Iranian Journal of Fuzzy Systems **Vol. 5, No. 2,** (2008) pp. 87-92

FUZZY SEMI-IDEAL AND GENERALIZED FUZZY QUOTIENT RING

Y. BINGXUE

ABSTRACT. The concepts of fuzzy semi-ideals of R with respect to $H \leq R$ and generalized fuzzy quotient rings are introduced. Some properties of fuzzy semi-ideals are discussed. Finally, several isomorphism theorems for generalized fuzzy quotient rings are established.

1. Preliminaries

Since the concept of fuzzy subgroups was introduced by Rosenfeld [6] in 1971, the literature of these fuzzy algebraic concepts has been growing very rapidly. Liu [4] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Subsequently, Mukherjee and Sen [5], and Zhang [7] further various fuzzy characterized ideals in rings. Based on the concept of fuzzy ideal, Kumbhojkar and Bapat [3] introduced the concept of fuzzy quotient ring and established the correspondence theorems for fuzzy ideals. In this paper, we first generalize the concepts of fuzzy ideals and fuzzy quotient rings and then establish several isomorphism theorems for generalized fuzzy quotient rings.

Unless specifically stated, R and R' always stand for rings with their null elements 0 and $0'$ respectively. By a fuzzy subset of a nonempty set U we mean a mapping from U to the closed interval [0, 1]. The set of all fuzzy subsets of U is denoted by $F(U)$ and we define $\sup\phi = 0$.

Definition 1.1. [1] Let $H \leq R$ and let S be a nonempty subset of R, if

$$x + y \in S, xy \in S, hx \in S, xh \in S, \forall x, y \in S, \forall h \in H,$$

then S is called a semi-ideal of R with respect to H. Particularly, S is called a semi-ideal of R if $H = R$.

Let A be a fuzzy subset of R. For all $r \in R$, we define the fuzzy subset $r + A$ of R as follows: $(r + A)(x) = A(x - r), \forall x \in R$.

Received: May 2006; Revised: July 2006; Accepted: September 2006

Key words and phrases: Fuzzy Semi-ideal, Generalized Fuzzy Quotient Ring, Homomorphism, Isomorphism.

2. **Fuzzy Semi-ideal**

Definition 2.1. Let $H \le R$ and let E be a fuzzy subset of R, if

$$E(h_1 + h_2) \ge E(h_1) \wedge E(h_2),\ E(h_1 h_2) \ge E(h_1) \wedge E(h_2),\ E(h_1 x) \ge E(x), E(x h_1) \ge E(x),$$
$$\forall h_1, h_2 \in H,\ \forall x \in R,$$

then E is called a fuzzy semi-ideal of R with respect to H. Particularly, E is called a fuzzy semi-ideal of R if $H = R$.

Obviously, if E is a fuzzy semi-ideal of R with respect to H, then for $x \in R$ we have that $E(0) \ge E(h)$.

A fuzzy ideal of R is a fuzzy semi-ideal of R with respect to R, but the converse is not true in general. For example, let Z denote the set of all integers and let $Z^{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z \right\}$ denote the matrix ring for the ordinary addition and multiplication of matrix. Then $H = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mid m \in Z \right\}$ is a subring of $Z^{2\times 2}$ and $I = \left\{ \begin{pmatrix} 0 & 2m \\ 0 & 2n \end{pmatrix} \mid m, n \in Z,\ n \ge 0 \right\}$ is a semi-ideal of $Z^{2\times 2}$ with respect to H. If A is a fuzzy subset of $Z^{2\times 2}$ such that

$$A(x) = \begin{cases} 1, x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 0.5, x \in I \setminus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ 0, x \in Z^{2\times 2} \setminus I \end{cases}.$$

Then A is a fuzzy semi-ideal of $Z^{2\times 2}$ with respect to H. But A is not a fuzzy ideal of $Z^{2\times 2}$.

Proposition 2.2. *Let $H \le R$ and let E be a fuzzy subset of R, then E is a fuzzy semi-ideal of R with respect to H if and only if $A_\lambda = \Phi$ or A_λ is a semi-ideal of R with respect to H for all $\lambda \in [0,1]$.*

Let $H \le R$ and let E be a fuzzy semi-ideal of R *with respect to* H, then $\overline{E}$ is called the kernel of E, where $\overline{E}(r) = E(r) \wedge E(-r), \forall r \in R$.

Proposition 2.3. *Let $H \le R$ and let E be a fuzzy semi-ideal of R with respect to H, then $\overline{E}$ is a fuzzy ideal of R with respect to H.*

Proposition 2.4. *Let* E *and* D *be fuzzy semi-ideals of* R*, then* $E \bigcap D, E+D, ED$ *are also fuzzy semi-ideals of* R*, and* $\overline{E \bigcap D} = \overline{E} \bigcap \overline{D},\ \overline{E+D} \supseteq \overline{E} + \overline{D},\ \overline{ED} \supseteq \overline{E}\,\overline{D}$.

Proposition 2.5. *Let* $H \le R$ *and let* E *be a fuzzy semi-ideal of* R *with respect to* H*, then for all* $h_1, h_2 \in H$*,* $h_1 + E \supseteq h_2 + E$ *holds in* H *if and only if*

$$E(h_2 - h_1) = E(0).$$

Proof. Sufficiency: If $E(h_2 - h_1) = E(0)$ holds, then for all $h \in H$, we have

$$\begin{aligned}(h_1 + E)(h) &= E(h - h_1) = E(h - h_2 + h_2 - h_1) \ge E(h - h_2) \wedge E(h_2 - h_1) \\ &= E(h - h_2) \wedge E(0) = E(h - h_2) = (h_2 + E)(h).\end{aligned}$$

So $h_1 + E \supseteq h_2 + E$ holds in H.

Necessity: If $h_1 + E \supseteq h_2 + E$ holds in H, then

$$E(h_2 - h_1) = (h_1 + E)(h_2) \ge (h_2 + E)(h_2) = E(0),$$

Hence $E(h_2 - h_1) = E(0)$. □

Corollary 2.6. *Let* $H \le R$ *and let* E *be a fuzzy semi-ideal of* R *with respect to* H*, then for all* $h_1, h_2 \in H$*,* $h_1 + E = h_2 + E$ *holds in* H *if and only if*

$$E(h_2 - h_1) = E(h_1 - h_2) = E(0).$$

3. Isomorphisms of Generalized Fuzzy Quotient Rings

Proposition 3.1. *Let* $H \le R$ *and let* E *be a fuzzy semi-ideal of* R *with respect to* H*, then for all* $h_1, h_2, x_1, x_2 \in H$*,*

$$\begin{cases} h_1 + E = x_1 + E \\ h_2 + E = x_2 + E \end{cases} \Rightarrow \begin{cases} (h_1 + h_2) + E = (x_1 + x_2) + E \\ h_1 h_2 + E = x_1 x_2 + E \end{cases} \text{ holds in } H.$$

Proof. For all $h \in H$ we have

$E(h_1 - x_1) = E(x_1 - h_1) = E(0),\ E(h_2 - x_2) = E(x_2 - h_2) = E(0),$

So, $E(h_1 + h_2 - x_1 - x_2) \ge E(h_1 - x_1) \wedge E(h_2 - x_2) = E(0)$,

$E(h_1 h_2 - x_1 x_2) = E((h_1 - x_1)h_2 + x_1(h_2 - x_2))$,

$\ge E(h_1 - x_1) \wedge E(h_2 - x_2) = E(0)$

i.e. $E(h_1 + h_2 - x_1 - x_2) = E(0)$, $E(h_1 h_2 - x_1 x_2) = E(0)$.

Similarly, $E(x_1 + x_2 - h_1 - h_2) = E(0)$, $E(x_1 x_2 - h_1 h_2) = E(0)$.

Hence $(h_1 + h_2) + E = (x_1 + x_2) + E$, $h_1 h_2 + E = x_1 x_2 + E$. □

Proposition 3.2 below, follows from the above Proposition 3.1.

Proposition 3.2. *Let $H \le R$ and let E be a fuzzy semi-ideal of R with respect to H, then $(H/E, +, \circ)$ forms a ring, where $H/E = \{h + E \mid h \in H\}$,*

$(h_1 + E) + (h_2 + E) = (h_1 + h_2) + E,\ (h_1 + E) \circ (h_2 + E) = h_1 h_2 + E,\ \forall h_1 h_2 \in H$.

The ring $(H/E, +, \circ)$ in Proposition 3.2 is called a generalized fuzzy quotient ring of H with respect to E.

Proposition 3.3. *Let $H \le R$ and let E be a fuzzy semi-ideal of R with respect to H, then $H/\overline{E}_0 \cap H \cong H/E$, where $\overline{E}_0 = \{r \in R \mid \overline{E}(r) = E(0)\}$.*

Proof. Let $f : H \to H/E$, $f(h) = h + E, \forall h \in H$, then f is an epimorphism and

$$\ker f = \{h \in H \mid h + E = E\} = \{h \in H \mid E(h) = E(-h) = E(0)\}$$
$$= \{h \in H \mid h \in \overline{E}_0\} = \overline{E}_0 \cap H.$$

From the homomorphism fundamental theorem we have $H/\overline{E}_0 \cap H \cong H/E$. □

Proposition 3.4. *Let $f : R \to R'$ be an epimorphism and let D be a fuzzy semi-ideal of R', then $f^{-1}(D)$ is a fuzzy semi-ideal of R and $R/f^{-1}(D) \cong R'/D$.*

Proof. For all $x, y \in R$, we have

$$f^{-1}(D)(x+y) = D(f(x+y)) = D(f(x) + f(y))$$
$$\ge D(f(x)) \wedge D(f(y)) = f^{-1}(D)(x) \wedge f^{-1}(D)(y).$$

Similarly, $f^{-1}(D)(xy) \ge f^{-1}(D)(x) \vee f^{-1}(D)(y)$.

Hence $f^{-1}(D)$ is a fuzzy semi-ideal of R.

Let $g : R/f^{-1}(D) \to R'/D$, $g(r + f^{-1}(D)) = f(r) + D$, then

$$r_1 + f^{-1}(D) = r_2 + f^{-1}(D) \Leftrightarrow f^{-1}(D)(r_1 - r_2) = f^{-1}(D)(r_2 - r_1) = f^{-1}(D)(0)$$
$$\Leftrightarrow D(f(r_1) - f(r_2)) = D(f(r_2) - f(r_1)) = D(0')$$
$$\Leftrightarrow f(r_1) + D = f(r_2) + D$$

Also, $g((r_1 + f^{-1}(D)) + (r_2 + f^{-1}(D))) = g(r_1 + r_2 + f^{-1}(D)) = f(r_1 + r_2) + D$

$$= f(r_1) + f(r_2) + D = (f(r_1) + D) + (f(r_2) + D)$$
$$= g(r_1 + f^{-1}(D)) + g(r_2 + f^{-1}(D))$$

So we have that g is an isomorphism. Hence $R/f^{-1}(D) \cong R'/D$. □

Proposition 3.5. *Let E and D be fuzzy semi-ideals of R such that $D \supseteq E$ and $D(0) = E(0)$, then $\overline{D}_0/E$ is an ideal of R/E and $\frac{R/E}{\overline{D}_0/E} \cong R/\overline{D}_0$, where* $\overline{D}_0 = \{r \in R \mid D(r) = D(-r) = D(0)\}$.

Proof. Let $f: R/E \to R/\overline{D}_0$ be such that $f(r+E) = r + \overline{D}_0$. Then we have

$$r_1 + E = r_2 + E \Rightarrow E(r_1 - r_2) = E(r_2 - r_1) = E(0)$$
$$\Rightarrow D(r_1 - r_2) = D(r_2 - r_1) \geq E(0) = D(0)$$
$$\Rightarrow r_1 - r_2 \in \overline{D}_0 \Rightarrow r_1 + \overline{D}_0 = r_2 + \overline{D}_0.$$

This shows that f is a mapping. Furthermore, we can prove that f is an epimorphism and $\ker f = \{r + E \mid r + \overline{D}_0 = \overline{D}_0\} = \{r + E \mid r \in \overline{D}_0\} = \overline{D}_0/E$. Hence $\frac{R/E}{\overline{D}_0/E} \cong R/\overline{D}_0$. □

Proposition 3.6. *Let E and D be fuzzy semi-ideals of R such that $D(0) = E(0)$, then $(\overline{D}_0 + \overline{E}_0)/E \cong \overline{D}_0/E \cap D$, where*
$\overline{D}_0 = \{r \in R \mid D(r) = D(-r) = D(0)\}$, $\overline{E}_0 = \{r \in R \mid E(r) = E(-r) = E(0)\}$.

Proof. It is clear that E and $E \cap D$ are fuzzy semi-ideals of R with respect to $\overline{D}_0 + \overline{E}_0$ and $\overline{D}_0$, respectively. Let

$$f: (\overline{D}_0 + \overline{E}_0)/E \to \overline{D}_0/E \cap D, \quad f(x + y + E) = x + E \cap D, \forall x \in \overline{D}_0, \forall y \in \overline{E}_0,$$

If $x_1 + y_1 + E = x + y + E$ (here $x_1, x \in \overline{D}_0$, $y_1, y \in \overline{E}_0$), then
$E(x + y - x_1 - y_1) = E(x_1 + y_1 - x - y) = E(0)$.

Hence, $E(x - x_1) = E(x - x_1 + y - y_1 + y_1 - y) \geq E(0) \wedge E(y_1 - y) = E(y_1 - y) = E(0)$.

Also, $D(x - x_1) = D(0)$. That is, $(E \cap D)(x - x_1) = (E \cap D)(0)$.

Similarly, $(E \cap D)(x_1 - x) = (E \cap D)(0)$. So $x + E \cap D = x_1 + E \cap D$.

This means that f is a mapping and further f is onto.
Conversely, if $x+E\bigcap D = x_1 + E\bigcap D$ (here $x_1, x \in \overline{D}_0$), then

$$(E\bigcap D)(x-x_1) = E(x-x_1)\wedge D(x-x_1) = E(x-x_1) = E(0).$$

So for all $y_1, y \in \overline{E}_0$, we have

$$E(x+y-x_1-y_1) \geq E(x-x_1)\wedge E(y-y_1) = E(0).$$

Similarly, $E(x_1+y_1-x-y)\geq E(0)$. i.e.

$$E(x+y-x_1-y_1) = E(x_1+y_1-x-y) = E(0).$$

Hence, $x_1+y_1+E = x+y+E$, which means that f is one-to-one.
It may also be verified that f keeps addition and multiplication operations, and so f is an isomorphism. Hence $(\overline{D}_0+\overline{E}_0)/E \cong \overline{D}_0/E\bigcap D$. □

Acknowledgments. This research was supported by Key Project of Chinese Ministry of Education (No.206089).

References

[1] Y. Bingxue, *Homomorphism and isomorphism of regular power ring,* Chinese Quarterly Journal of Mathematics, **2** (2000), 23-28.
[2] Y. Bingxue, *The structure of fuzzy power ring,* The Journal of Fuzzy Mathematics, **4** (2000), 953-961.
[3] H. V. Kumbhojkar and M. S. Bapat, *Correspondence theorem for fuzzy ideals,* Fuzzy Sets and Systems, **11** (1983), 213-219.
[4] W. Liu, *Fuzzy invariant subgroups and fuzzy ideals,* Fuzzy Sets and Systems, **8** (1982), 133-139.
[5] T. K. Mukherje and M. K. Sen, *On fuzzy ideals of rings,* Fuzzy Sets and Systems, **21** (1987), 99-104.
[6] A. Rosenfeld, *Fuzzy groups,* J. Math. Anal. Appl., **35** (1971), 512-517.
[7] Y. Zhang, *Prime L-fuzzy ideals and primary Prime L-fuzzy ideals,* Fuzzy Sets and Systems, **27** (1988), 345-350.

Yao Bingxue, Department of Mathematics, Hainan Normal University, Haikou, Hainan 571158, China and School of Mathematics Science, Liaocheng University, Liaocheng, Shandong 252059, China
Email address: **yaobingxue@lcu.edu.cn**