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RESIDUAL OF IDEALS OF AN L-RING

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ABSTRACT. The concept of right (left) quotient (or residual) of an ideal η by an ideal ν of an *L*-subring μ of a ring *R* is introduced. The right (left) quotients are shown to be ideals of μ. It is proved that the right quotient $[\eta :_r \nu]$ of an ideal η by an ideal ν of an *L*-subring μ is the largest ideal of μ such that $[\eta :_r \nu]\nu \subseteq \eta$. Most of the results pertaining to the notion of quotients (or residual) of an ideal of ordinary rings are extended to *L*-ideal theory of *L*-subrings.

1. **Introduction**

Various operations on fuzzy ideals of a ring have been introduced and discussed in the literature, which are carried over easily to *L*-ideal theory. However, a detailed investigation of the concept of quotient (or residual) of *L*-ideals is still awaited. An attempt in this direction has been made in [12], wherein the notion of quotient (residual) of *L*-ideals of a ring *R* is introduced and some basic results are obtained. However, some important results pertaining to the notion of quotients, which are useful for the development of the theory of Noetherian rings in the classical setting, are not extended to the theory of *L*-ideals.

In order to overcome this shortcoming, we study the *L*-ideal theory of an *L*-subring μ of a given ring *R*. Such type of studies have been initiated in our papers [13,14,15]. In papers [13,14], we have defined and discussed the maximal ideals of an *L*-subring. On the other hand, in paper [15], we have introduced the notion of prime ideals, semiprime ideals and primary ideals of an *L*-subring. In what follows, we shall consider an *L*-subring μ of a given ring *R* and call the system $L(\mu, R)$ an *L*-ring.

2. **Preliminaries**

In this section we recall some of the basic definitions and concepts used in the sequel. For details we refer to [10,11,12].

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Let X be a non-empty set and L be a lattice. By an L-subset of X, we mean a function from X to L. The set of all L-subsets of X is called the L-power set of X and is denoted by L^X. For $\mu, \nu \in L^X$, if $\nu(x) \leq \mu(x)$, $\forall \; x \in X$, we say that ν is contained in μ and write $\nu \subseteq \mu$. Throughout the paper, unless otherwise specified, R will denote an ordinary ring and L will denote a lattice. Also, Z^+ will denote the set of positive integers.

Definition 2.1. [12] Let L be a complete lattice and let $\{\mu_i\}_{i\in\lambda}$ be a family of L-subsets of X. Define $\bigcup_{i\in\lambda} \mu_i$ and $\bigcap_{i\in\lambda} \mu_i \in L^X$ as follows:

$$\left(\bigcup_{i\in\lambda} \mu_i\right)(x) = \underset{i\in\lambda}{\vee} \mu_i(x), \qquad \forall x \in R$$

$$\left(\bigcap_{i\in\lambda} \mu_i\right)(x) = \underset{i\in\lambda}{\wedge} \mu_i(x), \quad \forall x \in R.$$

$\bigcup_{i\in\lambda} \mu_i$ and $\bigcap_{i\in\lambda} \mu_i$ are respectively called the union and intersection of the family $\{\mu_i\}_{i\in\lambda}$.

Definition 2.2. (Definition 3.1.6 [12]) Let L be a lattice and R be a ring. Let $\mu \in L^R$. Then μ is called an L-subring of R if

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, $\quad \forall \; x, y \in R$, and

(2) $\mu(xy) \geq \mu(x) \wedge \mu(y)$, $\quad \forall \; x, y \in R$.

The set of all L-subrings of R is denoted by $L(R)$. It is obvious that if μ is an L-subring of R, then $\mu(x) \leq \mu(0)$, $\forall \; x \in R$. For convenience, we use the notation $L(\mu, R)$ for the L-subring μ of R and call it the L-ring $L(\mu, R)$.

Definition 2.3. (Definition 3.1.7[12]) Let L be a lattice and $\mu \in L^R$. Then μ is called an L-ideal of R if

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, $\quad \forall \; x, y \in R$, and

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$, $\forall \; x, y \in R$.

We denote the set of all L-ideals of R by $LI(R)$. It is obvious that if R has identity 1 and $\mu \in LI(R)$, then $\mu(x) \geq \mu(1)$.

Definition 2.4. (Definition 3.2.11[12]) Let $\nu \in L^R$ and $\mu \in L(R)$ with $\nu \subseteq \mu$. Then ν is called an L-ideal of μ (or in μ) if

(1) $\nu(x-y) \geq \nu(x) \wedge \nu(y)$, $\forall\ x, y \in R$, and

(2) $\nu(xy) \geq (\mu(x) \wedge \nu(y)) \vee (\nu(x) \wedge \mu(y))$, $\forall\ x, y \in R$

For convenience, ν is called an ideal of μ (or the *L*-ring $L(\mu, R)$). If ν is an ideal of the *L*-ring $L(\mu, R)$, then we write $\nu \triangleleft \mu$.

Definition 2.5. Let *L* be a lattice and $L(\mu, R)$ be an *L*-ring. If ν is an *L*-subring of *R* with $\nu \subseteq \mu$, then ν is called a subring of μ *(* or *L*-ring $L(\mu, R)$*)*.

Clearly if ν is a subring of μ, then $\nu(x^n) \geq \nu(x),\ \forall\ n \in Z^+$.

Definition 2.6. [12] Let *L* be a complete lattice and $\eta, \nu \in L^R$. Then we define $\eta + \nu$, $\eta\nu$ and $\eta o \nu$ by

$$(\eta + \nu)(x) = \vee \{\eta(y) \wedge \nu(z) \mid y, z \in R; x = y + z\}$$

$$(\eta\nu)(x) = \vee \left\{ \mathop{\wedge}_{i=1}^{n} (\eta(y_i) \wedge \nu(z_i)) \mid y_i, z_i \in R,\ i = 1,2,\ldots,n, \sum_{i=1}^{n} y_i z_i = x \right\}$$

$$(\eta o \nu)(x) = \vee \{\eta(y) \wedge \nu(z) \mid y, z \in R,\ yz = x\}.$$

Clearly if η and ν are subrings of an *L*-ring $L(\mu, R)$ with $\eta(0) = \nu(0)$, then $\eta \subseteq \eta + \nu$ and $\nu \subseteq \eta + \nu$.

Lemma 2.7. [12] *Let L be a complete lattice and* $\eta, \nu, \xi \in L^R$. *Then the following assertions hold:*

(1) $\eta o \nu \subseteq \eta\nu$,

(2) $\xi o(\eta + \nu) \subseteq \xi o \eta + \xi o \nu$,

(3) $(\xi + \eta) o \nu \subseteq \xi o \nu + \eta o \nu$,

(4) $(\xi \cap \eta) o \nu \subseteq (\xi o \nu) \cap (\eta o \nu)$,

(5) $\eta \subseteq \nu \Rightarrow \eta\xi \subseteq \nu\xi$ *and* $\eta o \xi \subseteq \nu o \xi$,

(6) $(\eta\nu)\xi = \eta(\nu\xi)$,

(7) $(\eta\nu)(x + y) \geq (\eta\nu)(x) \wedge (\eta\nu)(y),\ \forall\, x, y \in R$.

We recall the following elementary results from [15]. For the sake of completeness, we offer their proofs.

Lemma 2.8. [15] *Let L be a complete lattice and* $L(\mu, R)$ *be an L-ring. Let* η *be a subring of* μ *then*

(1) $\eta o \eta \subseteq \eta\eta \subseteq \eta$

(2) $\eta + \eta = \eta$

In particular, $\mu o \mu \subseteq \mu\mu \subseteq \mu$ *and* $\mu + \mu = \mu$.

Proof. The proof is obvious.

Lemma 2.9. (Theorem 3.2.15 [12]) *Let $L(\mu, R)$ be an L-ring. Then the intersection of two ideals of μ is an ideal of μ.*

Theorem 2.10. [15] *Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let $\eta \in L^R$ with $\eta \subseteq \mu$. Then η is an ideal of μ if and only if*

(1) $\eta(x-y) \geq \eta(x) \wedge \eta(y)$, $\forall\, x, y \in R$

(2) $\eta o \mu \subseteq \eta$ *and* $\mu o \eta \subseteq \eta$.

Proof. Suppose η is an ideal of μ. Let $z \in R$. Since η is an ideal of μ, we have

$$\eta(xy) \geq \eta(x) \wedge \mu(y), \quad \forall\, x, y \in R.$$

Thus

$$\begin{aligned}\eta(z) &\geq \vee \{\eta(x) \wedge \mu(y) \mid x, y \in R,\ z = xy\} \\ &= (\eta o \mu)(z)\end{aligned}$$

Hence $\eta o \mu \subseteq \eta$. Similarly $\mu o \eta \subseteq \eta$.
Conversely, suppose (1) and (2) hold. Let $x, y \in R$. Then

$$\begin{aligned}\eta(xy) \geq (\eta o \mu)(xy) &= \vee \{\eta(x_i) \wedge \mu(y_i) \mid x_i, y_i \in R,\ x_i y_i = xy\} \\ &\geq \eta(x) \wedge \mu(y).\end{aligned}$$

Similarly, $\eta(xy) \geq \mu(x) \wedge \eta(y)$. Thus η is an ideal of μ.

Theorem 2.11. [15] *Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let $\eta \in L^R$ with $\eta \subseteq \mu$. Then η is an ideal of μ if and only if*

(1) $\eta(x-y) \geq \eta(x) \wedge \eta(y)$, $\forall\, x, y \in R$,

(2) $\eta\mu \subseteq \eta$ *and* $\mu\eta \subseteq \eta$.

Proof. Let η be an ideal of μ. For $x \in R$ we may write

$$x = \sum_{i=1}^{n} y_i z_i, \text{ where } y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in R.$$

Since η is an ideal of μ, we have

$$\begin{aligned}\eta(x) &= \eta\left(\sum_{i=1}^{n} y_i z_i\right) \\ &\geq \bigwedge_{i=1}^{n} \eta(y_i z_i) \\ &\geq \bigwedge_{i=1}^{n} (\eta(y_i) \wedge \mu(z_i))\end{aligned}$$

Thus

$$\eta(x) \geq \vee \left\{ \overset{n}{\underset{i=1}{\wedge}} (\eta(y_i) \wedge \mu(z_i)) \mid y_i, z_i \in R, \sum_{i=1}^{n} y_i z_i = x \right\}$$

$$= (\eta\mu)(x).$$

Hence $\eta\mu \subseteq \eta$. Similarly, $\mu\eta \subseteq \eta$.

The converse follows from Lemma 2.7 and Theorem 2.10.

Theorem 2.12. [15] *Let L be a complete lattice and $L(\mu, R)$ be an L-ring. If η and ν are ideals of μ with $\eta(0)=\nu(0)$, then $\eta+\nu$ is an ideal of μ and $\eta \subseteq \eta+\nu$, $\nu \subseteq \eta+\nu$.*

Proof. Let $x, y \in R$. It is easy to show that

$$(\eta+\nu)(x-y) \geq (\eta+\nu)(x) \wedge (\eta+\nu)(y).$$

Since η is an ideal of μ, we have, by Theorem 2.10, $\mu o \eta \subseteq \eta$. Similarly, $\mu o \nu \subseteq \nu$. Now, by Lemma 2.7, we have

$$\mu o(\eta+\nu) \subseteq \mu o \eta + \mu o \nu \subseteq \eta + \nu.$$

Similarly, $(\eta+\nu) o \mu \subseteq \eta+\nu$. Thus, by Theorem 2.10, $\eta+\nu$ is an ideal of μ.

Theorem 2.13. [15] *Let L be a complete lattice and $L(\mu, R)$ be an L-ring. If η and ν are ideals of μ, then $\eta\nu$ is an ideal of μ.*

Proof. Obviously, $(\eta\nu)(-x) = (\eta\nu)(x)$, $\forall x \in R$. By Lemma 2.7, we have

$$(\eta\nu)(x+y) \geq (\eta\nu)(x) \wedge (\eta\nu)(y), \ \vee x, y \in R.$$

Since ν is an ideal of μ, by Theorem 2.11, we have $\nu\mu \subseteq \nu$. Thus, by Lemma 2.7, we have $(\eta\nu)\mu = \eta(\nu\mu) \subseteq \eta\nu$. Similarly, $\mu(\eta\nu) \subseteq \eta\nu$. Thus, by Theorem 2.11, $\eta\nu$ is an ideal of μ.

Lemma 2.14. *Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let η be a subring of μ. Let $\nu, \theta \in L^R$. Then $\nu o \theta \subseteq \eta$ if and only if $\nu\theta \subseteq \eta$.*

Proof. Suppose $\nu o \theta \subseteq \eta$. Let $x \in R$. Let $x = \sum_{i=1}^{n} y_i z_i$, $y_1, y_2, \ldots, y_n$, $z_1, z_2, \ldots, z_n \in R$.

Then

$$\eta(x) = \eta(y_1 z_1 + y_2 z_2 + \ldots + y_n z_n)$$

$$\geq \overset{n}{\underset{i=1}{\wedge}} \eta(y_i z_i) \qquad \text{(Since } \eta \text{ is a subring of } \mu\text{)}$$

$$\geq \overset{n}{\underset{i=1}{\wedge}} (\nu o \theta)(y_i z_i) \qquad \text{(Since } \nu o \theta \subseteq \eta\text{)}$$

$$\geq \overset{n}{\underset{i=1}{\wedge}} \{\nu(y_i) \wedge \theta(z_i)\}$$

Thus

$$\eta(x) \geq \vee\left\{\overset{n}{\underset{i=1}{\wedge}} \nu(y_i) \wedge \theta(z_i) \mid x = \sum_{i=1}^{n} y_i z_i,\ y_i, z_i \in R\right\}$$
$$= (\nu\theta)(x),$$

Hence $\nu\theta \subseteq \eta$. The converse follows from Lemma 2.7 (i).

Definition 2.15. [12] A Complete *Heyting algebra* L is a complete lattice such that for all A $\subseteq L$ and for all b $\in L$,

$$\vee\{a \wedge b \,|\, a \in A\} = (\vee\{\mathrm{a} \,|\, \mathrm{a} \in \mathrm{A}\}) \wedge \mathrm{b} \text{ and } \wedge\{a \vee b \,|\, a \in A\} = (\vee\{\mathrm{a} \,|\, \mathrm{a} \in \mathrm{A}\}) \vee \mathrm{b}.$$

Most of the proofs of our results are based on the completeness of the given lattice and the definition of Heyting algebra. This fact is specifically mentioned whenever it is used.

3. Quotient of Ideals

Definition 3.1. Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let η and ν be ideals of μ. The right quotient (residual) of η by ν, denoted by $[\eta :_r \nu]$, is defined by

$$[\eta :_r \nu] = \bigcup\{\xi \mid \xi \lhd \mu \textit{ and } \xi o \nu \subseteq \eta\}.$$

The left quotient of η by ν, denoted by $[\eta :_l \nu]$, is define by

$$[\eta :_1 \nu] = \bigcup\{\xi | \xi \lhd \mu \textit{ and } \nu o \xi \subseteq \eta\}.$$

If R is commutative, then $[\eta :_r \nu] = [\eta :_1 \nu]$. In this case it is called the quotient of η by ν and denoted by $[\eta : \nu]$.

Theorem 3.2. *Let L be a complete Heyting algebra and $L(\mu, R)$ be an L-ring. Let η, ν be ideals of μ. Then $[\eta :_r \nu]$ and $[\eta :_1 \nu]$ are ideals of μ. Also $\eta \subseteq [\eta :_r \nu] \subseteq \mu$ and $\eta \subseteq [\eta :_1 \nu] \subseteq \mu$.*

Proof. Let $x, y \in R$. Clearly, $[\eta :_r \nu](-x) = [\eta :_r \nu](x)$.
Now write $A = \{\xi | \xi \lhd \mu \textit{ and } \xi o \nu \subseteq \eta\}$. Suppose $\xi, \xi' \in A$. Then ξ and ξ' are ideals of μ such that $\xi o \nu \subseteq \eta$ and $\xi' o \nu \subseteq \eta$. By Theorem 2.12, $\xi + \xi'$ is an ideal of μ. Hence, by Lemma 2.7 and Lemma 2.8, it follows that

$$(\xi + \xi') o \nu \subseteq \xi o \nu + \xi' o \nu \subseteq \eta + \eta = \eta.$$

Thus $\xi + \xi' \in A$.

Now,

$$[\eta :_r \nu](x) \wedge [\eta :_r \nu](y) = \left[\vee_{\xi \in A} \xi(x)\right] \wedge \left[\vee_{\xi' \in A} \xi'(y)\right]$$

which, since L is complete Heyting algebra,

$$\begin{aligned} &= \vee \{\xi(x) \wedge \xi'(y) \mid \xi, \xi' \in A\} \\ &\leq \vee \{(\xi + \xi')(x + y) \mid \xi, \xi' \in A\} \\ &\leq [\eta :_r \nu](x + y) \qquad (\text{Since } \xi + \xi' \in A) \end{aligned}$$

Now, $$\begin{aligned} [\eta :_r \nu](xy) &= \left(\bigcup_{\xi \in A} \xi\right)(xy) \\ &= \vee_{\xi \in A} \xi(xy) \\ &\geq \vee_{\xi \in A} [\xi(x) \wedge \mu(y)] \\ &= \left(\vee_{\xi \in A} \xi(x)\right) \wedge \mu(y) \quad (\text{Since } L \text{ is complete Heyting algebra}) \\ &= [\eta :_r \nu](x) \wedge \mu(y) \end{aligned}$$

Similarly, $[\eta :_r \nu](xy) \geq [\eta :_r \nu](y) \wedge \mu(x)$. Thus $[\eta :_r \nu]$ is an ideal of μ. Clearly, $[\eta :_r \nu] \subseteq \mu$. Since η is an ideal of μ, we have, by Theorem 2.10, $\eta o \mu \subseteq \eta$. Thus, by Lemma 2.7, we have $\eta o \nu \subseteq \eta o \mu \subseteq \eta$. Hence $\eta \in A$ and therefore $\eta \subseteq [\eta :_r \nu]$. Similarly, $[\eta :_l \nu]$ is an ideal of μ and $\eta \subseteq [\eta :_l \nu] \subseteq \mu$.

Theorem 3.3. *Let L be a complete Heyting algebra. Let $L(\mu, R)$ be an L-ring and η, ν be ideals of μ. Then*

(1) $[\eta :_r \nu]$ is the largest ideal of μ with the property that $[\eta :_r \nu]\nu \subseteq \eta$.

(2) $[\eta :_l \nu]$ is the largest ideal of μ with the property that $\nu[\eta :_l \nu] \subseteq \eta$.

Proof. Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \nu \subseteq \eta\}$. Then $[\eta :_r \nu] = \bigcup_{\xi \in A} \xi$. Let $x \in R$ and $x = \sum_{i=1}^{m} u_i w_i$. Now

$$\eta(u_i w_i) \geq (\xi o \nu)(u_i w_i) \geq \xi(u_i) \wedge \nu(w_i), \quad \forall \xi \in A$$

Thus

$$\begin{aligned} \eta(u_i w_i) &\geq \vee_{\xi \in A} [\xi(u_i) \wedge \nu(w_i)] \\ &= \left[\vee_{\xi \in A} \xi(u_i)\right] \wedge \nu(w_i) \quad (\text{Since } L \text{ is a complete Heyting algebra}) \\ &= [\eta :_r \nu](u_i) \wedge \nu(w_i) \end{aligned}$$

Hence

$$\eta(x) = \eta\left(\sum_{i=1}^{m} u_i w_i\right) \geq \bigwedge_{i=1}^{m} \eta(u_i w_i) \geq \bigwedge_{i=1}^{m} \{[\eta :_r \nu](u_i) \wedge \nu(w_i)\}$$

Consequently

$$\eta(x) \geq \vee\left\{\bigwedge_{i=1}^{m} [\eta :_r \nu](u_i) \wedge \nu(w_i) \mid x = \sum_{i=1}^{m} u_i w_i\right\} = ([\eta :_r \nu]\nu)(x)$$

Hence $[\eta :_r \nu]\nu \subseteq \eta$.

Suppose λ is an ideal of μ such that $\lambda\nu \subseteq \eta$. By Lemma 2.7, $\lambda o \nu \subseteq \lambda\nu \subseteq \eta$. Thus $\lambda \in A$ and hence $\lambda \subseteq [\eta :_r \nu]$.

Theorem 3.4. *Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let η, ν and θ be ideals of μ . Then the following assertions hold.*

(1) If $\eta \subseteq \nu$, then $[\eta :_r \theta] \subseteq [\nu :_r \theta]$ and $[\theta :_r \nu] \subseteq [\theta :_r \eta]$,

(2) If $\eta \subseteq \nu$ then $[\nu :_r \eta] = \mu$,

(3) $[\eta :_r \eta] = \mu$,

(4) If $\eta(0) = \nu(0)$, then $[\eta :_r \nu] = [\eta :_r \eta + \nu]$.

Proof.

(1) Let $\eta \subseteq \nu$ .Write $A = \{\xi | \xi \lhd \mu \text{ and } \xi o \theta \subseteq \eta\}$ and $B = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \theta \subseteq \nu\}$.
Let $\xi \in A$. Then $\xi \lhd \mu$ and $\xi o \theta \subseteq \eta \subseteq \nu$. Thus $\xi \in B$ and hence $A \subseteq B$. Therefore

$$[\eta :_r \theta] = \bigcup_{\xi \in A} \xi \subseteq \bigcup_{\xi \in B} \xi = [\nu :_r \theta].$$

Similarly we can show that $[\theta :_r \nu] \subseteq [\theta :_r \eta]$.

(2) Let $\eta \subseteq \nu$. Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \eta \subseteq \nu\}$. Now $\mu \lhd \mu$ and since $\eta \lhd \mu$, we have $\mu o \eta \subseteq \eta \subseteq \nu$. Thus $\mu \in A$ and hence

$$\mu \subseteq \bigcup_{\xi \in A} \xi = [\nu :_r \eta] \subseteq \mu .$$

Therefore $[\nu :_r \eta] = \mu$.

(3) Obvious.

(4) Since $\eta(0) = \nu(0)$, by Theorem 2.12, $\eta + \nu$ is an ideal of μ and $\nu \subseteq \eta + \nu$. By (1), $[\eta :_r \eta + \nu] \subseteq [\eta :_r \nu]$.

Write $A=\{\xi \mid \xi \lhd \mu \, and \, \xi o \nu \subseteq \eta\}$ and $B=\{\xi \lhd \mu \, and \, \xi o(\eta+\nu) \subseteq \eta\}$. Let $\xi \in A$. Then $\xi \lhd \mu$ and $\xi o \nu \subseteq \eta$. Since $\eta \lhd \mu$, $\xi o \eta \subseteq \mu o \eta \subseteq \eta$. Therefore, by Lemma 2.7 and Lemma 2.8 , we have

$$\xi o(\eta+\nu) \subseteq \xi o \eta+\xi o \nu \subseteq \eta+\eta=\eta .$$

Hence $\xi \in B$. It follows that

$$[\eta :_r \nu]=\bigcup_{\xi \in A} \xi \subseteq \bigcup_{\xi \in B} \xi=[\eta :_r \eta+\nu] .$$

Consequently

$$[\eta :_r \eta+\nu]=[\eta :_r \nu] .$$

Similar results hold for left quotients.

Corollary 3.5. *Let L be a complete Heyting algebra and $L(\mu, R)$ be an L-ring. Let η and ν be ideals of μ Then*

(1) $[[\eta :_r \nu] :_r \eta]=\mu$,
(2) $[\eta :_r \eta\nu]=\mu$,
(3) $[[\eta :_r \eta]_r : \nu]=\mu$,
(4) $[\eta :_r (\eta \cap \nu)]=\mu$,
(5) $[(\eta \cap \nu) :_r \eta\nu]=\mu$.

Proof.

(1) Since $\eta \subseteq [\eta :_r \nu]$, by Theorem 3.4 (2), we have $[[\eta :_r \nu] :_r \eta]=\mu$.

(2) By Theorem 2.13 , $\eta\nu$ is an ideal of μ. Since η is an ideal of μ, by Lemma 2.7 and Lemma 2.11, we have

$$\eta\nu \subseteq \eta\mu \subseteq \eta .$$

Therefore, by Theorem 3.4 (2), we have $[\eta :_r \eta\nu]=\mu$.

(3) By Theorem 3.4 (3), $[\eta :_r \eta]=\mu$. Since $\nu \subseteq \mu=[\eta :_r \eta]$, by Theorem 3.4 (2), we have $[[\eta :_r \eta]_r : \mu]=\mu$.

(4) Since $\eta \cap \nu$ is an ideal of μ and $\eta \cap \nu \subseteq \eta$, by Theorem 3.4(2), we have $[\eta :_r (\eta \cap \nu)]=\mu$.

(5) $\eta \cap \nu$ and $\eta\nu$ are ideals of μ. Since $\eta\nu \subseteq \eta$ and $\eta\nu \subseteq \nu$, we have $\eta\nu \subseteq \eta \cap \nu$ .Therefore, by Theorem 3.4 (2), we have $[(\eta \cap \nu) :_r \eta\nu]=\mu$.

Corollary 3.6. *Let L be a complete Heyting algebra and $L(\mu, R)$ be L-ring. Let η and ν be ideals of μ, with $\eta(0)=\nu(0)$. Then*

(1) $[(\eta+\nu) :_r \eta]=\mu$,
(2) $[(\eta+\nu) :_r (\eta \cap \nu)]=\mu$,

(3) $[(\eta+\nu):_r \eta\nu]=\mu$.

Proof.

(1) By Theorem 2.12, $\eta+\nu$ is an ideal of μ and $\eta\subseteq\eta+\nu$. Thus by Theorem 3.4 (2), we have $[(\eta+\nu):_r \eta]=\mu$.

(2) Since $(\eta\cap\nu)\subseteq(\eta+\nu)$, by Theorem 3.4 (2), we have $[(\eta+\nu):_r (\eta\cap\nu)]=\mu$.

(3) Since $\eta\nu\subseteq\eta\subseteq\eta+\nu$, by Theorem 3.4 (2), we have $[(\eta+\nu):_r \eta\nu]=\mu$.

Theorem 3.7. *Let L be a complete Heyting algebra. Let $\eta_1,\eta_2,...,\eta_m$, ν and θ be ideals of μ, Then*

(1) $\left[\bigcap_{i=1}^{m}\eta_i :_r \nu\right]=\bigcap_{i=1}^{m}[\eta_i :_r \nu]$.

(2) $\left[\nu :_r \sum_{i=1}^{m}\eta_i\right]=\bigcap_{i=1}^{m}[\nu :_r \eta_i]$ *(provided $\eta_i(0)=\eta_j(0)\ \forall i,j$).*

Proof.

(1) Since $\bigcap_{j=1}^{m}\eta_j\subseteq\eta_i\ \ \forall i$, by Theorem 3.4(1), we have

$$\left[\bigcap_{i=1}^{m}\eta_i :_r \nu\right]\subseteq[\eta_i :_r \nu],\qquad \forall\ i.$$

Hence $\left[\bigcap_{i=1}^{m}\eta_i :_r \nu\right]\subseteq\bigcap_{i=1}^{m}[\eta_i :_r \nu]$.

Write $A=\{\xi \mid \xi\lhd\mu \text{ and } \xi o\nu\subseteq\eta_1\}$, $B=\{\xi \mid \alpha\lhd\mu \text{ and } \xi o\nu\subseteq\eta_2\}$,and $C=\{\xi \mid \xi\lhd\mu \text{ and } \xi o\nu\subseteq\eta_1\cap\eta_2\}$. Let $x\in R$. Now

$$\begin{aligned}([\eta_1 :_r \nu]\cap[\eta_2 :_r \nu])(x) &= \left[\left(\bigcup_{\xi\in A}\xi\right)\cap\left(\bigcup_{\xi\in B}\xi'\right)\right](x)\\
&=\left(\vee_{\xi\in A}\xi(x)\right)\wedge\left(\vee_{\xi'\in B}\xi'(x)\right)\\
&=\vee_{\xi'\in B}\left[\left(\vee_{\xi\in A}\xi(x)\right)\wedge\xi'(x)\right] \quad \text{(Since } L \text{ is a complete Heyting algebra)}\\
&=\vee_{\xi'\in B}\left\{\vee_{\xi\in A}\left(\xi(x)\wedge\xi'(x)\right)\right\} \quad \text{(Since } L \text{ is a complete Heyting algebra)}\\
&=\vee\{\xi(x)\wedge\xi'(x) \mid \xi\in A,\ \xi'\in B\}.\end{aligned}$$

Let $\xi \in A$ and $\xi' \in B$. Then ξ and ξ' are ideals of μ. Also $\xi o \nu \subseteq \eta_1$ and $\xi' o \nu \subseteq \eta_2$. Now, by Lemma 2.9, $\xi \cap \xi'$ is an ideal of μ and by Lemma 2.7, we have

$$(\xi \cap \xi') o \nu \subseteq (\xi o \nu) \cap (\xi' o \nu) \subseteq \eta_1 \cap \eta_2 .$$

Thus $\xi \cap \xi' \in C$. Hence

$$[\eta_1 \cap \eta_2 :_r \nu] = \bigcup_{\xi \in C} \xi \supseteq \bigcup \{\xi \cap \xi' \mid \xi \in A,\ \xi' \in B\}.$$

Therefore

$$\begin{aligned}[\eta_1 \cap \eta_2 :_r \nu](x) &\geq \vee \{(\xi \cap \xi')(x) \mid \xi \in A, \xi' \in B\} \\ &= \vee \{\xi(x) \wedge \xi'(x) \mid \xi \in A, \xi' \in B\} \\ &= ([\eta_i :_r \nu] \cap [\eta_2 :_r \nu])(x)\end{aligned}$$

Hence

$$[\eta_1 :_r \nu] \cap [\eta_2 :_r \nu] \subseteq [\eta_1 \cap \eta_2 :_r \nu].$$

Consequently

$$[\eta_1 \cap \eta_2 :_r \nu] = [\eta_1 :_r \nu] \cap [\eta_2 :_r \nu].$$

(2) By Theorem 2.12, $\eta_1 + \eta_2$ is an ideal of μ such that $\eta_1 \subseteq \eta_1 + \eta_2$ and $\eta_2 \subseteq \eta_1 + \eta_2$. Thus, by Theorem 3.4 (1), we have

$$[\nu :_r \eta_1 + \eta_2] \subseteq [\nu :_r \eta_1] \text{ and } [\nu :_r \eta_1 + \eta_2] \subseteq [\nu :_r \eta_2]$$

Hence $[\nu :_r \eta_1 + \eta_2] \subseteq [\nu :_r \eta_1] \cap [\nu :_r \eta_2]$.

Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \eta_1 \subseteq \nu\}$, $B = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \eta_2 \subseteq \nu\}$ and $C = \{\xi \mid \xi \lhd \mu \text{ and } \xi o(\eta_1 + \eta_2) \subseteq \nu\}$.

Let $x \in R$. Then $([\nu :_r \eta_1 + \eta_2])(x) = \vee_{\xi \in C} \xi(x)$. Now,

$$([\nu :_r \eta_1] \cap [\nu :_r \eta_2])(x) = [\nu :_r \eta_1](x) \wedge [\nu :_r \eta_2](x)$$

Which, since L is a complete Heyting algebra,

$$= \left(\vee_{\xi \in A} \xi(x)\right) \wedge \left(\vee_{\xi \in B} \xi'(x)\right) = \vee \{\xi(x) \wedge \xi'(x) \mid \xi \in A, \xi' \in B\}$$

Let $\xi \in A$, $\xi' \in B$. Then ξ and ξ' are ideals of μ. Also $\xi o \eta_1 \subseteq \nu$. $\xi' o \eta_1 \subseteq \nu$. Now. by Lemma 2.9, $\xi \cap \xi'$ is an ideal of μ. Also,

$$\begin{aligned}(\xi \cap \xi') o(\eta_1 + \eta_2) &\subseteq (\xi \cap \xi') o \eta_1 + (\xi \cap \xi') o \eta_2 \qquad \text{(by Lemma 2.7)} \\ &\subseteq \xi o \eta_1 + \xi' o \eta_2 \subseteq \nu + \nu = \nu \\ &\subseteq \nu + \nu = \nu .\end{aligned}$$

Thus $\xi \cap \xi' \in C$ and hence

$$[\nu :_r (\eta_1 + \eta_2)] = \bigcup_{\lambda \in C} \lambda \supseteq \bigcup \{\xi \cap \xi' \mid \xi \in A,\ \xi' \in B\}.$$

Therefore

$$[\nu :_r \eta_1 + \eta_2](x) \geq \vee \{\xi(x) \wedge \xi'(x) | \xi \in A, \xi' \in B\}$$

$$= ([\nu :_r \eta_1] \cap [\nu :_r \eta_2])(x)$$

Thus

$$[\nu :_r \eta_1 + \eta_2] = [\nu :_r \eta_1] \cap [\nu :_r \eta_2].$$

Theorem 3.8. *Let L be a complete Heyting algebra and* $L(\mu, R)$ *be an L-ring. Let* η, ν *and* θ *be ideals of* μ. *Then*

(1) $[\eta :_r \nu\theta] = [[\eta :_r \theta] :_r \nu]$

(2) $[\eta :_l \nu\theta] = [[\eta :_l \nu] :_l \theta]$.

Proof. By Theorem 2.13, $\nu\theta$ is an ideal of μ.
Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \nu \subseteq [\eta :_r \theta]\}$ and $B = \{\xi \mid \xi \lhd \mu \text{ and } \xi o(\nu\theta) \subseteq \eta\}$. Then

$$[[\eta :_r \theta] :_r \nu] = \cup \{\xi | \xi \in A\} \text{ and } [\eta :_r \nu\theta] = \cup[\xi \mid \xi \in B].$$

To show that $[\eta :_r \nu\theta] = [[\eta :_r \theta] :_r \nu]$, it is sufficient to show $A = B$.
Let $\xi \in A$. Then $\xi \lhd \mu$ and $\xi o \nu \subseteq [\eta :_r \theta]$. By Lemma 2.14, we have $\xi\nu \subseteq [\eta :_r \theta]$. By Theorem 3.3, $[\eta :_r \theta]\theta \subseteq \eta$. Thus by Theorem 2.7, we have

$$\xi(\nu\theta) = (\xi\nu)\theta \subseteq [\eta :_r \theta]\theta \subseteq \eta .$$

Hence, by Lemma 2.14, we have $\xi o(\nu\theta) \subseteq \eta$. Therefore $\xi \in B$. Thus $A \subseteq B$. Conversely, suppose that $\xi \in B$. Then $\xi \lhd \mu$ and $\xi o(\nu\theta) \subseteq \eta$. By Lemma 2.14, we have $\xi(\nu\theta) \subseteq \eta$. Hence, by Lemma 2.7, it follows that $(\xi\nu)\theta = \xi(\nu\theta) \subseteq \eta$. Since ξ and ν are ideals of μ, by Lemma 2.13, $\xi\nu$ is an ideal of μ. That is, $\xi\nu$ is an ideal of μ and $(\xi\nu)\theta \subseteq \eta$. By Theorem 3.3, $[\eta :_r \theta]$ is the largest ideal of μ such that $[\eta :_r \theta]\theta \subseteq \eta$. Therefore $\xi\nu \subseteq [\eta :_r \theta]$. Hence, by Lemma 2.14, we have $\xi o \nu \subseteq [\eta :_r \theta]$ and thus $\xi \in A$. Therefore, $B \subseteq A$ and consequently $A = B$.

Theorem 3.9. *Let L be a complete Heyting algebra and* $L(\mu, R)$ *be an L-ring. Let* η, ν *and* θ *be ideals of* μ. *Then*

(1) $[\eta :_r \nu] \subseteq [\eta\theta :_r \nu\theta]$

(2) $[\eta :_l \nu] \subseteq [\theta\eta :_l \theta\nu]$.

Proof. By Theorem 2.13, $\eta\theta$ and $\nu\theta$ are ideals of μ.
Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \nu \subseteq \eta\}$ and $B = \{\xi | \xi \lhd \mu \text{ and } \xi o(\nu\theta) \subseteq \eta\theta\}$. Then

$$[\eta :_r \nu] = \cup\{\xi \mid \xi \in A\} \text{ and } [\eta\theta :_r \nu\theta] = \cup\{\xi \mid \xi \in B\}.$$

Suppose $\xi \in A$. Then $\xi \lhd \mu$ and $\xi o \nu \subseteq \eta$. Therefore, by Lemma 2.14, $\xi\nu \subseteq \eta$ and hence, by Lemma 2.7, we have

$$\xi(\nu\theta) = (\xi\nu)\theta \subseteq \eta\theta .$$

Finally, by Lemma 2.14, $\xi o(\nu\theta) \subseteq \eta\theta$. So $\xi \in B$ and hence $A \subseteq B$. Therefore

$$[\eta :_r \nu] = \bigcup\{\xi \mid \xi \in A\} \subseteq \bigcup\{\xi \mid \xi \in B\} = [\eta\theta :_r \nu\theta].$$

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