A COMMON FIXED POINT THEOREM FOR ψ-WEAKLY COMMUTING MAPS IN $\mathcal{L}$-FUZZY METRIC SPACES

R. SAADATI, S. SEDGHI AND H. ZHOU

ABSTRACT. In this paper, a common fixed point theorem for ψ-weakly commuting maps in $\mathcal{L}$-fuzzy metric spaces is proved.

1. **Introduction and Preliminaries**

The notion of fuzzy sets was introduced by Zadeh [26] and various concepts of fuzzy metric spaces were considered in [7, 8, 14, 15]. Many authors have studied fixed point theory in fuzzy metric spaces. The most interesting references are [3, 4, 10, 11, 16, 18, 25].

In the sequel, we shall adopt the usual terminology, notation and conventions of $\mathcal{L}$-fuzzy metric spaces introduced by Saadati et al. [21, 22] and [1].

Definition 1.1. [10] Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a non-empty set called universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \longrightarrow L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.

Lemma 1.2. [6] *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1$ *and* $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice .*

Definition 1.3. [2] An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ on a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Classically, a triangular norm T on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, x) = x$, for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 1.4. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$; (boundary condition)
(ii) $(\forall (x, y) \in L^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$; (commutativity)

Received: June 2006; Revised: December 2006; Accepted: February 2007
Key words and phrases: $\mathcal{L}$-Fuzzy contractive mapping, Complete $\mathcal{L}$-fuzzy metric space, Common fixed point theorem, ψ-weakly commuting maps.

(iii) $(\forall(x,y,z)\in L^3)(\mathcal{T}(x,\mathcal{T}(y,z))=\mathcal{T}(\mathcal{T}(x,y),z))$; (associativity)
(iv) $(\forall(x,x',y,y')\in L^4)(x\leq_L x'$ and $y\leq_L y'\Rightarrow \mathcal{T}(x,y)\leq_L \mathcal{T}(x',y'))$. (monotonicity)

A t-norm can also be defined recursively as an $(n+1)$-ary operation ($n\in\mathbf{N}\setminus\{0\}$) by $\mathcal{T}^1=\mathcal{T}$ and

$$\mathcal{T}^n(x_{(1)},\cdots,x_{(n+1)})=\mathcal{T}(\mathcal{T}^{n-1}(x_{(1)},\cdots,x_{(n)}),x_{(n+1)})$$

for $n\geq 2$ and $x_{(i)}\in L$.

Definition 1.5. [5] A t-norm $\mathcal{T}$ on L^* is called *t-representable* if and only if there exist a t-norm T and a t-conorm S on $[0,1]$ such that, for all $x=(x_1,x_2),y=(y_1,y_2)\in L^*$,

$$\mathcal{T}(x,y)=(T(x_1,y_1),S(x_2,y_2)).$$

Definition 1.6. A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N}:L\to L$ satisfying $\mathcal{N}(0_\mathcal{L})=1_\mathcal{L}$ and $\mathcal{N}(1_\mathcal{L})=0_\mathcal{L}$. If $\mathcal{N}(\mathcal{N}(x))=x$, for all $x\in L$, then $\mathcal{N}$ is called an involutive negation.

If, for all $x\in[0,1]$, $N_s(x)=1-x$, we say that N_s is the standard negation on $([0,1],\leq)$.

Definition 1.7. The 3-tuple $(X,\mathcal{M},\mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t–norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2\times\,]0,+\infty[$ satisfying the following conditions for every x,y,z in X and t,s in $]0,+\infty[$:

(a) $\mathcal{M}(x,y,t)>_L 0_\mathcal{L}$;
(b) $\mathcal{M}(x,y,t)=1_\mathcal{L}$ for all $t>0$ if and only if $x=y$;
(c) $\mathcal{M}(x,y,t)=\mathcal{M}(y,x,t)$;
(d) $\mathcal{T}(\mathcal{M}(x,y,t),\mathcal{M}(y,z,s))\leq_L \mathcal{M}(x,z,t+s)$;
(e) $\mathcal{M}(x,y,\cdot):]0,\infty[\to L$ is continuous.

In this case $\mathcal{M}$ is called an *$\mathcal{L}$-fuzzy metric*. If $\mathcal{M}=\mathcal{M}_{M,N}$ is an intuitionistic fuzzy set (see Definition 1.3) then the 3-tuple $(X,\mathcal{M}_{M,N},\mathcal{T})$ is said to be an *intuitionistic fuzzy metric space*.

Example 1.8. [24] *Let (X,d) be a metric space. Set $\mathcal{T}(a,b)=(a_1b_1,\min(a_2+b_2,1))$ for all $a=(a_1,a_2)$ and $b=(b_1,b_2)$ in L^* and let M and N be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:*

$$\mathcal{M}_{M,N}(x,y,t)=(M(x,y,t),N(x,y,t))=(\frac{t}{t+md(x,y)},\frac{d(x,y)}{t+d(x,y)}),$$

in which $m>1$. Then $(X,\mathcal{M}_{M,N},\mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 1.9. [22] Let $X=\mathbf{N}$. Define $\mathcal{T}(a,b)=(\max(0,a_1+b_1-1),a_2+b_2-a_2b_2)$ for all $a=(a_1,a_2)$ and $b=(b_1,b_2)$ in L^* and let M and N be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t)=(M(x,y,t),N(x,y,t))=\begin{cases}(\frac{x}{y},\frac{y-x}{y}) & if\quad x\leq y\\ (\frac{y}{x},\frac{x-y}{x}) & if\quad y\leq x.\end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in]0, +\infty[$, we define the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_\mathcal{M}$ denote the family of all open subsets of X. Then $\tau_\mathcal{M}$ is called the *topology induced by the* $\mathcal{L}$*-fuzzy metric* $\mathcal{M}$.

Lemma 1.10. [9] *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x, y, t)$ is nondecreasing with respect to t, for all x, y in X.*

Proof. Let $t, s \in]0, +\infty[$ be such that $t < s$. Then $k = s - t > 0$ and

$$\mathcal{M}(x, y, t) = \mathcal{T}(\mathcal{M}(x, y, t), 1_\mathcal{L}) = \mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, y, k)) \leq_L \mathcal{M}(x, y, s).$$

□

Definition 1.11. A sequence $\{x_n\}_{n \in \mathbf{N}}$ in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a *Cauchy sequence*, if for each $\varepsilon \in L \setminus \{0_\mathcal{L}\}$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$,

$$\mathcal{M}(x_m, x_n, t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}_{n \in \mathbf{N}}$ is said to be *convergent* to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_\mathcal{L}$ whenever $n \to +\infty$ for every $t > 0$. A $\mathcal{L}$-fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent.

Definition 1.12. Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times]0, \infty[$ if

$$\lim_{n \to \infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times]0, \infty[$ converges to a point $(x, y, t) \in X \times X \times]0, \infty[$ i.e., $\lim_n \mathcal{M}(x_n, x, t) = \lim_n \mathcal{M}(y_n, y, t) = 1_\mathcal{L}$ and $\lim_n \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)$.

Lemma 1.13. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is a continuous function on $X \times X \times]0, \infty[$.*

Proof. The proof is same as for fuzzy metric spaces (see Proposition 1 of [20]). □

2. The Main Results

Definition 2.1. Let f and g be maps from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. The maps f and g are said to be weakly commuting if

$$\mathcal{M}(fgx, gfx, t) \geq_L \mathcal{M}(fx, gx, t)$$

for each x in X and $t > 0$.

Definition 2.2. Let f and g be maps from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. The maps f and g are said to be ψ-weakly commuting if there exists a positive real function $\psi : (0, \infty) \to (0, \infty)$ such that

$$\mathcal{M}(fgx, gfx, t) \geq_L \mathcal{M}(fx, gx, \psi(t))$$

for each x in X and $t > 0$.

Weak commutativity implies ψ-weak commutativity in $\mathcal{L}$-fuzzy metric spaces. However, ψ-weak commutativity implies weak commutativity only when $\psi(t) \geq t$.

Example 2.3. *Let $X = \mathbf{R}$. Let $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{M}_{M,N}$ be the intuitionistic fuzzy set on $X \times X \times]0, +\infty[$ defined as follows:*

$$\mathcal{M}_{M,N}(x, y, t) = \left((\exp(\frac{|x - y|}{t}))^{-1}, \frac{\exp(\frac{|x-y|}{t}) - 1}{\exp(\frac{|x-y|}{t})} \right),$$

for all $t \in \mathbf{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space. Define $f(x) = 2x - 1$ and $g(x) = x^2$. Then

$$\begin{aligned}
\mathcal{M}_{M,N}(fgx, gfx, t) &= \left((\exp(2\frac{|x - 1|^2}{t}))^{-1}, \frac{\exp(2\frac{|x-1|^2}{t}) - 1}{\exp(2\frac{|x-1|^2}{t})} \right) \\
&= \left((\exp(\frac{|x - 1|^2}{t/2}))^{-1}, \frac{\exp(\frac{|x-1|^2}{t/2}) - 1}{\exp(\frac{|x-1|^2}{t/2})} \right) = \mathcal{M}_{M,N}(fx, gx, t/2) \\
&<_{L^*} \left((\exp(\frac{|x - 1|^2}{t}))^{-1}, \frac{\exp(\frac{|x-1|^2}{t}) - 1}{\exp(\frac{|x-1|^2}{t})} \right) = \mathcal{M}_{M,N}(fx, gx, t)
\end{aligned}$$

Therefore, for $\psi(t) = t/2$, f and g are ψ-weakly commuting. But f and g are not weakly commuting since the exponential function is strictly increasing.

Theorem 2.4. *Let $(X, \mathcal{M}, \mathcal{T})$ be a left complete $\mathcal{L}$-fuzzy metric space and let f and g be ψ-weakly commuting self-mappings of X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$;

(b) *Either f or g is continuous;*

(c) $\mathcal{M}(fx, fy, t) \geq_L \mathcal{C}(\mathcal{M}(gx, gy, t))$, *where $\mathcal{C} : L \longrightarrow L$ is a continuous function such that $\mathcal{C}(a) >_L a$ for each $a \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$.*

Then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. By (a), choose a point x_1 in X such that $fx_0 = gx_1$. In general choose x_{n+1} such that $fx_n = gx_{n+1}$. Then for $t > 0$,

$$\begin{aligned}
\mathcal{M}(fx_n, fx_{n+1}, t) &\geq_L \mathcal{C}(\mathcal{M}(gx_n, gx_{n+1}, t)) = \mathcal{C}(\mathcal{M}(fx_{n-1}, fx_n, t)) \\
&>_L \mathcal{M}(fx_{n-1}, fx_n, t)
\end{aligned}$$

Thus $\{\mathcal{M}(fx_n, fx_{n+1}, t); n \geq 0\}$ is an increasing sequence in L and therefore, tends to a limit $a \leq_L 1_\mathcal{L}$. We claim that $a = 1_\mathcal{L}$. For if $a <_L 1_\mathcal{L}$, when $n \longrightarrow \infty$ in the above inequality we get $a \geq_L \mathcal{C}(a) >_L a$, a contradiction. Hence $a = 1_{\mathcal{L}}$, i.e.,

$$\lim_n \mathcal{M}(fx_n, fx_{n+1}, t) = 1_{\mathcal{L}}.$$

If we define

$$c_n(t) = \mathcal{M}(fx_n, fx_{n+1}, t) \tag{2.1}$$

then $\lim_{n\to\infty} c_n(t) = 1_{\mathcal{L}}$. Now, we prove that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Suppose that $\{fx_n\}$ is not a Cauchy sequence in $f(X)$. For convenience, let $y_n = fx_n$ for $n = 1, 2, 3, \cdots$. Then there is an $\epsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that for each integer k, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that

$$d_k(t) = \mathcal{M}(y_{n(k)}, y_{m(k)}, t) \leq \mathcal{N}(\epsilon) \quad \text{for} \quad k = 1, 2, \cdots. \tag{2.2}$$

We may assume that

$$\mathcal{M}(y_{n(k)}, y_{m(k)-1}, t) > \mathcal{N}(\epsilon), \tag{2.3}$$

by choosing $m(k)$ to be the smallest number exceeding $n(k)$ for which (2.2) holds. Using (2.1), we have

(2.4)

$$\mathcal{N}(\epsilon) \geq d_k(t) \geq \mathcal{T}(\mathcal{M}(y_{n(k)}, y_{m(k)-1}, t/2), \mathcal{M}(y_{m(k)-1}, y_{m(k)}, t/2)) \geq \mathcal{T}(c_k(t/2), \mathcal{N}(\epsilon))$$

Hence, $d_k(t) \longrightarrow \mathcal{N}(\epsilon)$ for every $t > 0$ as $k \longrightarrow \infty$.

We note that

$$\begin{aligned} & d_k(t) = \mathcal{M}(y_{n(k)}, y_{m(k)}, t) \\ \geq\ & \mathcal{T}^2(\mathcal{M}(y_{n(k)}, y_{n(k)+1}, t/3), \mathcal{M}(y_{n(k)+1}, y_{m(k)+1}, t/3), \mathcal{M}(y_{m(k)+1}, y_{m(k)}, t/3) \\ \geq\ & \mathcal{T}^2(c_k(t/3), \mathcal{C}(\mathcal{M}(y_{n(k)}, y_{m(k)}, t/3)), c_k(t/3)) \\ =\ & \mathcal{T}^2(c_k(t/3), \mathcal{C}(d_k(t/3)), c_k(t/3)). \end{aligned}$$

Thus, as $k \longrightarrow \infty$ in the above inequality we have

$$\mathcal{N}(\epsilon) \geq \mathcal{C}(\mathcal{N}(\epsilon)) > \mathcal{N}(\epsilon)$$

which is a contradiction. Thus, $\{fx_n\}_n$ is Cauchy and by the completeness of X, $\{fx_n\}_n$ converges to z in X. Also $\{gx_n\}_n$ converges to z in X. Let us suppose that the mapping f is continuous. Then $\lim_n ffx_n = fz$ and $\lim_n fgx_n = fz$. Further we have since f and g are ψ-weakly commuting

$$\mathcal{M}(fgx_n, gfx_n, t) \geq_L \mathcal{M}(fx_n, gx_n, \psi(t)).$$

On letting $n \to \infty$ in the above inequality we get $\lim_n gfx_n = fz$, by Lemma 1.13. We now prove that $z = fz$. Suppose $z \neq fz$ then $\mathcal{M}(z, fz, t) <_L 1_{\mathcal{L}}$. By (c)

$$\mathcal{M}(fx_n, ffx_n, t) \geq_L \mathcal{C}(\mathcal{M}(gx_n, gfx_n, t)).$$

Letting $n \to \infty$ in the above inequality we get

$$\mathcal{M}(z, fz, t) \geq_L \mathcal{C}(\mathcal{M}(z, fz, t)) >_L \mathcal{M}(z, fz, t),$$

a contradiction. Therefore, $z = fz$. Since $f(X) \subseteq g(X)$ we can find z_1 in X such that $z = fz = gz_1$. Now,

$$\mathcal{M}(ffx_n, fz_1, t) \geq_L \mathcal{C}(\mathcal{M}(gfx_n, gz_1, t)).$$

Taking limits as $n \to \infty$ we get

$$\mathcal{M}(fz, fz_1, t) \geq_L \mathcal{C}(\mathcal{M}(fz, gz_1, t)) = 1_{\mathcal{L}}$$

. Since $\mathcal{C}(1_{\mathcal{L}}) = 1_{\mathcal{L}}$, this implies that $fz = fz_1$, i.e., $z = fz = fz_1 = gz_1$. Also for any $t > 0$,

$$\mathcal{M}(fz, gz, t) = \mathcal{M}(fgz_1, gfz_1, t) \geq_L \mathcal{M}(fz_1, gz_1, \psi(t)) = 1_{\mathcal{L}}$$

which again implies that $fz = gz$. Thus z is a common fixed point of f and g.

Now, to prove uniqueness suppose $z' \neq z$ is another common fixed point of f and g. Then there exists $t > 0$ such that $\mathcal{M}(z, z', t) <_L 1_{\mathcal{L}}$, and

$$\begin{aligned}\mathcal{M}(z, z', t) &= \mathcal{M}(fz, fz', t) \geq_L \mathcal{C}(\mathcal{M}(gz, gz', t)) = \mathcal{C}(\mathcal{M}(z, z', t)) \\ &>_L \mathcal{M}(z, z', t)\end{aligned}$$

which is contradiction. Therefore, $z = z'$, i.e., z is a unique common fixed point of f and g. □

Example 2.5. *Consider Example 1.8 in which $X = [0, 1]$. Define $f(x) = 1$ and*

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}, \end{cases}$$

on X. It is evident that $f(X) \subseteq g(X)$, f is continuous and g is discontinuous. Define $\mathcal{C} : L^* \to L^*$ by $\mathcal{C}(a) = (\sqrt{a_1}, a_2^2)$, then

$$\mathcal{C}(a) = (\sqrt{a_1}, a_2^2) >_{L^*} (a_1, a_2) = a$$

for $0 < a_i < 1,\ i = 1, 2$ and

$$\mathcal{M}(fx, fy, t) \geq_{L^*} \mathcal{C}(\mathcal{M}(gx, gy, t))$$

for all x, y in X, f and g are ψ-weakly commuting. Thus all the conditions of last theorem are satisfied and 1 is a common fixed point of f and g.

Acknowledgements. The authors would like to thank the referees for their useful comments and suggestions.

References

[1] H. Adibi, Y. J. Cho, D. O'Regan and R. Saadati, *Common fixed point theorems in $\mathcal{L}$-fuzzy metric spaces*, Appl. Math. Comput., **182** (2006), 820-828.

[2] A. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.

[3] S. S. Chang, Y. J. Cho, B. S. Lee, J. S. Jung and S. M. Kang, *Coincidence point and minimization theorems in fuzzy metric spaces*, Fuzzy Sets and Systems, **88** (1997), 119-128.

[4] Y. J. Cho, H. K. Pathak, S. M. Kang and J. S. Jung, *Common fixed points of compatible maps of type (β) on fuzzy metric spaces*, Fuzzy Sets and Systems, **93** (1998), 99-111.

[5] G. Deschrijver , C. Cornelis and E. E. Kerre, *On the representation of intuitionistic fuzzy t-norms and t-conorms*, IEEE Transactions on Fuzzy Systems, **12** (2004), 45-61.

[6] G. Deschrijver and E. E Kerre, *On the relationship between some extensions of fuzzy set theory*, Fuzzy Sets and Systems, **33** (2003), 227-235.

[7] Z. K. Deng, *Fuzzy pseduo-metric spaces*, J. Math. Anal. Appl., **86** (1982), 74-95.

[8] M. A. Erceg, *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl., **69** (1979), 205-230.

[9] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64** (1994), 395-399.

[10] J. Goguen, *$\mathcal{L}$-fuzzy sets*, J. Math. Anal. Appl., **18** (1967), 145-174.

[11] V. Gregori and A. Sapena, *On fixed point theorem in fuzzy metric spaces*, Fuzzy Sets and Systems, **125** (2002), 245-252.

[12] O. Hadžić and E. Pap, *Fixed point theory in PM spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
[13] O. Hadžić and E. Pap, *New classes of probabilistic contractions and applications to random operators*, in: Y. J. Cho, J. K. Kim and S. M. Kong (Eds.), *Fixed point theory and application*, Nova Science Publishers, Hauppauge, NewYork, **4** (2003), 97-119.
[14] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems, **12** (1984), 215-229.
[15] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, **11** (1975), 326-334.
[16] D. Miheʘt, *A Banach contraction theorem in fuzzy metric spaces*, Fuzzy Sets and Systems, **144** (2004), 431-439.
[17] S. B. Hosseini, D. O'Regan and R. Saadati, *Some results on intuitionistic fuzzy spaces*, Iranian J. Fuzzy Systems, **4** (2007), 53-64.
[18] E. Pap, O. Hadzic and R. Mesiar, *A fixed point theorem in probabilistic metric spaces and an application*, J. Math. Anal. Appl., **202** (1996), 433-449.
[19] A. Razani and M. Shirdaryazdi, *Some results on fixed points in the fuzzy metric space*, J. Appl. Math. and Computing, **20** (2006), 401-408.
[20] J. Rodríguez López and S. Ramaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, **147** (2004), 273-283.
[21] R. Saadati, *Notes to the paper "Fixed points in intuitionistic fuzzy metric spaces" and its generalization to $\mathcal{L}$-fuzzy metric spaces*, Chaos, Solitons and Fractals, **35**(2008), 176-180.
[22] R. Saadati, A. Razani and H. Adibi, *A Common fixed point theorem in $\mathcal{L}$-fuzzy metric spaces*, Chaos, Solitons and Fractals, **33** (2007), 358-363.
[23] R. Saadati and J. H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos, Solitons and Fractals, **27** (2006), 331-344.
[24] R. Saadati and J. H. Park, *Intuitionistic fuzzy Euclidean normed spaces*, Commun. Math. Anal., **1(2)** (2006), 86-90.
[25] S. Sessa and B. Fisher, *On common fixed points of weakly commuting mappings and set valued mappings*, Internat. J. Math. Math. Sci., **9(2)** (1986), 323-329.
[26] L. A. Zadeh, *Fuzzy sets*, Inform. and control, **8** (1965), 338-353.

R. Saadati*, Faculty of Sciences, University of Shomal, Amol, P.O. Box 731, Iran
E-mail address: `rsaadati@eml.cc`

S. Sedghi, Department of Mathematics, Islamic Azad University-Ghaemshahr Branch, Ghaemshar, P.O. Box 163, Iran

H. Zhou, Department of Mathematics, Shijiazhuang Mechnical Engineering University, Shijiazhuang 050003, People's Republic of China

*Corresponding author