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SOME PROPERTIES OF NEAR SR-COMPACTNESS

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Abstract. In this paper, we study some properties of the near SR-compactness in L-topological spaces, where L is a fuzzy lattice. The near SR-compactness is a kind of compactness between Lowen's fuzzy compactness and SR-compactness, and it preserves desirable properties of compactness in general topological spaces.

1. Introduction

Compactness is one of the most important notions in topology. Various kinds of fuzzy compactness have been proposed [3,6,7,9], which are all generalizations of the classical compactness. There may well be another compactness to be discovered. In this regard we have introduced two new kinds of fuzzy compactness called SR-compactness[2] and near SR-compactness[4] in L-topological spaces, respectively. The present paper studies some properties of near SR-compactness. Near SR-compactness is hereditary for strongly semiclosed subsets, finitely additive, and is preserved under S-irresolute mapping. Every set with finite support is near SR-compact. Also the near SR-compact space is described with cover form and family of strongly semiclosed sets having a finite intersection property.

2. Preliminaries

In this paper, $L = L(\leq, \vee, \wedge, ')$ always denotes a fuzzy lattice, i.e., a completely distributive lattice with an order-reversing involution " $'$ ". 0 and 1 denote the smallest and the largest element in L, respectively. Let X be a nonempty crisp set, L^X the set of all L-subsets on X, and 0_X and 1_X respectively denote the smallest and the largest element in L^X. $M(L)$ will denote the set of all nonzero irreducible elements of L. Put $M^*(L^X) = \{x_\alpha : x \in X \text{ and } \alpha \in M(L)\}$. For each $\psi \subset L$, we define $\psi' = \{A' : A \in \psi\}$.

An L-topological space(L-ts) is a pair (L^X, δ) where δ is a subfamily of L^X which contains 0_X and 1_X, and is closed for any suprema and finite infima. δ is called an L-topology on X. Members of δ are called open L-sets and their complements are called closed L-sets.

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Definition 2.1. [8] Let L be a lattice and $\alpha \in L$. A set B of L is called a minimal set of α, if the following two conditions hold; (1) $\bigvee B = \alpha$; (2) for each $x \in B$ and every subset C of L with $\bigvee C \geq \alpha$, there is $z \in C$ such that $z \geq x$. In a fuzzy lattice, each element α has a greatest minimal set which we will denote by $\beta(\alpha)$.

Definition 2.2. [1] Let (L^X, δ) be an L-ts, $A \in L^X$. Then A is called a strongly semiopen set iff there is a $B \in \delta$ such that $B \leq A \leq B^{-o}$, and A is called a strongly semiclosed set iff there is a $B \in \delta'$ such that $B^{o-} \leq A \leq B$, where B^o and B^- are the interior and closure of B, respectively. In what follows $SSO(L^X)$ and $SSC(L^X)$ will denote the family of strongly semiopen sets and family of strongly semiclosed sets of an L-ts (L^X, δ), respectively.

Definition 2.3. [8] Let $(L^X, \delta), (L^Y, \tau)$ be two L-ts's and $f : L^X \to L^Y$ be a mapping induced by a crisp mapping $f : X \to Y$. We define $f : L^X \to L^Y$ and its inverse mapping $f^{-1} : L^Y \to L^X$ as follows:
$\forall A \in L^X, y \in Y, f(A)(y) = \bigvee\{A(x) : x \in X, f(x) = y\}$,
$\forall B \in L^Y, x \in X, f^{-1}(B)(x) = B(f(x))$.
Then $f : L^X \to L^Y$ is called a function of Zadeh's type.

3. Some Properties of Near SR-Compactness

Definition 3.1. [2] Let (L^X, δ) be an L-ts and $x_\lambda \in M^*(L^X)$. $A \in SSC(L^X)$ is called a strongly semiclosed R-neighborhood, or briefly, SSC-R-neighborhood of x_λ, if $x_\lambda \notin A$. $B \in L^X$ is called a strongly semi-R-neighborhood, or briefly, SS-R-neighborhood of x_λ, if x_λ has an SSC-R-neighborhood A satisfying $B \leq A$. The set of all SS-R-neighborhoods (SSC-R-neighborhoods) of x_λ is denoted by $\xi(x_\lambda)(\xi^-(x_\lambda))$.

Definition 3.2. [2] Let (L^X, δ) be an L-ts, $A \in L^X$ and $\alpha \in M(L)$. $\phi \subset SSC(L^X)$ is called an α-SS-remote neighborhood family of A (briefly, α-SS-RF of A) if for each x_α in A, there is $P \in \phi$ such that $P \in \xi(x_\alpha)$.

Definition 3.3. Let (L^X, δ) be an L-ts and $A \in L^X$. A is called near SR-compact if every α-SS-RF ϕ of A has a finite subfamily ψ of ϕ such that ψ is an α-SS-RF of $A(\alpha \in M(L))$. Specifically, when $A = 1_X$ is near SR-compact, we call (L^X, δ) a near SR-compact space.

Obviously, the following statements are valid:
SR-compactness [2] $\Rightarrow$ near SR-compactness $\Rightarrow$ fuzzy compactness [8]
The converse of these statements need not be true [4].

Theorem 3.4. *Let A be a near SR-compact set in L-ts (L^X, δ). Then for each $B \in SSC(L^X)$, $A \wedge B$ is near SR-compact.*

Proof. Let S be a constant α-net in $A \wedge B$. Then S is also a constant α-net in A. Since A is near SR-compact, by Theorem 2.2 of [4], S has an SS-cluster point x_α in A with height α. Clearly S is also a net in $B \in SSC(L^X)$. Since x_α is an SS-cluster point of S we have $x_\alpha \leq B$. Hence $x_\alpha \leq A \wedge B$, i.e., x_α is an SS-cluster point of S in $A \wedge B$. Thus $A \wedge B$ is near SR-compact. □

Corollary 3.5. *Let (L^X, δ) be a near SR-compact space and B a strongly semi-closed set in (L^X, δ). Then B is near SR-compact.*

Theorem 3.6. *Let A and B be two near SR-compact sets in an L-ts (L^X, δ). Then $A \vee B$ is also near SR-compact.*

Proof. Let $\phi \subset SSC(L^X)$ be an α-SS-RF of $A \vee B$ ($\alpha \in M(L)$). Then ϕ is an α-SS-RF of both A and B. Since A and B are both near SR-compact sets, there exist finite subfamilies ψ_1 and ψ_2 of ϕ such that ψ_1 and ψ_2 are α-SS-RF of A and B, respectively. Put $\psi = \psi_1 \cup \psi_2$. Clearly, ψ is a finite subfamily of ϕ, and also an α-SS-RF of $A \vee B$. Thus, $A \vee B$ is near SR-compact. □

Theorem 3.7. *Let (L^X, δ) be an L-ts and $A \in L^X$. If A has finite support, then A is near SR-compact.*

Proof. Let the support of $\sigma_o(A)$ be $\{x_1, ..., x_n\}, and suppose \phi$ is an α-SS-RF of $A (\alpha \in M(L))$. For each $i \leq n$, choose $P_i \in \phi$ so that $\alpha \not\leq P_i(x_i)$. Then the finite subfamily $\psi = \{P_1, ..., P_n\}$ of ϕ is an α-SS-RF of A. Thus A is near SR-compact. □

Definition 3.8. [2] Let (L^X, δ) and (L^Y, τ) be two L-ts's and $f : (L^X, \delta) \to (L^Y, \tau)$ a function of Zadeh's type. f is called an S-irresolute mapping if $f^{-1}(B) \in SSO(L^X)$ for each $B \in SSO(L^Y)$.

Theorem 3.9. *Let (L^X, δ) and (L^Y, τ) be L-ts's, A a near SR-compact set in (L^X, δ), and $f : (L^X, \delta) \to (L^Y, \tau)$ an S-irresolute mapping. Then $f(A)$ is near SR-compact in (L^Y, τ).*

Proof. Let $\phi \subset SSC(L^Y)$ be an α-SS-RF of $f(A)(\alpha \in M(L))$. To begin with, let us show that $f^{-1}(\phi) = \{f^{-1}(P) : P \in \phi\}$ is an α-SS-RF of A. Since f is an S-irresolute mapping, then $f^{-1}(\phi) \subset SSC(L^X)$. Let $x_\alpha \in A$; then $f(x_\alpha) = (f(x))_\alpha \in f(A)$ and since ϕ is an α-SS-RF of $f(A)$, there is a $P \in \phi$ with $P \in \xi((f(x))_\alpha)$, i.e., $(f(x))_\alpha \notin P$, or, equivalently, $P(f(x)) \ngeq \alpha$. By the definition of inverse mapping, $f^{-1}(P)(x) = P(f(x)) \ngeq \alpha$, hence $x_\alpha \notin f^{-1}(P)$. It follows that $f^{-1}(P) \in \xi(x_\alpha)$. Therefore $f^{-1}(\phi)$ is an α-SS-RF of A. Since A is near SR-compact, there exists a finite subfamily ψ of ϕ such that $f^{-1}(\psi)$ is an α-SS-RF of A. It is easily to show that ψ is an α-SS-RF of $f(A)$. Thus $f(A)$ is near SR-compact. □

Corollary 3.10. *Let (L^X, δ) be a near SR-compact space and $f : (L^X, \delta) \to (L^Y, \tau)$ an onto S-irresolute mapping. Then (L^Y, τ) is a near SR-compact space.*

Definition 3.11. [2] Let (L^X, δ) be an L-ts, r be a prime element of L and $r < 1$. $\mu \subset SSO(L^X)$ is called an r-S-cover of (L^X, δ) if for each $x \in X$, there is $U \in \mu$ such that $U(x) \not\leq r$.

Theorem 3.12. *An L-ts (L^X, δ) is a near SR-compact space iff for every r-S-cover μ there is a finite subfamily ν of μ such that ν is an r-S-cover, where r is a prime element of L and $r < 1$.*

Proof. Let (L^X, δ) be a near SR-compact space, μ be an r-S-cover and r a prime element and $r < 1$. Put $\phi = \mu'$, then $\phi \subset SSC(L^X)$ and for each $x \in X$ there is

$Q = U' \in \phi$ such that $U(x) \not\leq r$, i.e., $r' \not\leq Q(x)$. Since r is a prime element and $r < 1$, $r' \in M(L)$. By $x_{r'} \not\leq Q$ we have $Q \in \xi(x_{r'})$, hence ϕ is an r'-SS-RF. Since (L^X, δ) is near SR-compact, there is a finite subfamily ν of μ such that $\psi = \nu'$ is an r'-SS-RF, i.e., for each $x \in X$, there is a $V' \in \psi = \nu'$ such that $V' \in \xi(x_{r'})$, i.e., $x_{r'} \notin V$, or $r' \not\leq V'(x)$, equivalently, for each $x \in X$, there is a $V \in \nu$ such that $V(x) \not\leq r$. Thus μ has a finite subfamily ν which is an r-S-cover.

Conversely, suppose every r-S-cover has a finite subfamily and it is an r-S-cover. Let ϕ be an α-SS-RF of (L^X, δ), $\mu = \phi'$ and $r = \alpha'$. Since $\alpha \in M(L)$, r is a prime element and $r < 1$. With the method of dual above, it is easily to prove that μ is an r-S-cover. Suppose ν is a finite subfamily of μ such that ν is an r-S-cover. Put $\psi = \nu'$, then ψ is a finite subfamily of ϕ. We can easily prove that ψ is an α-SS-RF of (L^X, δ). Thus (L^X, δ) is near SR-compact. □

Definition 3.13. Let (L^X, δ) be an L-ts, r be a prime element of L, $r < 1$ and $\mu \subset L^X$. If for every finite subfamily ν of μ, there is an $x \in X$ such that $(\bigwedge \nu)(x) \geq r'$, then we say that μ has an r-finite intersection property.

Theorem 3.14. *An L-ts (L^X, δ) is a near SR-compact space iff for every $\mu \subset SSC(L^X)$ having an r-finite intersection property, there is an $x \in X$ such that $(\bigwedge \mu)(x) \geq r'$, where r is a prime element of L and $r < 1$.*

Proof. Let (L^X, δ) be a near SR-compact space. Suppose there is a prime element r of L, $r < 1$ and some $\mu \subset SSC(L^X)$ has an r-finite intersection property, for each $x \in X$ such that $(\bigwedge \mu)(x) \not\geq r'$. Then there exists $A \in \mu$ such that $A(x) \not\geq r'$, i.e., $A'(x) \not\leq r$. This shows μ' is an r-S-cover. By Theorem 3.12, there is a finite subfamily $\nu = \{A_1, ..., A_n\}$ of μ such that ν' is an r-S-cover. Hence for each $x \in X$, there is $A_i \in \nu$ such that $A_i'(x) \not\leq r$. And so $(\bigvee_{i=1}^{n} A_i')(x) \not\leq r$, i.e.

$$(\bigwedge \nu)(x) = (\bigwedge_{i=1}^{n} A_i)(x) \not\geq r',$$

which contradicts the assumption that μ has an r-finite intersection property.

Conversely, let μ be an r-S-cover, r a prime element and $r < 1$. If none of the finite subfamily ν of μ is an r-S-cover, then there is an $x \in X$ such that $B(x) \leq r$ for each $B \in \nu$. Hence $(\bigvee \nu)(x) \leq r$, or equivalently, $(\bigwedge \nu')(x) \geq r'$. This shows that $\mu' \subset SSC(L^X)$ has an r-finite intersection property. Hence there is an $x \in X$ such that $(\bigwedge \mu')(x) \geq r'$, i.e., $(\bigvee \mu)(x) \leq r$. This implies that μ is not an r-S-cover, a contradiction. By Theorem 3.12, (L^X, δ) is near SR-compact. □

Theorem 3.15. *A near SR-compact L-topological space (where $L = [0, 1]$) is SR-compact iff each strongly semiclosed set, viewed as a function, has a maximum.*

Proof. Necessity: Let (X, δ) be an SR-compact space and A a strongly semiclosed set in (X, δ). Then A is a SR-compact set by Theorem 4.10 of [2]. Hence, by Theorem 4.9 of [2], the set A, viewed as a function, has a maximum.

Sufficiency: Suppose (X, δ) is near SR-compact but not SR-compact. Then, by Theorem 4.8 of [2], there exists an α-net $S = \{S(n), n \in D\}$ without any SS-cluster point with height α . Therefore, for each crisp point $x \in X$ corresponding to the fuzzy point x_α, there exists $P_x \in \xi(x_\alpha)$ and $n(x) \in D$ such that $S(n) \in P_x$ holds whenever $n \geq n(x)$. Let $\phi = \{P_x : x \in X\}$. Then ϕ is an α-SS-RF. Since (X, δ) is a near SR-compact space, ϕ has a finite subfamily $\psi = \{P_{x_1}, ..., P_{x_k}\}$ so that ψ is an α-SS-RF. Let $A = P_{x_1} \wedge ... \wedge P_{x_k}$. Then A is strongly semiclosed by Theorem 2.5 of [1], and A, as a function, has no maximum. In fact, take $n_0 \in D$ so that $n_0 \geq n(x_i)(i = 1, 2, ..., k)$. Then $S(n) \leq A$ for each $n \geq n_0$, i.e., S is eventually in A. Since ψ is an α-SS-RF, $A(x) < \alpha$ for each $x \in X$. On the other hand, by definition of α-net we know that for any real number $\epsilon > 0$, there is an $x \in X$ such that $A(x) > \alpha - \epsilon$. Hence it is impossible that A, as a function, has a maximum. □

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