

MODULARITY OF AJMAL FOR THE LATTICES OF FUZZY IDEALS OF A RING

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ABSTRACT. In this paper, we construct two fuzzy sets using the notions of level subsets and strong level subsets of a given fuzzy set in a ring R . These fuzzy sets turn out to be identical and provide a universal construction of a fuzzy ideal generated by a given fuzzy set in a ring. Using this construction and employing the technique of strong level subsets, we provide the shortest and direct fuzzy set theoretic proof of the fact that the lattice $\mathfrak{I}(R)$ of all fuzzy ideals of a ring R is modular.

1. Introduction

Fuzzy algebraic substructures are important when viewed from a lattice theoretic point of view. Ajmal and Thomas initiated such types of studies in the year 1994 [3]. It was later independently established by Ajmal [1] that the set of all fuzzy normal subgroups of a group constitute a sublattice of the lattice of all fuzzy subgroups of a given group and is modular.

Tom Head also discussed the modularity of a set of fuzzy subgroups of a group in his pioneering paper [7], wherein he formulated the well-known metatheorem. However the assertion that the set of all fuzzy normal subgroups constitute a modular lattice, follows from subdirect product theorem, is false. In his proof, Tom Head used the fact that the Rep function commutes with the set product of two fuzzy normal subgroups and this set product is considered to be the join of the given fuzzy normal subgroups. In fact, the join of two fuzzy normal subgroups is their set product if and only if they have the same tip. Therefore what follows from the subdirect product theorem is the fact that the set of all fuzzy normal subgroups with the same tip is modular.

In a series of papers [1,2-5] various sublattices of the lattice \mathcal{L} of all fuzzy subgroups of a group G are constructed and examined. On the other hand, for the set of all fuzzy ideals of a ring R , the only notable attempt so far, has been made in the year 1995 [4], wherein it has been shown that the set of all fuzzy ideals $\mathfrak{I}(R)$ constitute a lattice under the ordering of fuzzy set inclusion. Moreover, the construction of a particular sublattice of $\mathfrak{I}(R)$, that is, the sublattice $\mathfrak{I}_{st}(R)$ of all

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fuzzy ideals, each of them having sup property and the same tip 't', is presented in [4]. In the same paper, this lattice is shown to be modular. More than the modularity of this lattice, its construction, which arises from a property of functions, is important.

Even after the emergence of metatheorem in 1995, a direct proof of modularity of the lattice of fuzzy ideals of a ring was published by Zhang [12] in 2002. The proof of modularity in this paper is unnecessarily long, complicated and tedious and the author tries to demonstrate the utility of nested sets to establish modularity. However, the fact is, that the underlying structure of the proof is exactly the same as that of modularity of the lattice of fuzzy normal subgroups given by Ajmal [1], whose work the author very generously refers to in his paper [12]. Another attempt for establishing the modularity of fuzzy ideals of a ring has been made by Q. Zhang jointly with G. Meng in the year 2000[11], wherein the authors prove that the sublattice $\mathfrak{I}_t(\mathbf{R})$ of all fuzzy ideals with the same tip t of a ring R is modular. This proof is also long and is again almost similar to the proof given by Ajmal and Thomas [3], wherein they prove that the sublattice $\mathcal{L}_{\text{fin}_t}$ of all fuzzy normal subgroups with finite range sets and same tip t is modular.

Zhang and Meng [11] considered the sup property to be an assumption and established the more general result that "the lattice $\mathfrak{I}_t(\mathbf{R})$ of all fuzzy ideals of a ring R with the same tip 't' is modular". On the other hand, in [9], the authors have arrived at the false result that "the lattice $\mathfrak{I}(\mathbf{R})$ is distributive". In fact the lattice $\mathfrak{I}(\mathbf{R})$ of ideals of a ring R is not distributive and has an obvious embedding in $\mathfrak{I}(\mathbf{R})$. The corrected result has already appeared in several papers [8,11].

In this paper, we construct two fuzzy sets using the notions of level subsets and strong level subsets of a given fuzzy set in a ring R. These fuzzy sets are shown to be identical and provide a universal construction of a fuzzy ideal generated by a given fuzzy set in a ring. Then, using this construction and employing the technique of strong level subsets, we prove that the lattice $\mathfrak{I}(\mathbf{R})$ of all ideals of a ring R is modular. The technique used in this paper to prove the modularity of $\mathfrak{I}(\mathbf{R})$ is different from that of Ajmal's proof of modularity of the lattice of fuzzy normal subgroups of a group appeared in [1].

2. Preliminaries

In this section we recall some definitions and results. Details may be found in [4].

A fuzzy set is a function from a non empty set X to the closed unit interval. We say that the fuzzy set θ is contained in the fuzzy set η if $\theta(x) \leq \eta(x)$ for every $x \in X$ and is denoted by $\theta \subset \eta$. We also use the notation ' \subset ' for ordinary set inclusion. It is well known that the fuzzy power set $F(X)$, that is the set of all fuzzy

sets in X , constitute a complete lattice under the ordering of fuzzy set inclusion ' \subset '. The suprema $\cup \theta_i$ and the infima $\cap \theta_i$ of a family of fuzzy sets $\{\theta_i\}$ in X are defined as $\cup \theta_i(x) = \sup\{\theta_i(x)\}$ and $\cap \theta_i(x) = \inf\{\theta_i(x)\}$ respectively. The notions of level subset θ_t and the strong level subset $\theta_t^>$ for $t \in [0,1]$ are defined as $\theta_t = \{x \in R : \theta(x) \geq t\}$ and $\theta_t^> = \{x \in R : \theta(x) > t\}$.

Definition 2.1. Let θ be a fuzzy set in a ring R . Then θ is said to be a fuzzy subring of R if for all $x, y \in R$

- (i) $\theta(x - y) \geq \min\{\theta(x), \theta(y)\}$,
- (ii) $\theta(xy) \geq \min\{\theta(x), \theta(y)\}$.

It can be easily verified that if θ is a fuzzy subring of R then $\theta(x) \leq \theta(0)$ and $\theta(x) = \theta(-x)$ for all $x \in R$.

Definition 2.2. Let θ be a fuzzy subring of a ring R . Then θ is called a fuzzy (left, right) ideal of R if

$$(\theta(xy) \geq \theta(y), \theta(xy) \geq \theta(x)), \theta(xy) \geq \max\{\theta(x), \theta(y)\} \text{ For all } x, y \in R.$$

Proposition 2.3. Let θ be a fuzzy set of a ring R . Then the following are equivalent:

- (i) θ is a fuzzy ideal of R ,
- (ii) each level subset θ_t is an ideal of R , for $t \in [0, t_0]$ where $t_0 = \sup \theta$,
- (iii) each strong level subset $\theta_t^>$ is an ideal of R , for $t \in [0, t_0[$ where $t_0 = \sup \theta$.

Proposition 2.4. The intersection $\cap \theta_i$ of any family $\{\theta_i\}$ of fuzzy ideals of a ring R is a fuzzy ideal of R .

Definition 2.5. Let θ be a fuzzy set in a ring R . Then the fuzzy ideal generated by θ is defined to be the least fuzzy ideal of R which contains θ and is denoted by $\langle \theta \rangle$, i.e.

$$\langle \theta \rangle = \bigcap_{\theta \subset \theta_i} \{\theta_i : \theta_i \in \mathcal{I}(R)\}.$$

Here we would like to point out that the existence of such a fuzzy ideal is ensured in view of Proposition 2.4.

Proposition 2.6. The set of all fuzzy ideals $\mathfrak{I}(\mathbf{R})$ of a ring \mathbf{R} is a complete lattice under the ordering of fuzzy set inclusion, where the infima and the suprema of a family of fuzzy ideals of \mathbf{R} are defined as the intersection of the family and the ideal generated by their union respectively.

Lemma 2.7. Let θ and η be fuzzy sets in \mathbf{R} . Then

- (i) $(\theta \cup \eta)_t^> = \theta_t^> \cup \eta_t^>$ for all $t \in [0,1]$,
- (ii) $(\theta \cap \eta)_t^> = \theta_t^> \cap \eta_t^>$ for all $t \in [0,1]$,
- (iii) $\theta \subset \eta$ if and only if $\theta_t^> \subset \eta_t^>$ for all $t \in [0,1]$.

3. Modularity

In this section, we obtain a fuzzy analogue of a well-known result of classical ring theory that the set of all ideals of a ring forms a modular lattice. We begin with the construction of a fuzzy ideal generated by a given fuzzy set in a ring \mathbf{R} .

Theorem 3.1. Let θ be a fuzzy set in a ring \mathbf{R} . Define fuzzy sets $\iota(\theta)$ and $\bar{\iota}(\theta)$ in \mathbf{R} by

$$\iota(\theta)(x) = \sup_{r < \sup \theta} \{r : x \in \langle \theta_r \rangle\}$$

and

$$\bar{\iota}(\theta)(x) = \sup_{r < \sup \theta} \{r : x \in \langle \theta_r^> \rangle\}.$$

Then $\iota(\theta) = \bar{\iota}(\theta) = \langle \theta \rangle$.

Proof. Let $t_0 = \sup \theta$. Firstly we show that $\bar{\iota}(\theta) \in \mathfrak{I}(\mathbf{R})$. To prove this, we claim

$$\bar{\iota}(\theta)_t^> = \langle \theta_t^> \rangle \text{ for all } t \in [0, t_0[.$$

Let $t \in [0, t_0[$. Then for $x \in \bar{\iota}(\theta)_t^>$, we have

$$\bar{\iota}(\theta)(x) = \sup_{r < \sup \theta} \{r : x \in \langle \theta_r^> \rangle\} > t.$$

Thus there exists 's' such that $t < s \leq t_0$ and $x \in \langle \theta_s^> \rangle$. But $\langle \theta_s^> \rangle \subset \langle \theta_r^> \rangle$ and hence $x \in \langle \theta_r^> \rangle$. So that $\bar{\iota}(\theta)_t^> \subset \langle \theta_r^> \rangle$. For the reverse inclusion, let $x \in \langle \theta_r^> \rangle$. Then

$$x = \sum_{i=1}^k a_i^r x_i b_i^s + \sum_{i=k+1}^p n_i x_i$$

where $x_i \in \theta_i^>$ for all $i = 1, 2, \dots, p$; $a_i, b_i \in R$ for all $i = 1, 2, \dots, k$; and $n_i \in Z$ for all $i = k+1, \dots, p$. Note that here $r, s = 0$ or 1 such that r and s can be taken as zero only one at a time and $a_i^0 x_i b_i^s = x_i b_i, a_i^r x_i b_i^0 = a_i x_i$ respectively. Let $t_1 = \min\{\theta(x_1), \theta(x_2), \dots, \theta(x_p)\}$. Then $t_1 > t$. Next choose $t_1 > t_2 > t$. This implies that $x_i \in \theta_{t_2}^>$ for all $i = 1, 2, \dots, p$. Consequently $x \in \langle \theta_{t_2}^> \rangle$ and hence

$$\sup_{r < \sup \theta} \{r : x \in \langle \theta_r^> \rangle\} \geq t_2 > t.$$

By the definition of $\check{i}(\theta)$, it follows that $x \in \check{i}(\theta)_t^>$. Thus $\langle \theta_t^> \rangle \subset \check{i}(\theta)_t^>$. Therefore $\check{i}(\theta)_t^> = \langle \theta_t^> \rangle$ for all $t \in [0, t_0[$. Thus, by Proposition 2.3, we have $\check{i}(\theta)_t^> \in \mathfrak{I}(R)$. It can be easily verified that $\check{i}(\theta)$ is the smallest fuzzy ideal of R containing θ and $\check{i}(\theta) = \check{i}(\theta)$. This completes the proof of the theorem. \square

The following result is straightforward:

Lemma 3.2. Let θ and η be fuzzy sets in a ring R . The

- (i) $\langle \theta \rangle(0) = \sup_{x \in R} \{\theta(x)\}$,
- (ii) $\langle \theta \rangle_t^> = \langle \theta_t^> \rangle$ for all $t \in [0, t_0[$, where $t_0 = \langle \theta \rangle(0)$.

In the following theorem we use the notations ‘ \wedge ’ and ‘ \vee ’ for the infima and suprema respectively in the lattice $\mathfrak{I}(R)$ of all fuzzy ideals of a ring R .

Theorem 3.3. The lattice $\mathfrak{I}(R)$ of fuzzy ideals of a ring R is modular.

Proof. Let $\theta, \eta, \psi \in \mathfrak{I}(R)$ be such that $\eta \subset \theta$. In view of modular inequality, it suffices to prove that

$$\theta \wedge (\eta \vee \psi) \subset \eta \vee (\theta \wedge \psi).$$

If not, then for some $z \in R$,

$$\theta \wedge (\eta \vee \psi)(z) > \eta \vee (\theta \wedge \psi)(z).$$

Setting $t = \eta \vee (\theta \wedge \psi)(z)$, we get

$$z \notin (\eta \vee (\theta \wedge \psi))_t^>, \theta(z) > t \text{ and } \eta \vee \psi(z) > t.$$

So there exist t_1 and $t_2 \in]0,1]$ such that $\theta(z) > t_1 > t$ and $\eta \vee \psi(z) > t_2 > t$. Let $r = \min\{t_1, t_2\}$. Then $\theta(z) > r > t$ and $\eta \vee \psi(z) > r > t$. Now $\eta \vee \psi = \langle \eta \cup \psi \rangle$. Therefore $z \in \theta_r^>$ and $z \in \langle \eta \cup \psi \rangle_r^>$. However, by Lemmas 3.2 and 2.7, we get

$\langle \eta \cup \psi \rangle_r^> = \langle \eta_r^> \cup \psi_r^> \rangle$. Thus $z \in \theta_r^> \cap \langle \eta_r^> \cup \psi_r^> \rangle$. We claim that

$$\theta_r^> \cap \langle \eta_r^> \cup \psi_r^> \rangle = \langle \eta_r^> \cup (\theta_r^> \cap \psi_r^>) \rangle.$$

Since $\theta_r^>$ and $\langle \eta \cup \psi \rangle_r^>$ are nonempty subsets of \mathbf{R} , we have $r < \theta(0)$ and $r < \langle \eta \cup \psi \rangle(0)$. Consequently by Proposition 2.3, $\theta_r^>$ and $\langle \eta \cup \psi \rangle_r^>$ are ideals of \mathbf{R} . Since $\eta \subset \theta$, so by Lemma 2.7 $\eta_r^> \subset \theta_r^>$. Also, $\mathbf{I}(\mathbf{R})$, the lattice of all ideals of \mathbf{R} is modular, so we have

$$\theta_r^> \cap (\eta_r^> + \psi_r^>) = \eta_r^> + (\theta_r^> \cap \psi_r^>).$$

Since in $\mathbf{I}(\mathbf{R})$, $\langle \eta_r^> \cup \psi_r^> \rangle = \eta_r^> + \psi_r^>$ and $\eta_r^> + (\theta_r^> \cap \psi_r^>) = \langle \eta_r^> \cup (\theta_r^> \cap \psi_r^>) \rangle$, our claim is established. Consequently $z \in \langle \eta_r^> \cup (\theta_r^> \cap \psi_r^>) \rangle$. Also, since $r > t$, we obtain:

$$\langle \eta \cup (\theta \cap \psi) \rangle_r^> \subset \langle \eta \cup (\theta \cap \psi) \rangle_t^>.$$

But by Lemma 3.2 and Lemma 2.7, we get

$$\langle \eta \cup (\theta \cap \psi) \rangle_r^> = \langle \eta_r^> \cup (\theta_r^> \cap \psi_r^>) \rangle.$$

Therefore $z \in \langle \eta \cup (\theta \cap \psi) \rangle_t^>$. This contradicts the fact $z \notin \langle \eta \cup (\theta \cap \psi) \rangle_t^>$. Consequently

$$\theta \wedge (\eta \vee \psi) \subset \eta \vee (\theta \wedge \psi).$$

Hence $\mathfrak{I}(\mathbf{R})$ is modular and this completes the proof.

The main result of Zhang [11] follows as a corollary of the above theorem.

Corollary 3.4. $\mathfrak{I}_t(\mathbf{R})$, the set of all fuzzy ideals of a ring \mathbf{R} having tip t is a modular lattice.

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