Calculate  $\bar{x}$  from  $\bar{B}$  by Definition 2.6.
- 3- If there exists  $i \in I^1$  such that  $\bar{a}_{ij} > \bar{x}_j$ ,  $\forall j \in J$  then stop. Problem 2 is infeasible (see Theorem 2.5).
- 4- If there exists  $i \in I^2$  and  $j \in J$  such that  $\bar{b}_{ij} < 0$  then stop. Problem 2 is infeasible (see Theorem 2.5).

- 5- If there exists  $i \in I^1$  and  $j \in J$  such that  $\bar{a}_{ij} \leq 0$  then remove the  $i$ th row of the matrix  $\bar{A}$  (see the part (a) of the Corollary 2.13).
- 6- If there exists  $j' \in J$  such that  $\bar{a}_{ij'} > \bar{x}_{j'}$ ,  $\forall i \in I^1$  then remove the  $j'$ th column of the matrix  $\bar{A}$  (see Theorem 3.4) and set  $x(e_0)_{j'} = 0$  if  $c_{j'} > 0$ .
- 7- For each  $i \in I^1$ , if  $J_i^* \neq \emptyset$  then remove the  $i$ th row of the matrix  $\bar{A}$  (see Corollary 3.7).
- 8- If  $c_j < 0$ , remove column  $j \in J$  from  $\bar{A}$ .
- 9- If  $j_2$  dominates  $j_1$  then remove the column  $j_1$  from  $\bar{A}$ ,  $\forall j_1, j_2 \in J$  (see Corollary 3.10) and set  $x(e_0)_{j_1} = 0$ .
- 10- Let  $J_i^{new} = \{j \in \bar{J}_i : \bar{a}_{ij} \leq \bar{x}_j\}$  and  $J_I^{new} = J_1^{new} \times J_2^{new} \times \dots \times J_m^{new}$ . First obtain the vectors  $x(e)$ ,  $\forall e \in J_I^{new}$  of  $\bar{A}$  using Definition 2.12, and then  $x(e_0)$  by pairwise comparison between the vectors  $x(e)$ .
- 11- Find  $x^*$  by Lemma 3.1.

**Numerical Example:**

Consider the problem below:

$$\min \quad 2x_1 - x_2 + 3x_3 + 2.5x_4 - x_5 - 6x_6 + 3x_7 + 2x_8$$

$$\begin{bmatrix} 0 & 0.16 & 0.37 & 0.95 & 0.17 & 0.07 & 0.77 & 0.14 \\ 0.08 & 0.51 & 0.26 & 0.1 & 0.3 & 0.4 & 0.3 & 0.35 \\ 0.99 & 0.59 & 0.28 & 0.34 & 0.34 & 0.74 & 0.19 & 0.21 \\ 0.83 & 0.75 & 0.25 & 0.35 & 0.2 & 0.5 & 0.2 & 0.2 \\ 0.73 & 0.84 & 0.94 & 0.44 & 0.54 & 0.84 & 0.99 & 0.44 \\ 0.37 & 0.7 & 0.55 & 0.4 & 0.2 & 0.2 & 0.73 & 0.24 \end{bmatrix} \bullet x \geq \begin{bmatrix} 0.4 \\ 0.5 \\ 0.57 \\ 0.6 \\ 0.72 \\ 0.6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0.94 & 0.69 & 0.49 & 0.5 & 0.51 & 0.87 & 0.43 \\ 0.02 & 1 & 0.82 & 0.59 & 0.89 & 0.76 & 0.74 & 0.52 \\ 0.1 & 0.73 & 0.4 & 0.37 & 0.45 & 0.7 & 0.69 & 0.2 \\ 0.1 & 0.1 & 0.3 & 0.2 & 0.25 & 0.49 & 0.58 & 0.01 \end{bmatrix} \bullet x \leq \begin{bmatrix} 0.7 \\ 0.76 \\ 0.6 \\ 0.5 \end{bmatrix}$$

$$x \in [0,1]^n$$

**Step 1.** The matrices,  $\bar{A}, \bar{B}$  are as follows:

$$\bar{A} = \begin{bmatrix} 0.8 & 0.64 & 0.43 & -0.15 & 0.63 & 0.73 & 0.03 & 0.66 \\ 0.92 & 0.49 & 0.74 & 0.9 & 0.7 & 0.6 & 0.7 & 0.65 \\ 0.15 & 0.55 & 0.86 & 0.8 & 0.8 & 0.4 & 0.95 & 0.93 \\ 0.37 & 0.45 & 0.95 & 0.85 & 1 & 0.7 & 1 & 1 \\ 0.71 & 0.6 & 0.5 & 1 & 0.9 & 0.6 & 0.45 & 1 \\ 0.83 & 0.5 & 0.65 & 0.8 & 1 & 1 & 0.47 & 0.96 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 1.4 & 0.46 & 0.71 & 0.91 & 0.9 & 0.89 & 0.53 & 0.97 \\ 1.5 & 0.52 & 0.7 & 0.93 & 0.63 & 0.76 & 0.78 & 1 \\ 1.1 & 0.47 & 0.8 & 0.83 & 0.75 & 0.5 & 0.51 & 1 \\ 0.9 & 0.9 & 0.7 & 0.8 & 0.75 & 0.51 & 0.42 & 0.99 \end{bmatrix}$$

**Step 2.**  $\bar{x} = [0.9 \ 0.46 \ 0.7 \ 0.8 \ 0.63 \ 0.5 \ 0.42 \ 0.97]$

**Step 3.** There is no  $i \in I^1$  such that  $\bar{a}_{ij} > \bar{x}_j$ ,  $\forall j \in J$  therefore we can go to step4.

**Step 4.** There is no  $i \in I^2$  and  $j \in J$  such that  $\bar{b}_{ij} < 0$  hence we can perform step5.

**Step 5.** Since  $\bar{a}_{14} < 0$ , hence the first row of the matrix  $\bar{A}$  is removed and  $\bar{A}$  becomes

$$\bar{A} = \begin{bmatrix} 0.92 & 0.49 & 0.74 & 0.9 & 0.7 & 0.6 & 0.7 & 0.65 \\ 0.15 & 0.55 & 0.86 & 0.8 & 0.8 & 0.4 & 0.95 & 0.93 \\ 0.37 & 0.45 & 0.95 & 0.85 & 1 & 0.7 & 1 & 1 \\ 0.71 & 0.6 & 0.5 & 1 & 0.9 & 0.6 & 0.45 & 1 \\ 0.83 & 0.5 & 0.65 & 0.8 & 1 & 1 & 0.47 & 0.96 \end{bmatrix}$$

**Step 6.** The fifth and seventh columns are removed and we have  $x(e_0)_7 = 0$ . Also, matrix  $\bar{A}$  becomes

$$\bar{A} = \begin{bmatrix} 0.92 & 0.49 & 0.74 & 0.9 & 0.6 & 0.65 \\ 0.15 & 0.55 & 0.86 & 0.8 & 0.4 & 0.93 \\ 0.37 & 0.45 & 0.95 & 0.85 & 0.7 & 1 \\ 0.71 & 0.6 & 0.5 & 1 & 0.6 & 1 \\ 0.83 & 0.5 & 0.65 & 0.8 & 1 & 0.96 \end{bmatrix}$$

**Step 7.** Since  $J_3^* \neq \emptyset, J_4^* \neq \emptyset$ , we can delete the second and third row. We get

$$\bar{A} = \begin{bmatrix} 0.92 & 0.49 & 0.74 & 0.9 & 0.6 & 0.65 \\ 0.71 & 0.6 & 0.5 & 1 & 0.6 & 1 \\ 0.83 & 0.5 & 0.65 & 0.8 & 1 & 0.96 \end{bmatrix}$$

**Step 8.** Since  $c_2, c_6 < 0$  then, we can remove the second and fifth columns and we get

$$\bar{A} = \begin{bmatrix} 0.92 & 0.74 & 0.9 & 0.65 \\ 0.71 & 0.5 & 1 & 1 \\ 0.83 & 0.5 & 0.8 & 0.96 \end{bmatrix}$$

**Step 9.** In the above matrix, the first and fourth columns dominate the second and third respectively. After removing the second and third columns, matrix  $\bar{A}$  becomes

$$\bar{A} = \begin{bmatrix} 0.92 & 0.65 \\ 0.71 & 1 \\ 0.83 & 0.96 \end{bmatrix}$$

And we have  $x(e_0)_3 = x(e_0)_4 = 0$ .

**Step 10.** In the new matrix, we have  $J_2^{new} = \{8\}$ ,  $J_5^{new} = \{1\}$  and  $J_6^{new} = \{1,8\}$ . For  $e_1 = (8,1,1)$ ,  $x(e_1)_1 = 0.83$  and  $x(e_1)_8 = 0.65$ . Also,  $e_2 = (8,1,8)$  results in  $x(e_2)_1 = 0.71$  and  $x(e_2)_8 = 0.96$ . Hence the minimal solutions are  $x(e_1) = (0.83, 0, 0, 0, 0, 0, 0, 0, 0.65)$  and  $x(e_2) = (0.71, 0, 0, 0, 0, 0, 0, 0, 0.96)$ .

**Step 11.** Since  $x(e_1)$  optimizes the problem with objective function  $c^{+t}x$  then  $x(e_1) = x(e_0)$  and also  $x^* = (0.83, 0.46, 0, 0, 0.63, 0.5, 0, 0, 0.65)$ .

## 5. Conclusion

In this paper we studied the linear optimization problem with fuzzy relational inequalities constraints defined by max-average operator. First, we discuss the feasibility region characterization and then, introduce a new simplification technique to solve the usual difficulty of finding the minimal solutions that optimize the problem with objective function  $c^+x$ . An algorithm together with simplification operations to accelerate the problem resolution is also presented together with an illustrative numerical example.

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