COUNTABLY NEAR PS-COMPACTNESS IN L-TOPOLOGICAL SPACES

S. Z. BAI

Abstract. In this paper, the concept of countably near PS-compactness in L-topological spaces is introduced, where L is a completely distributive lattice with an order-reversing involution. Countably near PS-compactness is defined for arbitrary L-subsets and some of its fundamental properties are studied.

1. Introduction

We know that since fuzzy topological spaces were introduced by Chang [5], many authors have established various kinds of fuzzy compactness [1-3,5,6,8-10,12-14]. Originally fuzzy compactness [5,6,9] was defined only for the whole fuzzy topological space rather than for arbitrary fuzzy subsets. Then Wang [12] introduced a new concept based on the fuzzy nets of Pu and Liu [11] and called it nice compactness (N-compactness), which is defined for any fuzzy subset and has more advantages. Zhao [14] generalized N-compactness theory to L-topological spaces(where L is a completely distributive lattice with an order-reversing involution). This has all the advantages of Wang's theory, and has been extensively used (e.g. [7]) to further the study of fuzzy compactness. In [1-3], we studied SR-compactness, PS-compactness and near PS-compactness in L-topological spaces, respectively. Every PS-compact set is not only SR-compact but also near PS-compact [2,3]; every SR-compact set is N-compact [1], and every near PS-compact set is Lowen's fuzzy compact [9,3]. That the converse of the above statements need not be true, is shown by Examples 5.6 and 5.7 in [3].

In this paper we use near PS-compactness as a background for the introduction of the concept of countably near PS-compactness of arbitrary L-subsets in L-topological spaces. It is an "L-good extension [8,13]" of general countably PS-compactness and hereditary for pre-semiclosed subsets. Moreover, it is finitely additive and can be preserved under PS-irresolute mapping and described with the cover forms and finite intersection properties.

Received: December 2005; Revised: September 2006; Accepted: September 2006

Key words and phrases: L-topology, Pre-semiclosed set, Remote-neighborhood, Countably near PS-compact set.

This work is supported by the NNSFs of China, NSF of Guangdong Province and STF of Jiangmen City(No. 60473009, 60542001, 021358, [2005]102).

2. Preliminaries

In this paper, L always denotes a completely distributive lattice with an order-reversing involution " $'$ ", 0 and 1 denote the least and the greatest elements in L respectively, X denotes a nonempty crisp set and L^X denotes the set of all L-subsets on X. $r \in L$ is called a prime element of L, if $a \wedge b \leq r$ implies $a \leq r$ or $b \leq r$, where $a, b \in L$ [8,13]. The set of all prime elements of L which are not 1 is denoted by $pr(L)$. $\alpha \in L$ is called a union-irreducible element of L, if for arbitrary $a, b \in L$ with $\alpha \leq a \vee b$, either $\alpha \leq a$ or $\alpha \leq b$ [8,13]. $M(L)$ and $M^*(L^X)$ denote the set of all nonzero union-irreducible elements of L and L^X respectively. Clearly, $r \in pr(L)$ iff $r' \in M(L)$. For each $\psi \subset L$, we define $\psi' = \{A' : A \in \psi\}$. We will denote L-topological space by L-ts, and $\varepsilon_r(A) = \{x \in X : A(x) \geq r\}$.

Let (L^X, δ) be an L-ts. $A \in L^X$ is called a pre-semiopen set iff $A \leq (A^-)_o$, and A is called a pre-semiclosed set iff $A \geq (A^o)_-$, where A^o, A^-, A_o and A_- are the interior, closure, semi-interior and semiclosure of A, respectively. $PSO(L^X)$ and $PSC(L^X)$ will always denote the family of pre-semiopen sets and family of pre-semiclosed sets of an L-ts (L^X, δ), respectively. It is clear that every semiopen set is pre-semiopen and every preopen set is pre-semiopen in L-ts. That none of the converses need be true is shown by Example 3.3 in [4].

Definition 2.1. [2] Let (L^X, δ) be an L-ts and $x_\lambda \in M^*(L^X)$.$A \in PSC(L^X)$ is called a pre-semiclosed remote-neighborhood, or briefly, PSC-RN of x_λ , if $x_\lambda \notin A$. The set of all PSC-RNs of x_λ is denoted by $\zeta(x_\lambda)$.

Definition 2.2. [2] Let (L^X, δ) be an L-ts, $A \in L^X$ and $\alpha \in M(L)$, $\phi \subset PSC(L^X)$ is called an α-PSC-remote neighborhood family of A (briefly α-PSC-RF of A) if, for each x_α in A, there is $P \in \phi$ such that $P \in \zeta(x_\alpha)$.

Definition 2.3. [3] Let (L^X, δ) be an L-ts and $A \in L^X$. A is called near PS-compact if every α-PSC-RF ϕ of A has a finite subfamily ψ of ϕ such that ψ is an α-PSC-RF of $A (\alpha \in M(L))$. Specifically, when $A = 1_X$ is near PS-compact, we call (L^X, δ) a near PS-compact space.

3. Countably Near PS-compactness and its Characterizations

Definition 3.1. Let (L^X, δ) be an L-ts and $A \in L^X$. A is called a countably near PS-compact set if every countable α-PSC-RF Φ of A has a finite subfamily Ψ of Φ such that Ψ is an α-PSC-RF of $A (\alpha \in M(L))$. Specifically, when $A = 1_X$ is countably near PS-compact, we call (L^X, δ) a countably near PS-compact space.

Clearly, every near PS-compact set is countably near PS-compact.

Definition 3.2. [2] Let (L^X, δ) be an L-ts, $A \in L^X$, $\mu \subset PSO(L^X)$, $r \in pr(L)$. μ is called an r-PS-cover of A if for each $x \in \varepsilon_{r'}(A)$, there is $U \in \mu$ such that $U(x) \not\leq r$.

Theorem 3.3. *Let (L^X, δ) be an L-ts, $r \in pr(L)$. $A \in L^X$ is a countably near PS-compact set iff every countable r-PS-cover μ of A has a finite subfamily ν of μ such that ν is an r-PS-cover of A.*

Proof. Let A be a countably near PS-compact set, μ be a countable r-PS-cover of A and $r \in pr(L)$. Put $\Phi = \mu'$, then $\Phi \subset PSC(L^X)$ and for each $x \in \varepsilon_{r'}(A)$ there is $Q = U' \in \Phi$ such that $U(x) \not\leq r$, i.e., $r' \not\leq Q(x)$. Now $r \in pr(L)$, implies that $r' \in M(L)$ and since $x_{r'} \not\leq Q$, we have $Q \in \zeta(x_{r'})$. Hence Φ is an r'-PSC-RF of A. Since A is countably near PS-compact, there is a finite subfamily ν of μ such that $\Psi = \nu'$ is an r'-PSC-RF of A, i.e. for each $x \in \varepsilon_{r'}(A)$, there is $V \in \nu$ such that $V' \in \zeta(x_{r'})$, i.e. $(x_{r'}) \notin V'$ or, equivalently, $r' \not\leq V'(x)$; for each $x \in \varepsilon_{r'}(A)$, there is $V \in \nu$ such that $V(x) \not\leq r$. Thus μ has a finite subfamily ν which is an r-PS-cover of A.

Conversely, suppose every countable r-PS-cover of A has a finite subfamily which is an r-PS-cover of A. Let Φ be an α-PSC-RF of A, $\mu = \Phi'$ and $r = \alpha'$. Since $\alpha \in M(L)$, hence $r \in pr(L)$. For each $x \in \varepsilon_\alpha(A) = \varepsilon_{r'}(A)$, i.e. $x_\alpha \in A$, there is $P \in \Phi$ such that $p \in \zeta x_\alpha$, i.e. $x_\alpha \notin P$, or $\alpha \not\leq P(x)$. So $P'(x) \not\leq \alpha'$. Put $U = P'$, then $U \in \mu$ and $U(x) \not\leq r$. Hence μ is a countable r-PS-cover of A. By supposition, Φ has a finite subfamily Ψ such that Ψ' is an r-PS-cover of A. So for each $x \in \varepsilon_{r'}(A) = \varepsilon_\alpha(A)$, i.e. $x_\alpha \in A$, there is $Q \in \Psi$ such that $Q'(x) \not\leq r$. Hence, $r' = \alpha \not\leq Q(x)$, i.e. $Q \in \zeta(x_\alpha)$. This shows that Ψ is an α-PSC-RF of A. Thus A is countably near PS-compact. □

Definition 3.4. Let (L^X, δ) be an L-ts, $A \in L^X$, $r \in pr(L)$ and $\Phi \subset L^X$. If for every finite subfamily Ψ of Φ, there is $x \in \varepsilon_{r'}(A)$ such that $(\bigwedge \Psi)(x) \geq r'$, then we say that Φ has an r-finite intersection property in A.

Theorem 3.5. *Let (L^X, δ) be an L-ts, $A \in L^X$ and $r \in pr(L)$. A is a countably near PS-compact set iff for every countable subfamily $\Phi \subset PSC(L^X)$ which has the r-finite intersection property in A, there is $x \in \varepsilon_{r'}(A)$ such that $(\bigwedge \Phi)(x) \geq r'$.*

Proof. Let A be a countably near PS-compact set. Suppose there is an $r \in pr(L)$ and some countable subfamily $\Phi \subset PSC(L^X)$ which has an r-finite intersection property in A, for each $x \in \varepsilon_{r'}(A)$ such that $(\bigwedge \Phi)(x) \not\geq r'$. Then there exists $P \in \Phi$ such that $P(x) \not\geq r'$, i.e., $P'(x) \not\leq r$. This shows that Φ' is a countable r-PS-cover of A. By Theorem 3.3, there is a finite subfamily $\Psi = \{P_1, ..., P_n\}$ of Φ such that Ψ' is an r-PS-cover of A. Hence for each $x \in \varepsilon_{r'}(A)$, there is $P_i \in \Psi$ such that $P_i'(x) \not\leq r$. And so

$$(\bigvee_{i=1}^{n} P_i')(x) \not\leq r \text{ i.e. } (\bigwedge \Psi)(x) = (\bigwedge_{i=1}^{n} P_i)(x) \not\geq r',$$

which contradicts the fact that Φ has an r-finite intersection property in A.

Conversely, let μ be a countable r-PS-cover of A and $r \in pr(L)$. If none of the finite subfamily ν of μ is an r-PS-cover of A, then there exists $x \in \varepsilon_{r'}(A)$ such that $B(x) \leq r$ for each $B \in \nu$ and so $(\bigvee \nu)(x) \leq r$ or, equivalently, $(\bigwedge \nu')(x) \geq r'$. This shows that the subfamily $\mu' \subset PSC(L^X)$ has an r-finite intersection property in A. Hence there is $x \in \varepsilon_{r'}(A)$ such that $(\bigwedge \mu')(x) \geq r'$, i.e. $(\bigvee \mu)(x) \leq r$. This implies that μ is not a countable r-PS-cover of A, a contradiction. By Theorem 3.3, A is countably near PS-compact. □

Lemma 3.6. [2] *Let* (X, δ) *be a crisp topological space and* $A \subset X$*. If* A *is pre-semiopen in* (X, δ) *then* χ_A *is pre-semiopen in the* L*-ts* $(L^X, \omega_L(\delta))$*, where* $(L^X, \omega_L(\delta))$ *is* L*-ts topologically generated by a crisp topological space* (X, δ)*.*

Definition 3.7. Let (L^X, δ) be an L-ts, $A \in L^X$ and $r \in pr(L)$. Put $l_r(A) = \{x \in X : A(x) \not\leq r\}$, and $l_r(\delta) = \{l_r(A) : A \in \delta\}$, then $l_r(\delta)$ is a crisp topology on X, called the r-cut topology of δ [13]. Also, $l_r(PSO(L^X)) = \{l_r(A) : A \in PSO(L^X)\}$ is the family of pre-semiopen sets in r-cut topological space $(X, l_r(\delta))$.

We say that a topological space (X, δ) is countably PS-compact iff every countable PS-cover of X has a finite subcover.

Theorem 3.8. *Let* (L^X, δ) *be an* L*-ts. Then* $A \in L^X$ *is countably near PS-compact iff the subset* $\varepsilon_{r'}(A)$ *of* $(X, l_r(\delta))$ *is countably PS-compact for each* $r \in pr(L)$*.*

Proof. Necessity. For any $r \in pr(L)$, suppose that μ is a countable PS-cover of $\varepsilon_{r'}(A)$. Then there exists a countable family $\psi \in PSO(L^X)$ such that $\mu = l_r(\psi) = \{l_r(U) : U \in \psi\}$. For each $x \in \varepsilon_{r'}(A)$, there is $U \in \psi$ such that $x \in l_r(U)$, i.e. $U(x) \not\leq r$. Hence ψ is a countable r-PS-cover of A. Since A is countably near PS-compact, from Theorem 3.3 there is a finite subfamily ν of ψ such that ν is an r-PS-cover of A. We now prove that a finite subfamily $l_r(\nu)$ is a PS-cover of $\varepsilon_{r'}(A)$. In fact, for each $x \in \varepsilon_{r'}(A)$, by ν is an r-PS-cover of A there is $U \in \nu$ such that $U(x) \not\leq r$, i.e. $x \in l_r(U) \in l_r(\nu)$. This shows that $l_r(\nu)$ is indeed a PS-cover of $\varepsilon_{r'}(A)$, and thus $\varepsilon_{r'}(A)$ is countably PS-compact.

Sufficiency. Suppose that μ is a countable r-PS-cover of $A(r \in pr(L))$. Then for each $x \in \varepsilon_{r'}(A)$, there exists $U \in \mu$ such that $U(x) \not\leq r$, i.e. $x \in l_r(U)$. Hence $l_r(\mu) = \{l_r(U) : U \in \mu\}$ is a countable PS-cover of $\varepsilon_{r'}(A)$. Since $\varepsilon_{r'}(A)$ is countably PS-compact, there is a finite subfamily ν of μ such that $l_r(\nu)$ is a PS-cover of $\varepsilon_{r'}(A)$. It is not difficult to see that ν is an r-PS-cover of A. By Theorem 3.3, A is countably near PS-compact. □

Theorem 3.9. *Let* $(L^X, \omega_L(\delta))$ *be an* L*-ts topologically generated by a crisp topological space* (X, δ)*. Then* $(L^X, \omega_L(\delta))$ *is countably near PS-compact iff* (X, δ) *is countably PS-compact.*

Proof. Necessity. Suppose that μ is a countable PS-cover of (X, δ). Then, by Lemma 3.6, $\chi_\mu = \{\chi_E : E \in \mu\}$ is a family of semi-pre-open sets of $(L^X, \omega_L(\delta))$. For each $r \in pr(L)$ we will prove that χ_μ is a countable r-PS-cover of 1_X. In fact, for each $x \in \varepsilon_{r'}(1_X) = X$, there exists $E \in \mu$ such that $x \in E$, and so $\chi_E(x) = 1 \not\leq r$. Hence, χ_μ is a countable r-PS-cover of 1_X. Since $(L^X, \omega_L(\delta))$ is countably near PS-compact, there is a finite subcover ν of μ such that χ_ν is an r-PS-cover of 1_X, i.e. for each $x \in X$, there exists $E \in \nu$ such that $\chi_E(x) \not\leq r$, and so $x \in E$. This shows that ν is a PS-cover of X, and hence (X, δ) countably PS-compact.

Sufficiency. For any $r \in pr(L)$, suppose that μ is a countable r-PS-cover of 1_X. Then for each $x \in \varepsilon_{r'}(1_X) = X$, there exists $U \in \mu$ such that $U(x) \not\leq r$, i.e. $x \in l_r(U)$, where $l_r(U)$ is pre-semiopen set in (X, δ). Hence, $l_r(\mu) = \{l_r(U) : U \in \mu\}$ is a countable PS-cover of (X, δ). Since (X, δ) is countably PS-compact, there is a finite subcover ν of μ such that $l_r(\nu)$ is a cover of (X, δ). Thus, for each

$x \in X = \varepsilon_{r'}(1_X)$, there exists $U \in \nu$ such that $x \in l_r(U) \in l_r(\nu)$, i.e. $U(x) \not\leq r$. This shows that ν is an r-PS-cover of 1_X, and hence $(L^X, \omega_L(\delta))$ is countably near PS-compact. □

4. Some Other Properties

Theorem 4.1. *Let A be a countably near PS-compact set in L-ts (L^X, δ). Then for each $B \in PSC(L^X)$, $A \wedge B$ is countably near PS-compact.*

Proof. Let Φ be a countable α-PSC-RF of $A \wedge B$ $(\alpha \in M(L))$and put $\Phi_1 = \Phi \cup \{B\}$. Then Φ is a countable α-PSC-RF of A. In fact, for each $x_\alpha \in A$, if $x_\alpha \in B$ then $x_\alpha \in A \wedge B$. Hence, there is $P \in \Phi \subset \Phi_1$ such that $P \in \zeta(x_\alpha)$. If $x_\alpha \notin B$, then $B \in \Phi_1$ and $B \in \zeta(x_\alpha)$. Thus, Φ is indeed a countable α-PSC-RF of A. Since A is a countably near PS-compact set, there exists a finite subfamily Ψ_1 of Φ_1 such that Ψ_1 is an α-PSC-RF of A. Let $\Psi = \Psi_1 - \{B\}$, then Ψ is a finite subfamily of Φ, and Ψ is an α-PSC-RF of $A \wedge B$. In fact, $x_\alpha \in A \wedge B$ implies $x_\alpha \in A$ and hence from the definition of Ψ_1, there exists $P \in \Psi_1$ with $P \in \zeta(x_\alpha)$. However, $x_\alpha \in B$ so $P \neq B$, and thus $P \in \Psi_1 - \{B\} = \Psi$. Hence, $A \wedge B$ is countably near PS-compact. □

Corollary 4.2. *Let (L^X, δ) be a countably near PS-compact space and $B \in PSC(L^X)$. Then B is countably near PS-compact.*

Theorem 4.3. *Let A and B be two countably near PS-compact sets in L-ts (L^X, δ). Then $A \vee B$ is countably near PS-compact.*

Proof. Let Φ be a countable α-PSC-RF of $A \vee B$ $(\alpha \in M(L))$. Then Φ is not only a countable α-PSC-RF of A, but also a countable α-PSC-RF of B. Since A is countably near PS-compact, there is a finite subfamily Ψ_1 of Φ such that Ψ_1 is an α-PSC-RF of A. Similarly, since B is countably near PS-compact, there is a finite subfamily Ψ_2 of Φ such that Ψ_2 is an α-PSC-RF of B. Put $\Psi = \Psi_1 \vee \Psi_2$. Then Ψ is a finite subfamily of Φ and Ψ is an α-PSC-RF of $A \vee B$. Thus, $A \vee B$ is countably near PS-compact. □

Definition 4.4. [2] Let (L^X, δ) and (L^Y, τ) be two L-ts's and $f : (L^X, \delta) \to (L^Y, \tau)$ an L-mapping. f is called a PS-irresolute mapping if $f^{-1}(B) \in PSO(L^X)$ for each $B \in PSO(L^Y)$.

Theorem 4.5. *Let $f : (L^X, \delta) \to (L^Y, \tau)$ be an PS-irresolute mapping and A be a countably near PS-compact set in (L^X, δ). Then $f(A)$ is countably near PS-compact in (L^Y, τ).*

Proof. Let Φ be a countable α-PSC-RF of $f(A)(\alpha \in M(L))$. To begin with, we show that $f^{-1}(\Phi) = \{f^{-1}(P) : P \in \Phi\}$ is a countable α-PSC-RF of A. Since f is a PS-irresolute mapping, $f^{-1}(\Phi) \subset PSC(L^X)$. Let $x_\alpha \in A$; then $f(x_\alpha) = (f(x))_\alpha \in f(A)$ and since ϕ is a countable α-PSC-RF of $f(A)$, there is $P \in \Phi$ with $P \in \zeta((f(x))_\alpha)$, i.e., $(f(x))_\alpha \notin P$ or, equivalently, $P(f(x)) \not\geq \alpha$. By the definition of inverse mapping, $f^{-1}(P)(x) = P(f(x)) \not\geq \alpha$, hence $x_\alpha \notin f^{-1}(P)$, i.e., $f^{-1}(P) \in \zeta(x_\alpha)$. Therefore $f^{-1}(\Phi)$ is a countable α-PSC-RF of A. Since A is countably near PS-compact, there is a finite subfamily Ψ of Φ such that $f^{-1}(\Psi)$ is an α-PSC-RF of A. It is easy to show that Ψ is an α-PSC-RF of $f(A)$. Thus $f(A)$ is countably near PS-compact. □

References

[1] S. Z. Bai, *The SR-compactness in L-fuzzy topological spaces*, Fuzzy Sets and Systems, **87** (1997), 219-225.
[2] S. Z. Bai, *L-fuzzy PS-compactness*, IJUFKS, **10** (2002), 201-209.
[3] S. Z. Bai, *Near PS-compact L-subsets*, Information Sciences, **115** (2003), 111-118.
[4] S. Z. Bai, *Pre-semiclosed sets and PS-convergence in L-fuzzy topological spaces*, J. Fuzzy Math. **9** (2001), 497-509.
[5] C. L. Chang, *Fuzzy topological spaces*, J.Math.Anal.Appl. **24** (1968), 182-190.
[6] B. Hutton, *Products of fuzzy topological spaces*, Topology Appl. **11** (1980), 59-67.
[7] Y. M. Liu and M. K. Luo, *Induced spaces and fuzzy Stone-Cech compactifications*, Scientia Sinica (A), **30** (1987), 1034-1044.
[8] Y. M. Liu and M. K. Luo, *Fuzzy topology*, World Scientific Publishing, Singapore, 1998.
[9] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J.Math.Anal.Appl. **56** (1976), 621-633.
[10] F. G. Shi, *Countable compactness and Lindeloff property of L-fuzzy sets*, Iranian journal of fuzzy systems, **1** (2004), 79-88.
[11] B. M. Pu and Y. M. Liu, *Fuzzy topological,I.Neighborhood structure of a fuzzy point and Moore-Smith convergence0*, J. Math. Anal. Appl. **76** (1980), 571-599.
[12] G. J. Wang, *A new fuzzy compactness defined by fuzzy nets*, J.Math.Anal.Appl. **94** (1983), 1-23.
[13] G. J. Wang, *Theory of L-fuzzy topological spaces*, Shaanxi Normal University, Xian, 1988.
[14] D. S. Zhao, *The N-compactness in L-fuzzy topological spaces*, J. Math. Anal. Appl. **128** (1987), 64-79.

Shi-Zhong Bai, Department of Mathematics, Wuyi University, Guangdong 529020, P.R.China
E-mail address: shizhongbai@yahoo.com