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REDEFINED FUZZY SUBALGEBRAS OF BCK/BCI-ALGEBRAS

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ABSTRACT. Using the notion of *anti fuzzy points* and its *besideness* to and *non-quasi- coincidence* with a fuzzy set, new concepts in anti fuzzy subalgebras in BCK/BCI-algebras are introduced and their properties and relationships are investigated.

1. Introduction

The concept of fuzzy sets was first initiated by Zadeh [5]. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. In this paper, we introduce the concept of an anti fuzzy subalgebra of BCK/BCI-algebras by using the notion of *anti fuzzy points* and its *besideness* to and *non-quasi-coincidence* with a fuzzy set, and investigate their inter-relations and related properties.

2. Preliminaries

A fuzzy set $\mathcal{A}$ in X of the form

$$\mathcal{A}(x) := \begin{cases} t \in [0,1) & \text{if } y = x, \\ 1 & \text{if } y \neq x \end{cases}$$

is called an *anti fuzzy point* with support x and value t and is denoted by x_t. A fuzzy set $\mathcal{A}$ in X is said to be *non-unit* if there exists $x \in X$ such that $\mathcal{A}(x) < 1$.

A fuzzy set $\mathcal{A}$ in a BCK/BCI-algebra X is called an *anti-fuzzy subalgebra* of X if it satisfies

$$(\forall x, y \in X)\,(\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}). \tag{1}$$

3. Redefined Fuzzy Subalgebras

Definition 3.1. An anti-fuzzy point x_t is said to *beside* (resp. *be non-quasi coincident with*) a fuzzy set $\mathcal{A}$, denoted by $x_t \lessdot \mathcal{A}$ (resp. $x_t \Upsilon \mathcal{A}$), if $\mathcal{A}(x) \leq t$ (resp. $\mathcal{A}(x) + t < 1$). We say that $\lessdot$ (resp. Υ) is a *beside relation* (resp. *non-quasi coincident with relation*) between anti-fuzzy points and fuzzy sets.

If $x_t \lessdot \mathcal{A}$ or $x_t \Upsilon \mathcal{A}$ (resp. $x_t \lessdot \mathcal{A}$ and $x_t \Upsilon \mathcal{A}$), we say that $x_t \lessdot \vee\Upsilon\, \mathcal{A}$ (resp. $x_t \lessdot \wedge\Upsilon\, \mathcal{A}$).

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Proposition 3.2. Let $\mathcal{A}$ be a fuzzy set in a BCK/BCI-algebra X. Then $\mathcal{A}$ satisfies condition (2) if and only if it satisfies the following condition.

$$(\forall x, y \in X)\,(\forall t_1, t_2 \in [0,1))\,(x_{t_1}, y_{t_2} \lessdot \mathcal{A} \;\Rightarrow\; (x * y)_{\max\{t_1,t_2\}} \lessdot \mathcal{A}). \qquad (2)$$

Proof. Assume that $\mathcal{A}$ satisfies condition (2). Let $x, y \in X$ and $t_1, t_2 \in [0,1)$ satisfy $x_{t_1}, y_{t_2} \lessdot \mathcal{A}$. Then $\mathcal{A}(x) \leq t_1$ and $\mathcal{A}(y) \leq t_2$. From (2) it follows that

$$\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq \max\{t_1, t_2\}.$$

Hence $(x * y)_{\max\{t_1,t_2\}} \lessdot \mathcal{A}$.

Conversely, suppose that condition (3.2) is valid. Since $x_{\mathcal{A}(x)} \lessdot \mathcal{A}$ and $y_{\mathcal{A}(y)} \lessdot \mathcal{A}$ for all $x, y \in X$, it follows from (3.2) that

$$(x * y)_{\max\{\mathcal{A}(x),\mathcal{A}(y)\}} \lessdot \mathcal{A}$$

so that $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}$. This completes the proof. □

Note that if $\mathcal{A}$ is a fuzzy set in X such that $\mathcal{A}(x) \geq 0.5$ for all $x \in X$, then the set $\{x_t \mid x_t \lessdot \wedge \Upsilon\, \mathcal{A}\}$ is empty. In what follows, unless otherwise specified, α and β will denote any one of $\lessdot$, Υ, $\lessdot \vee \Upsilon$, and $\lessdot \wedge \Upsilon$ and $x_t\overline{\alpha}\mathcal{A}$ will mean that $x_t\alpha\mathcal{A}$ does not hold.

Definition 3.3. A fuzzy set $\mathcal{A}$ in a BCK/BCI-algebra X is called an $(\alpha, \beta)^*$*-fuzzy subalgebra* of X, where $\alpha \neq \lessdot \wedge \Upsilon$, if it satisfies the following implication:

$$(\forall x, y \in X)\,(\forall t_1, t_2 \in [0,1))\,(x_{t_1}\alpha\mathcal{A}, y_{t_2}\alpha\mathcal{A} \;\Rightarrow\; (x * y)_{\max\{t_1,t_2\}}\beta\mathcal{A}). \qquad (3)$$

Example 3.4. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\mathcal{A}$ be a fuzzy set in X defined by $\mathcal{A}(0) = 0.4$, $\mathcal{A}(a) = 0.3$, and $\mathcal{A}(b) = \mathcal{A}(c) = 0.7$. It is easy to verify that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Theorem 3.5. In a BCK/BCI-algebra, every $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra.

Proof. Let $\mathcal{A}$ be a $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of a BCK/BCI-algebra X. Let $x, y \in X$ and $t_1, t_2 \in [0,1)$ satisfy $x_{t_1} \lessdot \mathcal{A}$ and $y_{t_2} \lessdot \mathcal{A}$. Then $x_{t_1} \lessdot \vee\Upsilon\, \mathcal{A}$ and $y_{t_2} \lessdot \vee\Upsilon\, \mathcal{A}$, implying that $(x * y)_{\max\{t_1,t_2\}} \lessdot \vee\Upsilon\, \mathcal{A}$. Hence $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

The converse of Theorem 3.5 is not true in general. For example, the $(\lessdot, \lessdot\vee\Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X in Example 3.4 is not a $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X since $a_{0.5} \lessdot \vee\Upsilon\, \mathcal{A}$ and $c_{0.2} \lessdot \vee\Upsilon\, \mathcal{A}$, but $(a * c)_{\max\{0.5,0.2\}} = b_{0.5}\overline{\lessdot \vee \Upsilon}\,\mathcal{A}$.

Obviously, any $(\lessdot, \lessdot)^*$-fuzzy subalgebra is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra, but the converse is not necessarily true. For example, the $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X in Example 3.4 is not a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X since $a_{0.38} \lessdot \mathcal{A}$ and

$a_{0.34} \lessdot \mathcal{A}$, but $(a*a)_{\max\{0.34,0.38\}} = 0_{0.38}\overline{\lessdot}\mathcal{A}$. Also, a $(\lessdot, \lessdot\vee\Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X may not be a $(\Upsilon, \lessdot\vee\Upsilon)^*$-fuzzy subalgebra. For example, the $(\lessdot, \lessdot\vee\Upsilon)^*$-fuzzy subalgebra $\mathcal{A}$ of X in Example 3.4 is not a $(\Upsilon, \lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X since $a_{0.6}\Upsilon\mathcal{A}$ and $b_{0.1}\Upsilon\mathcal{A}$, but $(a*b)_{max\{0.6,0.1\}} = c_{0.6}\overline{\lessdot\vee\Upsilon}\,\mathcal{A}$.

Theorem 3.6. Let $\mathcal{A}$ be a fuzzy set in a BCK/BCI-algebra X. Then the left diagram shows the relationship between $(\alpha,\beta)^*$-fuzzy subalgebras of X, where α, β are one of $\lessdot$ and Υ. Also we have the right diagram.

$$\begin{tikzcd}
 & (\alpha,\alpha\vee\beta)^* & \\
(\alpha,\beta)^* \arrow[ur] & & (\alpha,\alpha)^* \arrow[ul] \\
 & (\alpha,\alpha\wedge\beta)^* \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\lessdot\vee\Upsilon, \lessdot\vee\Upsilon)^* & \\
(\lessdot\vee\Upsilon, \Upsilon)^* \arrow[ur] & & (\lessdot\vee\Upsilon, \lessdot)^* \arrow[ul] \\
 & (\lessdot\vee\Upsilon, \lessdot\wedge\Upsilon)^* \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. The proof is easy. □

Proposition 3.7. Let $\mathcal{A}$ be a non-unit fuzzy set in a BCK/BCI-algebra X. If $\mathcal{A}$ is an $(\alpha,\beta)^*$-fuzzy subalgebra of X, then $\mathcal{A}(0) < 1$.

Proof. Assume that $\mathcal{A}(0) = 1$. Since $\mathcal{A}$ is non-unit, there exists $x \in X$ such that $\mathcal{A}(x) = t < 1$. If $\alpha = \lessdot$ or $\alpha = \lessdot\vee\Upsilon$, then $x_t\alpha\mathcal{A}$, but $(x*x)_{\max\{t,t\}} = 0_t\overline{\beta}\mathcal{A}.$, which is a contradiction. If $\alpha = \Upsilon$, then $x_0\alpha\mathcal{A}$ because $\mathcal{A}(x) + 0 = t + 0 = t < 1$. On the other hand, $(x*x)_{\max\{0,0\}} = 0_0\overline{\beta}\mathcal{A}$, which is a contradiction. Hence $\mathcal{A}(0) < 1$. □

For a fuzzy set $\mathcal{A}$ in a BCK/BCI-algebra X, we denote

$$X^* := \{x \in X \mid \mathcal{A}(x) < 1\}.$$

Theorem 3.8. Let $\mathcal{A}$ be a non-unit fuzzy set in a BCK/BCI-algebra X. If $\mathcal{A}$ is an $(\alpha,\beta)^*$-fuzzy subalgebra of X where (α,β) is one of the following:

$$\bullet\ (\lessdot,\lessdot), \quad \bullet\ (\lessdot,\Upsilon), \quad \bullet\ (\Upsilon,\lessdot), \quad \bullet\ (\Upsilon,\Upsilon),$$

then the set X^* is a subalgebra of X.

Proof. (i) Assume that $\mathcal{A}$ is a $(\lessdot,\lessdot)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. Assume that $\mathcal{A}(x*y) = 1$. Note that $x_{\mathcal{A}(x)} \lessdot \mathcal{A}$ and $y_{\mathcal{A}(y)} \lessdot \mathcal{A}$. But, since $\mathcal{A}(x*y) = 1 > \max\{\mathcal{A}(x), \mathcal{A}(y)\}$, we get $(x*y)_{\{\mathcal{A}(x),\mathcal{A}(y)\}}\overline{\lessdot}\mathcal{A}$. This is a contradiction, and so $\mathcal{A}(x*y) < 1$ which shows that $x*y \in X^*$. Hence X^* is a subalgebra of X.

(ii) Assume that $\mathcal{A}$ is a $(\lessdot,\Upsilon)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. If $\mathcal{A}(x*y) = 1$, then

$$\mathcal{A}(x*y) + \max\{\mathcal{A}(x), \mathcal{A}(y)\} \geq 1.$$

Hence $(x*y)_{\max\{\mathcal{A}(x),\mathcal{A}(y)\}}\overline{\Upsilon}\mathcal{A}$, which is a contradiction since $x_{\mathcal{A}(x)} \lessdot \mathcal{A}$ and $y_{\mathcal{A}(y)} \lessdot \mathcal{A}$. Thus $\mathcal{A}(x*y) < 1$, and so $x*y \in X^*$. Therefore X^* is a subalgebra of X.

(iii) Assume that $\mathcal{A}$ is a $(\Upsilon,\lessdot)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. Thus $x_0\Upsilon\mathcal{A}$ and $y_0\Upsilon\mathcal{A}$. If $\mathcal{A}(x*y) = 1$, then $\mathcal{A}(x*y) =$

$1 > 0 = \max\{0,0\}$. Therefore $(x*y)_{\max\{0,0\}}\overline{\lessdot}\mathcal{A}$, which is a contradiction. Hence $\mathcal{A}(x*y) < 1$, and so $x*y \in X^*$.

(iv) Assume that $\mathcal{A}$ is a $(\Upsilon,\Upsilon)^*$-fuzzy subalgebra of X. Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. If $\mathcal{A}(x*y) = 1$, then $\mathcal{A}(x*y) + \max\{0,0\} = 1$ and so $(x*y)_{\max\{0,0\}}\overline{\Upsilon}\mathcal{A}$. This is impossible, hence $\mathcal{A}(x*y) < 1$, i.e., $x*y \in X^*$. This completes the proof. □

Corollary 3.9. Let $\mathcal{A}$ be a non-unit fuzzy set in a BCK/BCI-algebra X. If $\mathcal{A}$ is an $(\alpha,\beta)^*$-fuzzy subalgebra of X where (α,β) is one of the following:

- $(\lessdot, \lessdot\wedge\Upsilon)$,
- $(\lessdot, \lessdot\vee\Upsilon)$,
- $(\Upsilon, \lessdot\wedge\Upsilon)$,
- $(\Upsilon, \lessdot\vee\Upsilon)$,
- $(\lessdot\vee\Upsilon, \lessdot\vee\Upsilon)$,
- $(\lessdot\vee\Upsilon, \lessdot\wedge\Upsilon)$,

then the set X^* is a subalgebra of X.

Proof. By Theorem 3.6, it is enough to prove the corollary for the cases:

$$\text{(i) } (\lessdot, \lessdot\vee\Upsilon) \quad\text{and}\quad \text{(ii) } (\Upsilon, \lessdot\vee\Upsilon).$$

(i) Let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$, and so $\mathcal{A}(x) = t_1$ and $\mathcal{A}(y) = t_2$ for some $t_1, t_2 \in [0,1)$. It follows that $x_{t_1}\lessdot\mathcal{A}$ and $y_{t_2}\lessdot\mathcal{A}$ so that $(x*y)_{\max\{t_1,t_2\}}\lessdot\vee\Upsilon\,\mathcal{A}$, i.e., $(x*y)_{\max\{t_1,t_2\}}\lessdot\mathcal{A}$ or $(x*y)_{\max\{t_1,t_2\}}\Upsilon\mathcal{A}$. If $(x*y)_{\max\{t_1,t_2\}}\lessdot\mathcal{A}$, then $\mathcal{A}(x*y) \le \max\{t_1,t_2\} < 1$ and thus $x*y \in X^*$. If $(x*y)_{\max\{t_1,t_2\}}\Upsilon\mathcal{A}$, then $\mathcal{A}(x*y) \le \mathcal{A}(x*y) + \max\{t_1,t_2\} < 1$. Hence $x*y \in X^*$.

For the case (ii), let $x, y \in X^*$. Then $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$, which imply that $x_0\Upsilon\mathcal{A}$ and $y_0\Upsilon\mathcal{A}$. Since $\mathcal{A}$ is a $(\Upsilon, \lessdot\vee\Upsilon)^*$-fuzzy subalgebra, $(x*y)_0 = (x*y)_{\max\{0,0\}}\lessdot\vee\Upsilon\,\mathcal{A}$, i.e., $(x*y)_0\lessdot\mathcal{A}$ or $(x*y)_0\Upsilon\mathcal{A}$. If $(x*y)_0\lessdot\mathcal{A}$, then $\mathcal{A}(x*y) = 0 < 1$. If $(x*y)_0\Upsilon\mathcal{A}$, then $\mathcal{A}(x*y) = \mathcal{A}(x*y) + 0 < 1$. Therefore $x*y \in X^*$. This completes the proof. □

Theorem 3.10. Let $\mathcal{A}$ be a non-unit fuzzy set in a BCK/BCI-algebra X. Then every $(\Upsilon,\Upsilon)^*$-fuzzy subalgebra of X is constant on X^*.

Proof. Let $\mathcal{A}$ be a non-unit $(\Upsilon,\Upsilon)^*$-fuzzy subalgebra of X. Assume that $\mathcal{A}$ is not constant on X^*. Then there exists $y \in X^*$ such that $t_y = \mathcal{A}(y) \neq \mathcal{A}(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. If $t_y < t_0$, then $\mathcal{A}(y) + (1-t_0) = t_y + 1 - t_0 < 1$ and so $y_{1-t_0}\Upsilon\mathcal{A}$. Since

$$\mathcal{A}(y*y) + (1-t_0) = \mathcal{A}(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

we have $(y*y)_{\max\{1-t_0,1-t_0\}}\overline{\Upsilon}\mathcal{A}$, which is a contradiction. If $t_y > t_0$., we choose $t_1, t_2 \in [0,1)$ such that $t_1 < 1 - t_y < t_2 < 1 - t_0$. Then $\mathcal{A}(0) + t_2 = t_0 + t_2 < 1$ and $\mathcal{A}(y) + t_1 = t_y + t_1 < 1$. Thus $0_{t_2}\Upsilon\mathcal{A}$ and $y_{t_1}\Upsilon\mathcal{A}$. Now since

$$\mathcal{A}(y*0) + \max\{t_1,t_2\} = \mathcal{A}(y) + t_2 = t_y + t_2 > 1,$$

we get $(y*0)_{\max\{t_1,t_2\}}\overline{\Upsilon}\mathcal{A}$, which is also a contradiction. Therefore $\mathcal{A}$ is a constant on X^*. □

Theorem 3.11. Let $\mathcal{A}$ be a fuzzy set in a BCK/BCI-algebra X. Then $\mathcal{A}$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X if and only if there exists a subalgebra S of X such that

$$\mathcal{A}(x) := \begin{cases} t \in [0,1) & \text{if } x \in S, \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

Proof. Let $\mathcal{A}$ be a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. Then by Proposition 3.7 and Theorems 3.10 and 3.8 we get that $\mathcal{A}(x) < 1$ for all $x \in X$ and X^* is a subalgebra of X, and

$$\mathcal{A}(x) := \begin{cases} \mathcal{A}(0) & \text{if } x \in X^*, \\ 1 & \text{otherwise} \end{cases}$$

Conversely, let S be a subalgebra of X which satisfies (3.11). Assume that $x_s \Upsilon \mathcal{A}$ and $y_r \Upsilon \mathcal{A}$ for some $s, r \in [0,1)$. Then $\mathcal{A}(x) + s < 1$ and $\mathcal{A}(y) + r < 1$, and so $\mathcal{A}(x) \neq 1$ and $\mathcal{A}(y) \neq 1$. Thus $x, y \in S$ and so $x * y \in S$. It follows that $\mathcal{A}(x * y) + \max\{s, r\} = t + \max\{s, r\} < 1$ so that $(x * y)_{\max\{s,r\}} \Upsilon \mathcal{A}$. Therefore $\mathcal{A}$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.12. Let S be a subalgebra of a BCK/BCI-algebra X and let $\mathcal{A}$ be a fuzzy set in X such that

(i) $(\forall x \in X \setminus S)$ $(\mathcal{A}(x) = 1)$,
(ii) $(\forall x \in S)$ $(\mathcal{A}(x) \leq 0.5)$.

Then $\mathcal{A}$ is a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $t_1, t_2 \in [0,1)$ be such that $x_{t_1} \Upsilon \mathcal{A}$ and $y_{t_2} \Upsilon \mathcal{A}$,; i.e. $\mathcal{A}(x) + t_1 < 1$ and $\mathcal{A}(y) + t_2 < 1$. If $x * y \notin S$, then $x \in X \setminus S$ or $y \in X \setminus S$, i.e., $\mathcal{A}(x) = 1$ or $\mathcal{A}(y) = 1$. It follows that $t_1 < 0$ or $t_2 < 0$. This is a contradiction, and so $x * y \in S$. Hence $\mathcal{A}(x * y) \leq 0.5$. If $\max\{t_1, t_2\} < 0.5$, then $\mathcal{A}(x * y) + \max\{t_1, t_2\} < 1$ and thus $(x * y)_{\max\{t_1,t_2\}} \Upsilon \mathcal{A}$. If $\max\{t_1, t_2\} \geq 0.5$, then $\mathcal{A}(x * y) \leq 0.5 \leq \max\{t_1, t_2\}$ and so $(x * y)_{\max\{t_1,t_2\}} \lessdot \mathcal{A}$. Therefore $(x * y)_{\max\{t_1,t_2\}} \lessdot \vee \Upsilon \mathcal{A}$. This completes the proof. □

Theorem 3.13. Let $\mathcal{A}$ be a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of a BCK/BCI-algebra X such that $\mathcal{A}$ is not constant on X^*. Then $\mathcal{A}(x) \leq 0.5$ for all $x \in X^*$.

Proof. Assume that $\mathcal{A}(x) > 0.5$ for all $x \in X$. Since $\mathcal{A}$ is not constant on X^*, there exists $x \in X^*$ such that $t_x = \mathcal{A}(x) \neq \mathcal{A}(0) = t_0$. Then either $t_0 > t_x$ or $t_0 < t_x$. For the first case, choose $\delta < 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. It follows that $x_\delta \Upsilon \mathcal{A}$,

$$\mathcal{A}(x * x) = \mathcal{A}(0) = t_0 > \delta = \max\{\delta, \delta\},$$

$$\mathcal{A}(x * x) + \max\{\delta, \delta\} = \mathcal{A}(0) + \delta = t_0 + \delta > 1$$

so that $(x * x)_{\max\{\delta,\delta\}} \overline{\lessdot \vee \Upsilon} \mathcal{A}$., which is a contradiction. For the second case, we can choose $\delta < 0.5$ such that $t_x + \delta > 1 > t_0 + \delta$. Then $0_\delta \Upsilon \mathcal{A}$ and $x_0 \Upsilon \mathcal{A}$, but $(x * 0)_{\max\{0,\delta\}} = x_\delta \overline{\lessdot \vee \Upsilon} \mathcal{A}$ since $\mathcal{A}(x * 0) = \mathcal{A}(x) > 0.5 > \delta = \max\{0, \delta\}$ and $\mathcal{A}(x * 0) + \max\{0, \delta\} = \mathcal{A}(x) + \delta = t_x + \delta > 1$. This again leads to a contradiction. Therefore $\mathcal{A}(x) \leq 0.5$ for some $x \in X$. We now show that $\mathcal{A}(0) \leq 0.5$. Assume that $\mathcal{A}(0) = t_0 > 0.5$. Since there exists $x \in X$ such that $\mathcal{A}(x) = t_x \leq 0.5$, we have

$t_0 > t_x$. Choose $t_1 < t_0$ such that $t_x + t_1 < 1 < t_0 + t_1$. Then $\mathcal{A}(x) + t_1 = t_x + t_1 < 1$, and so $x_{t_1} \Upsilon \mathcal{A}$. Now we get

$$\mathcal{A}(x * x) + \max\{t_1, t_1\} = \mathcal{A}(0) + t_1 = t_0 + t_1 > 1,$$

$$\mathcal{A}(x * x) = \mathcal{A}(0) = t_0 > t_1 = \max\{t_1, t_1\}.$$

Hence $(x * x)_{\max\{t_1,t_1\}} \overline{\lessdot \vee \Upsilon} \mathcal{A}$, a contradiction. Therefore $\mathcal{A}(0) \leq 0.5$. Finally suppose that $t_x = \mathcal{A}(x) > 0.5$ for some $x \in X^*$. Let t be such that $0 < t < 0.5$ and $t_x > 0.5 + t$. Therefore $\mathcal{A}(x) + 0 < 1$ and $\mathcal{A}(0) + (0.5 - t) < 1$ implying that $x_0 \Upsilon \mathcal{A}$ and $0_{0.5-t} \Upsilon \mathcal{A}$. However, since $\mathcal{A}(x * 0) = \mathcal{A}(x) > 0.5 - t = \max\{0, 0.5 - t\}$ and $\mathcal{A}(x * 0) + \max\{0, 0.5 - t\} = \mathcal{A}(x) + 0.5 - t > 0.5 + t + 0.5 - t = 1$, we have $(x * 0)_{\max\{0,0.5-t\}} \overline{\lessdot \vee \Upsilon} \mathcal{A}$. This is also a contradiction. Hence $\mathcal{A}(x) \leq 0.5$ for all $x \in X^*$. □

We give a characterization of a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra.

Theorem 3.14. Let $\mathcal{A}$ be a fuzzy set in a BCK/BCI-algebra X. Then $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X if and only if it satisfies the following inequality:

$$(\forall x, y \in X)\,(\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}). \tag{1}$$

Proof. Assume that $\mathcal{A}$ is $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. Let $x, y \in X$ be such that $\max\{\mathcal{A}(x), \mathcal{A}(y)\} > 0.5$. Then $\mathcal{A}(x*y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}$. If not, then $\mathcal{A}(x * y) < t < \max\{\mathcal{A}(x), \mathcal{A}(y)\}$ for some $t \in (0.5, 1)$. It follows that $x_t \lessdot \mathcal{A}$ and $y_t \lessdot \mathcal{A}$,. However, $(x*y)_{\max\{t,t\}} = (x*y)_t \overline{\lessdot \vee \Upsilon} \mathcal{A}$ which is a contradiction. Hence $\mathcal{A}(x*y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}$ whenever $\max\{\mathcal{A}(x), \mathcal{A}(y)\} > 0.5$. If $\max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq 0.5$, then $x_{0.5} \lessdot \mathcal{A}$ and $y_{0.5} \lessdot \mathcal{A}$ implying that $(x * y)_{0.5} = (x * y)_{\max\{0.5,0.5\}} \lessdot \vee \Upsilon \mathcal{A}$. Now, if $\mathcal{A}(x*y) > 0.5$, then $\mathcal{A}(x*y) + 0.5 > 0.5 + 0.5 = 1$, a contradiction. Therefore $\mathcal{A}(x * y) \leq 0.5$. Hence $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$ for all $x, y \in X$.

Conversely assume that $\mathcal{A}$ satisfies (1). Let $x, y \in X$ and $t_1, t_2 \in [0, 1)$ be such that $x_{t_1} \lessdot \mathcal{A}$ and $y_{t_2} \lessdot \mathcal{A}$. Then $\mathcal{A}(x) \leq t_1$ and $\mathcal{A}(y) \leq t_2$. Suppose that $\mathcal{A}(x * y) > \max\{t_1, t_2\}$. If $\max\{\mathcal{A}(x), \mathcal{A}(y)\} > 0.5$ then

$$\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} = \max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq \max\{t_1, t_2\}.$$

This is a contradiction, and so $\max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq 0.5$. It follows that

$$\mathcal{A}(x * y) + \max\{t_1, t_2\} < 2\mathcal{A}(x * y) \leq 2 \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \leq 1$$

so that $(x * y)_{\max\{t_1,t_2\}} \Upsilon \mathcal{A}$. Hence $(x * y)_{\max\{t_1,t_2\}} \lessdot \vee \Upsilon \mathcal{A}$, and consequently $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.15. For any subset S of a BCK/BCI-algebra X, let χ_S denote the characteristic function of S. Then the function $\chi_S^c : X \to [0, 1]$ defined by $\chi_S^c(x) = 1 - \chi_S(x)$ for all $x \in X$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X if and only if S is a subalgebra of X.

Proof. Assume that χ_S^c is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X and let $x, y \in S$. Then $\chi_S^c(x) = 1 - \chi_S(x) = 0$ and $\chi_S^c(y) = 1 - \chi_S(y) = 0$. Hence $x_0 \lessdot \chi_S^c$ and $y_0 \lessdot \chi_S^c$,. It follows that $(x * y)_0 = (x * y)_{\max\{0,0\}} \lessdot \vee \Upsilon \chi_S^c$. Thus $\chi_S^c(x * y) \leq 0$ or $\chi_S^c(x * y) + 0 < 1$. If $\chi_S^c(x * y) \leq 0$, then $1 - \chi_S(x * y) = 0$, i.e., $\chi_S(x * y) = 1$.

Hence $x * y \in S$. If $\chi_S^c(x * y) + 0 < 1$, then $\chi_S(x * y) > 0$. Thus $\chi_S(x * y) = 1$, and so $x * y \in S$. Therefore S is a subalgebra of X.

Conversely, suppose that S is a subalgebra of X. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$, and thus

$$\chi_S^c(x * y) = \max\{\chi_S^c(x), \chi_S^c(y)\} \leq \max\{\chi_S^c(x), \chi_S^c(y), 0.5\}.$$

If any one of x and y does not belong to S, then $\chi_S^c(x) = 1$ or $\chi_S^c(y) = 1$. Hence $\chi_S^c(x*y) \leq \max\{\chi_S^c(x), \chi_S^c(y)\} \leq \max\{\chi_S^c(x), \chi_S^c(y), 0.5\}$. Hence by Theorem 3.14, χ_S^c is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.16. A fuzzy set $\mathcal{A}$ in a BCK/BCI-algebra X is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X if and only if the set

$$L(\mathcal{A}; t) := \{x \in X \mid \mathcal{A}(x) \leq t\},\ t \in [0.5, 1)$$

is a subalgebra of X.

Proof. Assume that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X and let $x, y \in L(\mathcal{A}; t)$. Then $\mathcal{A}(x) \leq t$ and $\mathcal{A}(y) \leq t$, and so $x_t \lessdot \mathcal{A}$ and $y_t \lessdot \mathcal{A}$. It follows from Theorem 3.14 that

$$\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \leq \max\{t, 0.5\} = t$$

so that $x * y \in L(\mathcal{A}; t)$. Hence $L(\mathcal{A}; t)$ is a subalgebra of X.

Conversely let $\mathcal{A}$ be a fuzzy set in X such that the set $L(\mathcal{A}; t) := \{x \in X \mid \mathcal{A}(x) \leq t\}$ is a subalgebra of X for all $t \in [0.5, 1)$. If there exist $x, y \in X$ such that $\mathcal{A}(x * y) > \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$, then we can take $t \in (0, 1)$ such that

$$\max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} < t < \mathcal{A}(x * y).$$

Thus $x, y \in L(\mathcal{A}; t)$ and $t > 0.5$, and so $x * y \in L(\mathcal{A}; t)$, i.e., $\mathcal{A}(x * y) \leq t$. This is a contradiction and therefore $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$ for all $x, y \in X$. Now it follows from Theorem 3.14, that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

For any fuzzy set $\mathcal{A}$ in X and $t \in [0, 1)$, we denote

$$\mathcal{A}_t := \{x \in X \mid x_t \Upsilon \mathcal{A}\} \quad \text{and} \quad [\mathcal{A}]_t := \{x \in X \mid x_t \lessdot \vee \Upsilon \mathcal{A}\}.$$

Obviously $[\mathcal{A}]_t = L(\mathcal{A}; t) \cup \mathcal{A}_t$.

Theorem 3.17. A fuzzy set $\mathcal{A}$ in a BCK/BCI-algebra X is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X if and only if $[\mathcal{A}]_t$ is a subalgebra of X for all $t \in [0, 1)$.

Proof. Let $\mathcal{A}$ be a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X and let $x, y \in [\mathcal{A}]_t$ for $t \in [0, 1)$. Then $x_t \lessdot \vee \Upsilon \mathcal{A}$ and $y_t \lessdot \vee \Upsilon \mathcal{A}$; that is, $\mathcal{A}(x) \leq t$ or $\mathcal{A}(x) + t < 1$, and $\mathcal{A}(y) \leq t$ or $\mathcal{A}(y) + t < 1$. Since $\mathcal{A}(x * y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$, by Theorem 3.14 we have $\mathcal{A}(x * y) \leq \max\{t, 0.5\}$. If not, then $x_t \overline{\lessdot \vee \Upsilon} \mathcal{A}$ or $y_t \overline{\lessdot \vee \Upsilon} \mathcal{A}$, a contradiction. If $t \geq 0.5$, then $\mathcal{A}(x * y) \leq \max\{t, 0.5\} = t$ and so $x * y \in L(\mathcal{A}; t) \subseteq [\mathcal{A}]_t$. If $t < 0.5$, then $\mathcal{A}(x * y) \leq \max\{t, 0.5\} = 0.5$ and thus $\mathcal{A}(x * y) + t < 0.5 + 0.5 = 1$. Hence $(x * y)_t \Upsilon \mathcal{A}$, and so $x * y \in \mathcal{A}_t \subseteq [\mathcal{A}]_t$. Therefore $[\mathcal{A}]_t$ is a subalgebra of X.

Conversely, let $\mathcal{A}$ be a fuzzy set in X and $t \in [0, 1)$ be such that $[\mathcal{A}]_t$ is a subalgebra of X. If possible, let $\max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} < t < \mathcal{A}(x * y)$ for some $t \in (0.5, 1)$. Then $x, y \in L(\mathcal{A}; t) \subseteq [\mathcal{A}]_t$, which implies that $x*y \in [\mathcal{A}]_t$. Hence $\mathcal{A}(x*y) \leq$

t or $\mathcal{A}(x*y)+t<1$, a contradiction. Therefore $\mathcal{A}(x*y) \le \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$ for all $x, y \in X$. It follows from Theorem 3.14, that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.18. Let $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebras of a BCK/BCI-algebra X. Then $\mathcal{A} := \bigcap_{i\in\Lambda} \mathcal{A}_i$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Proof. By Theorem 3.14 we have $\mathcal{A}_i(x*y) \le \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}$, and so

$$\begin{aligned} \mathcal{A}(x*y) &= \inf_{i\in\Lambda} \mathcal{A}_i(x*y) \\ &\le \inf_{i\in\Lambda} \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\} \\ &= \max\{\inf_{i\in\Lambda} \mathcal{A}_i(x), \inf_{i\in\Lambda} \mathcal{A}_i(y), 0.5\} \\ &= \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}. \end{aligned}$$

It follows that $\mathcal{A}$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.19. Let $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of $(\lessdot, \lessdot)^*$-fuzzy subalgebras of a BCK/BCI-algebra X. Then $\mathcal{A} := \bigcup_{i\in\Lambda} \mathcal{A}_i$ is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X.

Proof. Let $x_t \lessdot \mathcal{A}$ and $y_r \lessdot \mathcal{A}$, where $t, r \in [0, 1)$. Then $\mathcal{A}(x) \le t$ and $\mathcal{A}(y) \le r$. Thus for all $i \in \Lambda$, we have $\mathcal{A}_i(x) \le t$ and $\mathcal{A}_i(y) \le r$ and so $\mathcal{A}_i(x*y) \le \max\{t, r\}$. Therefore $\mathcal{A}(x*y) \le \max\{t, r\}$, which implies that $(x*y)_{\max\{t,r\}} \lessdot \mathcal{A}$. □

The following is an open problem: *Is the union of two $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebras of a BCK/BCI-algebra X a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X?*

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