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FUZZY IDEALS AND FUZZY LIMIT STRUCTURES

Y. YUE AND J. FANG

ABSTRACT. In this paper, we establish the theory of fuzzy ideal convergence on completely distributive lattices and give characterizations of some topological notions. We also study fuzzy limit structures and discuss the relationship between fuzzy co-topologies and fuzzy limit structures.

1. Introduction

Since Chang [2] introduced fuzzy set theory to topology, many researchers have tried successfully to generalize the theory of general topology to a fuzzy setting using crisp methods. The fundamental idea of a topology itself being fuzzy, first appeared in 1980 in [4] and again in 1991 in [15], in which a topology was an L-subset of a traditional powerset. This was followed by L-subsets of L^X in 1985 in the independent and parallel generalizations of Kubiak [7] and Šostak [11]. In [16], we studied topological molecular lattices in the Kubiak-Šostak sense ; we called these lattices fuzzy topological molecular lattices (FTML) and established fuzzy remote neighborhood systems.

In posets, the concepts of filter and ideal are dual to each other, and both these concepts are very useful when studying problems concerning ordered structures. Since an FTML is, in fact, an ordered structure, it is natural to use these two tools to study its properties. It is well known that the convergence of ideal (or filter) is an very important part in fuzzy topology. One aim of this paper is to establish the convergence theory of fuzzy ideals in FTML.

In addition, limit structures provide a good tool for interpreting topological structure and play an important role in fuzzy topology. In the framework of L-topology, K. C. Min [6] introduced fuzzy limit spaces using prefilters and in the framework of fuzzifying topology, Xu [13] introduced fuzzifying topological limit structures and characterized fuzzifying topologies by filter convergence structures. Yang [14] studied ideals on completely distributive lattices and Li [8] established limit structures on completely distributive lattices using the usual ideals. Another purpose of this paper is to study fuzzy limit structures using fuzzy ideals in the framework of FTML.

This paper is organized as follows: In section 2, the concept of a fuzzy ideal and its properties are introduced. In section 3, we study convergence of fuzzy ideals and give characterizations of some topological notions. In section 4, we discuss

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relations between fuzzy co-topologies and fuzzy limit structures and find that we can use fuzzy limit structures to interpret fuzzy co-topology. In section 5, we prove, from a categorical point of view, that **FLimML** is a topological category over $\mathbf{CD}^{op}$ and **FTML** is a bireflective full subcategory of **FlimML**.

2. Preliminaries

Let a, b be elements in a complete lattice L. An element $a \in L$ is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of all coprimes of L is denoted by $M(L)$. Let $e|a$ denote the set $\{b \in L | e \not\leq b, b \geq a\}$ and $\beta^*(e)$ denote the standard minimal set of e for $e \in M(L)$. For basic results about completely distributive lattices we refer to [3, 9].

Definition 2.1. [12] Let L_1 and L_2 be two complete lattices. A map $f : L_1 \to L_2$ is called a generalized order-homomorphism, or briefly, a GOH, if
(1) $f(a) = 0$ if and only if $a = 0$;
(2) f is union-preserving;
(3) $f^{\vdash}$ is union-preserving, where $f^{\vdash}(b) = \bigvee\{a \in L_1 | f(a) \leq b\}$ for all $b \in L_2$.

Definition 2.2. [7, 11, 16] Let L be a completely distributive lattice. A fuzzy co-topology is a map $\eta : L \to [0, 1]$ such that-
(FCT1) $\eta(1) = \eta(0) = 1$;
(FCT2) $\eta(u \vee v) \geq \eta(u) \wedge \eta(v)$ for all $u, v \in L$;
(FCT3) $\eta(\bigwedge\limits_{j\in J} u_j) \geq \bigwedge\limits_{j\in J} \eta(u_j)$ for every family $\{u_j | j \in J\} \subseteq L$.

If η is a fuzzy co-topology, then we say that (L, η) is a fuzzy topological molecular lattice (FTML, for short). The value $\eta(u)$ can be interpreted as the degree of closeness of $u \in L$. A continuous map between two FTMLs (L_1, η) and (L_2, δ) is a GOH $f : L_1 \to L_2$ such that $\eta(f^{\vdash}(u)) \geq \delta(u)$ for all $u \in L_2$. The category of FTMLs and their continuous GOHs is called the Kubiak-Šostak extension of Wang's **TML**, denoted by **FTML**.

Definition 2.3. [16] A fuzzy remote neighborhood system is a set $R = \{R_e | e \in M(L)\}$ of maps $R_e : L \to [0, 1]$ such that:
(FRN1) $R_e(1) = 0, R_e(0) = 1$;
(FRN2) $R_e(u) > 0 \Rightarrow e \not\leq u$;
(FRN3) $R_e(u \vee v) = R_e(u) \wedge R_e(v)$;
(FRN4) $R_e(u) = \bigvee\limits_{v\in e|u} \bigwedge\limits_{a\not\leq v} R_a(v)$.

The pair (L, R) is called a fuzzy remote neighborhood space (TFRNS, for short). A continuous map between fuzzy remote neighborhood spaces (L_1, R) and (L_2, S) is a GOH $f : L_1 \to L_2$ such that $S_{f(e)}(u) \leq R_e(f^{\vdash}(u))$ for all $e \in M(L_1)$ and $u \in L_2$. The category of TFRNSs and their continuous GOHs is denoted by **TFRNS**.

Suppose $\eta : L \to [0, 1]$ is a fuzzy co-topology. Let $R^{\eta}_e : L \to [0, 1]$ be defined as follows:

$$R^{\eta}_e(u) = \begin{cases} \bigvee_{v\in e|u} \eta(v), & e \not\leq u, \\ 0, & e \leq u. \end{cases}$$

Then we have the following lemmas.

Lemma 2.4. [16] *(1)* $R^\eta = \{R^\eta_e | e \in M(L)\}$ *is a fuzzy remote neighborhood system.*
(2) $\eta(u) = \bigwedge_{e \not\leq u} R^\eta_e(u)$*, for all* $u \in L$*.*

Lemma 2.5. [16] *A GOH* $f : (L_1, \eta_1) \to (L_2, \eta_2)$ *is continuous if and only if* $f : (L_1, R^{\eta_1}) \to (L_2, R^{\eta_2})$ *is continuous.*

In the following discussion, the superscript η of R^η_e is usually omitted if no confusion arises.

3. Fuzzy Ideals

In this section, we define the concept of a fuzzy ideal and discuss its properties.

Definition 3.1. A map $\mathcal{I} : L \to [0, 1]$ is called a fuzzy ideal on L if $\mathcal{I}$ satisfies the following conditions:
(FID1) $\mathcal{I}(1) = 0$ and $\mathcal{I}(0) = 1$;
(FID2) $\mathcal{I}(u \vee v) = \mathcal{I}(u) \wedge \mathcal{I}(v)$.

Remark 3.2. If we replace $[0, 1]$ with $\{0, 1\}$ in the above definition, then a fuzzy ideal is just the usual real ideal.

Remark 3.3. Let (L, η) be an FTML. Then R_e is a fuzzy ideal on L. If $e \in M(L)$, then $\hat{e} : L \to [0, 1]$ is a fuzzy ideal on L, where

$$\hat{e}(u) = \begin{cases} 1, & e \not\leq u, \\ 0, & e \leq u. \end{cases}$$

Obviously, $R_e \leq \hat{e}$ for all $e \in M(L)$.

Definition 3.4. Suppose a map $\mathcal{B} : L \to [0, 1]$ satisfies the following conditions:
(FIB1) $\mathcal{B}(1) = 0$ and $\bigvee_{v \in L} \mathcal{B}(v) = 1$; (FIB2) $\mathcal{B}(u \vee v) \geq \mathcal{B}(u) \wedge \mathcal{B}(v)$
then $\mathcal{B}$ is called a fuzzy ideal base on L.

Lemma 3.5. *Let* $\mathcal{B}$ *be a fuzzy ideal base on L and define* $\mathcal{I}_\mathcal{B} : L \to [0, 1]$ *as follows:* $\mathcal{I}_\mathcal{B}(u) = \bigvee_{v \geq u} \mathcal{B}(v)$*. Then* $\mathcal{I}_\mathcal{B}$ *is a fuzzy ideal on L.*

Proof. (FID1) is trivial and $\mathcal{I}_\mathcal{B}(u \vee v) \leq \mathcal{I}_\mathcal{B}(u) \wedge \mathcal{I}_\mathcal{B}(v)$ is obvious from the definition of $\mathcal{I}_\mathcal{B}$. In order to prove (FID2), it suffices to show that $\mathcal{I}_\mathcal{B}(u \vee v) \geq \mathcal{I}_\mathcal{B}(u) \wedge \mathcal{I}_\mathcal{B}(v)$. Let $r < \mathcal{I}_\mathcal{B}(u) \wedge \mathcal{I}_\mathcal{B}(v)$. Clearly, $r < \mathcal{I}_\mathcal{B}(u)$ and $r < \mathcal{I}_\mathcal{B}(v)$. Then there exists $a \in L$ such that $a \geq u$ and $r \leq \mathcal{B}(a)$. Similarly, there is some $b \in L$ such that $b \geq v$ and $r \leq \mathcal{B}(b)$. Hence $a \vee b \geq u \vee v$ and $r \leq \mathcal{B}(a) \wedge \mathcal{B}(b) \leq \mathcal{B}(a \vee b)$. Therefore, $r \leq \bigvee_{d \geq u \vee v} \mathcal{B}(d) = \mathcal{I}_\mathcal{B}(u \vee v)$. From the arbitrariness of r, we have $\mathcal{I}_\mathcal{B}(u \vee v) \geq \mathcal{I}_\mathcal{B}(u) \wedge \mathcal{I}_\mathcal{B}(v)$. □

Lemma 3.6. *Let* $\mathcal{I}$ *be a fuzzy ideal on L and define* $\mathcal{I}_{[t]} = \{u | \mathcal{I}(u) \geq t\}$ *for* $t \in [0, 1]$*,* $\mathcal{I}_t = \{u | \mathcal{I}(u) > t\}$ *for* $t \in [0, 1)$*. Then we have:*
(1) $\mathcal{I}_{[t]}$ *and* $\mathcal{I}_t$ *are usual real ideals;*
(2) $\mathcal{I}_{[t]} = \bigwedge_{s<t} \mathcal{I}_{[s]} = \bigwedge_{s<t} \mathcal{I}_s$ *and* $\mathcal{I}_t = \bigvee_{s>t} \mathcal{I}_{[s]} = \bigvee_{s>t} \mathcal{I}_s$*;*
(3) $\mathcal{I} = \bigvee_{t \in [0,1]} (t \wedge \mathcal{I}_{[t]})$*,* $\mathcal{I} = \bigvee_{t \in [0,1)} (t \wedge \mathcal{I}_t)$*.*

Lemma 3.7. *Let $f : L \to L_1$ be a GOH and $\mathcal{I}$ be a fuzzy ideal on L. Define $f[\mathcal{I}] : L_1 \to [0,1]$ by $f[\mathcal{I}](u) = \bigvee_{f^{\vdash}(u) \leq v} \mathcal{I}(v) = \mathcal{I}(f^{\vdash}(u))$. Then $f[\mathcal{I}]$ is a fuzzy ideal on L_1.*

Lemma 3.8. *Let $Fidl(L)$ denote all fuzzy ideals on L. If $1 \in M(L)$, then $Fidl(L)$ is a complete lattice.*

Proof. It is easy to check that $\bigwedge_{t\in T} \mathcal{I}^t$ defined by $(\bigwedge_{t\in T} \mathcal{I}^t)(u) = \bigwedge_{t\in T}(\mathcal{I}^t(u))$ is just the infimum of $\{\mathcal{I}^t\}_{t\in T}$ and it is routine to verify that $\bigvee_{t\in T} \mathcal{I}^t$ defined by

$$\bigvee_{t\in T} \mathcal{I}^t(u) = \bigvee\{\wedge_{i=1}^{i=n} \mathcal{I}^{t_i}(u_i) \mid u = \vee_{i=1}^{i=n} u_i,\ 1 \leq i \leq n,\ u_i \in L\ n \in N\ \}$$

is the supremum of $\{\mathcal{I}^t\}_{t\in T}$. □

As the following example shows, if $1 \notin M(L)$, then $Fidl(L)$ is not necessarily a complete lattice.

Example 3.9. Let L be the diamond lattice, i.e., $L = \{0, a, b, 1\}$ with $a \vee b = 1$ and $a \wedge b = 0$. Then we have $1 \notin M(L)$. Define $\mathcal{I}_1 : L \to [0,1]$ and $\mathcal{I}_2 : L \to [0,1]$ as follows:

$$\mathcal{I}_1(u) = \begin{cases} 1, & u = \{0, a\} \\ 0, & u \in \{b, 1\}, \end{cases}$$

and

$$\mathcal{I}_2(u) = \begin{cases} 1, & u = \{0, b\}, \\ 0, & u \in \{a, 1\}, \end{cases}$$

Then it is easy to verify that $\mathcal{I}_1$ and $\mathcal{I}_2$ are two fuzzy ideals on L, and there is no fuzzy ideal on L bigger than both $\mathcal{I}_1$ and $\mathcal{I}_2$.

Definition 3.10. Let I be a fuzzy ideal on L. If there is no other fuzzy ideal L bigger than $\mathcal{I}$, then $\mathcal{I}$ is called a maximal fuzzy ideal.

By Definition 3.10, both $\mathcal{I}_1$ and $\mathcal{I}_2$ in Example 3.9 are maximal fuzzy ideals.

4. **Convergence of Fuzzy Ideals and its Applications**

The purpose of this section is to introduce the convergence theory of fuzzy ideals and discuss its applications.

Definition 4.1. Let (L, η) be an FTML and $\mathcal{I}$ be a fuzzy ideal on L. If $R_e \leq \mathcal{I}$, then we say that e is a limit point of $\mathcal{I}$ (or $\mathcal{I}$ converges to e): in symbols, $\mathcal{I} \to e$. If $\bigvee_{A\vee P=1}(R_e(A) \wedge \mathcal{I}(P)) = 0$, then we say that e is a cluster point of $\mathcal{I}$ or $\mathcal{I}$ accumulates to e (briefly $\mathcal{I}\infty e$). We denote the union of all limit points of $\mathcal{I}$ by $lim\mathcal{I}$ and the union of all cluster points by $ad\mathcal{I}$.

Remark 4.2. From the above definition, we have $R_e \to e$. If $1 \in M(L)$, then $\mathcal{I}\infty e$ for all $\mathcal{I} \in Fidl(L)$ and $e \in M(L)$, i.e., $ad\mathcal{I} = 1$ for all $\mathcal{I} \in Fidl(L)$.

Theorem 4.3. *The following statements are true:*
(1) $\mathcal{I} \to e \Leftrightarrow e \leq lim\mathcal{I}$;
(2) $\mathcal{I} \infty e \Leftrightarrow e \leq ad\mathcal{I}$;
(3) If $e \geq u$*, then* $\mathcal{I} \to e$*(resp.* $\mathcal{I}\infty e$*)* $\Longrightarrow \mathcal{I} \to u$*(resp.* $\mathcal{I}\infty u$*); (4) If* $\mathcal{I}_1 \leq \mathcal{I}_2$*, then* $lim\mathcal{I}_1 \leq lim\mathcal{I}_2$ *and* $ad\mathcal{I}_2 \leq ad\mathcal{I}_1$*;*
(5) $\mathcal{I} \to e \Longrightarrow \mathcal{I}\infty e$.

Proof. We only prove (1) and (5), the others are trivial and omitted.

(1) $\mathcal{I} \to e \Longrightarrow e \leq lim\mathcal{I}$ is obvious. Let $e \leq lim\mathcal{I}$ and $\lambda < R_e(u) = \bigvee_{v\in e|u} \bigwedge_{a\not\leq v} R_a(v)$. Then there exists $v \in L$ such that $e \not\leq v \geq u$ and $\lambda \leq \bigwedge_{a\not\leq v} R_a(v)$. Since $e \not\leq v$ and $e = \bigvee \beta^*(e)$, there exists $r \in \beta^*(e)$ such that $r \not\leq v \geq u$. Hence $\lambda \leq R_r(v) \leq R_r(u)$. Furthermore, by $r \in \beta^*(e)$ and $e \leq lim\mathcal{I}$, there is some $\mu \in M(L)$ such that $\mathcal{I} \to \mu$ and $r \leq \mu$. Thus $R_r \leq R_\mu \leq \mathcal{I}$. Therefore, $\lambda \leq \mathcal{I}(u)$, as desired.

(5) Let $\mathcal{I} \to e$, i.e., $R_e \leq \mathcal{I}$. Then we have

$$\begin{aligned}\bigvee_{A\vee P=1} (R_e(A) \wedge \mathcal{I}(P)) &\leq \bigvee_{A\vee P=1} (\mathcal{I}(A) \wedge \mathcal{I}(P)) \\ &= \bigvee_{A\vee P=1} \mathcal{I}(A \vee P) \\ &= \mathcal{I}(1) = 0,\end{aligned}$$

i.e., $\mathcal{I}\infty e$. □

Theorem 4.4. *For any* $\mathcal{I} \in Fidl(L)$*,* $e \leq ad\mathcal{I}$ *if and only if there exists* $\mathcal{I}^* \in Fidl(L)$ *such that* $\mathcal{I} \leq \mathcal{I}^*$ *and* $\mathcal{I}^* \to e$*.*

Proof. Let $e \leq ad\mathcal{I}$. From Theorem 4.3 (2), we have $\mathcal{I}\infty e$, i.e., $\bigvee_{A\vee P=1}(R_e(A) \wedge \mathcal{I}(P)) = 0$. Define $\mathcal{I}^* : L \to [0, 1]$ as follows:

$$\forall u \in L,\ \mathcal{I}^*(u) = \bigvee\{R_e(u_1) \wedge \mathcal{I}(u_2) \mid u = u_1 \vee u_2\}.$$

Then we have $\mathcal{I}^* \in Fidl(L)$, $R_e \leq \mathcal{I}^*$ and $\mathcal{I} \leq \mathcal{I}^*$. Hence $I^* \to e$.

Conversely, if there exists $\mathcal{I}^* \in Fidl(L)$ such that $\mathcal{I} \leq \mathcal{I}^*$ and $\mathcal{I}^* \to e$, then we have $e \leq lim\mathcal{I}^* \leq ad\mathcal{I}^* \leq ad\mathcal{I}$ from (2), (4) and (5) of Theorem 4.3. □

Theorem 4.5. *If* $\mathcal{I}$ *is a maximal fuzzy ideal on* L*, then* $ad\mathcal{I} = lim\mathcal{I}$*.*

Proof. We need to prove $adI \leq lim\mathcal{I}$. Let $e \leq ad\mathcal{I}$. From Theorem 4.4, we know that there exists $\mathcal{I}^* \in Fidl(L)$ such that $\mathcal{I} \leq \mathcal{I}^*$ and $\mathcal{I}^* \to e$. Since $\mathcal{I}$ is a maximal fuzzy ideal, we have $\mathcal{I} = \mathcal{I}^*$. Hence $\mathcal{I} \to e$, i.e., $e \leq lim\mathcal{I}$, as desired. □

From Theorem 4.4 and Theorem 4.5, we have the following Corollary.

Corollary 4.6. *For an FTML* (L, η)*, the following conditions are equivalent:*
(1) Every fuzzy ideal has cluster points;
(2) Every maximal fuzzy ideal has limit points.

Definition 4.7. Let $e \in M(L)$ and $a \in L$. If $R_e(a) = 0$, then e is called an adherence point of a.

It is easy to verify the following theorem.

Theorem 4.8. *e is an adherence point of a if and only if there exists a fuzzy ideal $\mathcal{I}$ such that $\mathcal{I}(a) = 0$ and $\mathcal{I} \to e$.*

Definition 4.9. An FTML (L, η) is called T_2 if $\forall a, b \in M(L)$ with $a \wedge b = 0$(a and b are disjoint) implies $\bigvee_{P \vee Q = 1}(R_a(P) \wedge R_b(Q)) > 0$.

Theorem 4.10. *(L, η) is T_2 if and only if for each fuzzy ideal $\mathcal{I}$ on L, $lim\mathcal{I}$ contains no disjoint points.*

Proof. Assume that $lim\mathcal{I}$ contains two disjoint points a and b, i.e., $\mathcal{I} \to a$, $\mathcal{I} \to b$ and $a \wedge b = 0$. Hence

$$\bigvee_{P \vee Q = 1}(R_a(P) \wedge R_b(Q)) \le \bigvee_{P \vee Q = 1}(\mathcal{I}(P) \wedge \mathcal{I}(Q)) = \bigvee_{P \vee Q = 1}\mathcal{I}(P \vee Q) = 0$$

This is in contradiction to $\bigvee_{P \vee Q = 1}(R_a(P) \wedge R_b(Q)) > 0$.

Conversely, suppose that there exist $a \in M(L)$ and $b \in M(L)$ with $a \wedge b = 0$ such that $\bigvee_{P \vee Q = 1}(R_a(P) \wedge R_b(Q)) = 0$. Then $R_b \infty a$. From Theorem 4.4, there exists $\mathcal{I}^* \in Fidl(L)$ such that $R_b \le \mathcal{I}^*$ and $\mathcal{I}^* \to a$. Hence $\mathcal{I}^* \to a$ and $\mathcal{I}^* \to b$. Therefore, $lim\mathcal{I}^*$ contains two disjoint points. This contradicts the given condition. □

Theorem 4.11. *A GOH $f : (L, \eta) \to (L_1, \eta_1)$ is continuous if and only if $f[\mathcal{I}] \to f(e)$ when $\mathcal{I} \to e$ (or $f(lim\mathcal{I}) \le limf[\mathcal{I}]$).*

Proof. *Let $\mathcal{I} \to e$. Then $R_e \le \mathcal{I}$. Hence $R_e(f^{\vdash}(u)) \le \mathcal{I}(f^{\vdash}(u)) = f[\mathcal{I}](u)$. Since $f : (L, \eta) \to (L_1, \eta_1)$ is continuous, we have $R_{f(e)}(u) \le R_e(f^{\vdash}(u))$. Thus $R_{f(e)}(u) \le f[\mathcal{I}](u)$. Therefore, $f[\mathcal{I}] \to f(e)$.*

Conversely, we want to prove $R_{f(e)}(u) \le R_e(f^{\vdash}(u))$. By $R_e \to e$, we have $f[R_e] \to f(e)$. In other words, $R_{f(e)}(u) \le f[R_e](u) = R_e(f^{\vdash}(u))$.

5. **Fuzzy Limit Structures**

In this section, we mainly study fuzzy limit structures using fuzzy ideals and discuss the relationship between fuzzy limit structures and fuzzy co-topologies.

Definition 5.1. A fuzzy limit structure on L is a subset $FLimS$ of $Fidl(L) \times M(L)$ satisfying the following conditions:
(L1) $(\hat{e}, e) \in FLimS$ for all $e \in M(L)$;
(L2) If $a \le b$ and $(\mathcal{I}, b) \in FLimS$, then $(\mathcal{I}, a) \in FLimS$;
(L3) If $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $(\mathcal{I}_1, e) \in FLimS$, then $(\mathcal{I}_2, e) \in FLimS$;
(L4) If $(\mathcal{I}_1, e) \in FLimS$ and $(\mathcal{I}_2, e) \in FLimS$, then $(\mathcal{I}_1 \wedge \mathcal{I}_2, e) \in FLimS$.

The pair $(L, FLimS)$ is called a fuzzy limit molecular lattice. For two limit molecular lattices $(L, FLimS_L)$ and $(N, FLimS_N)$, a GOH $f : (L, FLimS_L) \to (N, FLimS_N)$ is called limit continuous if $(\mathcal{I}, e) \in FLimS_L$ always implies that $(f[\mathcal{I}], f(e)) \in FLimS_N$. Let **FLimML** denote the category of fuzzy limit molecular lattices and limit continuous GOHs.

Lemma 5.2. *Let* (L, η) *be an FTML and* $FLimS^\eta = \{(\mathcal{I}, e) | \mathcal{I} \to e\}$. *Then* $FLimS^\eta$ *is a fuzzy limit structure.*

Lemma 5.3. *Let* $(L, FLimS)$ *be a fuzzy limit molecular lattice and* $\eta^{FLimS} : L \to [0, 1]$ *be defined by*

$$\eta^{FLimS}(a) = \bigwedge_{e \not\leq a} \bigwedge_{(\mathcal{I},e) \in FLimS} \mathcal{I}(a).$$

Then η^{FLimS} *is a fuzzy co-topology on* L.

Proof. (FCT1) is obvious from the definition of η^{FLimS}. Now we prove (FCT2) and (FCT3)

(FCT2): Let $\lambda \leq \eta^{FLimS}(a) \wedge \eta^{FLimS}(b)$, i.e., $\lambda \leq \eta^{FLimS}(a)$ and $\lambda \leq \eta^{FLimS}(b)$. For each $e \in M(L)$ with $e \not\leq a \vee b$ and for each $\mathcal{I} \in Fidl(L)$ with $(\mathcal{I}, e) \in FLimS$, we have $e \not\leq a$ and $e \not\leq b$. Hence $\lambda \leq \mathcal{I}(a)$ and $\lambda \leq \mathcal{I}(b)$. Thus $\lambda \leq \mathcal{I}(a) \wedge \mathcal{I}(b) = \mathcal{I}(a \vee b)$, as desired.

(FCT3): Let $\lambda \leq \bigwedge_{t \in T} \eta^{FLimS}(a_t)$. Then $\lambda \leq \eta^{FLimS}(a_t)$ for all $t \in T$. Let $e \in M(L)$ and $\mathcal{I} \in Fidl(L)$ be such that $e \not\leq \bigwedge_{t \in T} a_t$ and $(\mathcal{I}, e) \in FLimS$. Then there exists $t_0 \in T$ such that $e \not\leq a_{t_0}$. Hence $\lambda \leq \mathcal{I}(a_{t_0}) \leq \mathcal{I}(\bigwedge_{t \in T} a_t)$ and FCT3 follows. □

From the above two lemmas we may conclude that we can construct a fuzzy limit structure $FLimS^\eta$ from a fuzzy co-topology η as well as a fuzzy co-topology η^{FLimS} from a given fuzzy limit structure $FLimS$. In what follows, we study their relationship.

Theorem 5.4. *Let* (L, η) *be an FTML. Then* $\eta^{FLimS^\eta} = \eta$.

Proof. From the definition of η^{FLimS^η} and Lemma 2.4, we have the following computation:

$$\begin{aligned} \eta^{FLimS^\eta}(a) &= \bigwedge_{e \not\leq a} \bigwedge_{(\mathcal{I},e) \in FLimS^\eta} \mathcal{I}(a) \\ &= \bigwedge_{e \not\leq a} R_e^\eta(a) \\ &= \eta(a). \end{aligned}$$

This is to say that $\eta^{FLimS^\eta} = \eta$. □

Theorem 5.5. *Let* $(L, FLimS)$ *be a fuzzy limit molecular lattice. Then* $FLimS \subseteq FLimS^{\eta^{FLimS}}$.

Proof. Let $(\mathcal{I}, e) \in FLimS$. We want to Prove that $(\mathcal{I}, e) \in FLimS^{\eta^{FLimS}}$. It suffices to Show that $R_e^{\eta^{FLimS}} \leq \mathcal{I}$. In fact, we have

$$\begin{aligned} R_e^{\eta^{FLimS}}(u) &= \bigvee_{v \in e|u} \eta^{FLimS}(v) \\ &= \bigvee_{v \in e|u} \bigwedge_{a \not\leq v} \bigwedge_{(\mathcal{I}^*, a) \in FLimS} \mathcal{I}^*(v) \\ &\leq \bigvee_{v \in e|u} \mathcal{I}(v) \\ &= \mathcal{I}(u). \end{aligned}$$

□

Definition 5.6. Let $(L, FLimS)$ be a fuzzy limit molecular lattice. $FLimS$ is said to be topological generated if there exists a fuzzy co-topology η such that $FLimS = FLimS^{\eta}$.

Remark 5.7. If $FLimS$ is topological generated, then $FLimS^{\eta^{FLimS}} = FLimS$ according to Theorem 5.4.

Theorem 5.8. *$f : (L_1, \eta_1) \to (L_2, \eta_2)$ is continuous if and only if $f : (L_1, FLimS^{\eta_1}) \to (L_2, FLimS^{\eta_2})$ is limit continuous.*

Proof. In fact, this theorem is just Theorem 4.11. □

Theorem 5.9. *If $f : (L_1, FLimS_1) \to (L_2, FLimS_2)$ is limit continuous, then $f : (L_1, \eta^{FLimS_1}) \to (L_2, \eta^{FLimS_2})$ is continuous.*

Proof. We need to show that $\eta^{FLimS_2}(u) \leq \eta^{FLimS_1}(f^{\vdash}(u))$ for all $u \in L_2$. Since $f : (L_1, FLimS_1) \to (L_2, FLimS_2)$ is limit continuous, we have $(f[\mathcal{I}], f(e)) \in FLimS_2$ whenever $(\mathcal{I}, e) \in FLimS_1$. Therefore,

$$\begin{aligned} \eta^{FLimS_2}(u) &= \bigwedge_{w \not\leq u} \bigwedge_{(\mathcal{I}_1, w) \in FLimS_2} \mathcal{I}_1(u) \\ &\leq \bigwedge_{f(e) \not\leq u} \bigwedge_{(f[\mathcal{I}], f(e)) \in FLimS_2} f[\mathcal{I}](u) \\ &\leq \bigwedge_{e \not\leq f^{\vdash}(u)} \bigwedge_{(\mathcal{I}, e) \in FLimS_1} \mathcal{I}(f^{\vdash}(u)) \\ &= \eta^{FLimS_1}(f^{\vdash}(u)), \end{aligned}$$

as desired. □

6. Category of FLimML

Let **CD** denote the category of completely distributive lattices with complete lattice morphisms as morphisms. We know that the category of completely distributive lattices with GOHs as its morphisms is the dual category of **CD**. For definitions and preliminaries of category theory, please refer to [1, 5, 10].

Theorem 6.1. **FLimML** *is a topological category over* $\mathbf{CD}^{op}$.

Proof. We only prove that it fulfills the initial lift property. Let $\{f_j : N \to (L_j, FLimS_j)\}_{j\in J}$ (f_j is GOH for all $j \in J$) be a source in **LimML** and let $FLimS \subseteq Fidl(N) \times M(N)$ be defined by

$$(\mathcal{I}, e) \in FLimS \text{ if and only if } (f_j[\mathcal{I}], f_j(e)) \in FLimS_j \text{ for all } j \in J$$

Then it is routine to show that $FLimS_N$ is the unique **FLimML**-structure on N which is initial with respect to the source $\{f_j : N \to (L_j, FLimS_j)\}_{j\in J}$. □

Let **TFLimML** denote the category of all topological fuzzy limit molecular lattices and their limit continuous maps. It is easy to verify that **TFLimML** is a full subcategory of **FLimML**. Moreover, we have the following theorem.

Theorem 6.2. **TFLimML** *is a bireflective full subcategory of* **FLimML**.

Proof. Assume that $(L, FLimS)$ is an FLimML. We assert that its **TFlimML**-reflection is defined by $id_L : (L, FLimS) \to (L, FLimS^{\eta^{FLimS}})$. This can be proved as follows:

(1) It is obvious that $(L, FLimS^{\eta^{FLimS}})$ is TFlimML. (2) It is trivial by Theorem 5.5 that $id_L : (L, FLimS) \to (L, FLimS^{\eta^{FLimS}})$ is limit continuous.

(3) We now prove that for each TFLimML $(N, FLimS^*)$ and each GOH $f : L \to N$, the continuity of $f : (L, FLimS) \to (N, FLimS^*)$ implies the continuity of $f : (L, FLimS^{\eta^{FLimS}}) \to (N, FLimS^*)$. In fact, from Theorem 5.9, we have $f : (L, \eta^{FLimS}) \to (N, \eta^{FLimS^*})$ is continuous if $f : (L, FLimS) \to (N, FLimS^*)$ is limit continuous. Hence $f : (L, FLimS^{\eta^{FLimS}}) \to (N, FLimS^{\eta^{FLimS^*}})$ is limit continuous from Theorem 4.8, i.e., $f : (L, FLimS^{\eta^{FLimS}}) \to (N, FLimS^*)$ is limit continuous on account of Remark 5.7. □

The following theorem follows from Theorems 5.4, 5.8, 5.9 and Remark 5.7. theorem.

Theorem 6.3. **TFLimML** *is isomorphic to* **FTML**.

Corollary 6.4. **FTML** *is a bireflective full subcategory of* **FlimML**.

Question 6.5. In Theorem 6.1, It is shown that **FLimML** is a topological category over $\mathbf{CD}^{op}$. Is **FLimML** M-topological, Monotopological or topologically algebraic?

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Yueli Yue*, Department of Mathematics, Ocean University of China, Qingdao, 266071, P. R. China

E-mail address: yueliyue@163.com

Jinming Fang, Department of Mathematics, Ocean University of China, Qingdao, 266071, P. R. China

E-mail address: jinming-fang@163.com

*Corresponding author