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# t-BEST APPROXIMATION IN FUZZY NORMED SPACES

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ABSTRACT. The main purpose of this paper is to find t-best approximations in fuzzy normed spaces. We introduce the notions of t-proximinal sets and F-approximations and prove some interesting theorems. In particular, we investigate the set of all t-best approximations to an element from a set.

## 1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh [9] in 1965. since then, many mathematicians have studied fuzzy normed spaces from several angles ([6], [1], [2]) and, in 2001, Veeramani introduced the concept of t-best approximations in fuzzy metric spaces. In this paper we consider the set of all t-best approximations on fuzzy normed spaces and prove several theorems pertaining to this set.

**Definition 1.1.** A binary operation  $* : [0,1] \times [0,1] \longrightarrow [0,1]$  is said to be a continuous t-norm if ([0,1],\*) is a topological monoid with unit 1 such that  $a * b \le c * d$  whenever  $a \le c$  and  $b \le d$   $(a,b,c,d \in [0,1])$ .

**Definition 1.2.** [6] The 3-tuple (X, N, \*) is said to be a fuzzy normed space if X is a vector space, \* is a continuous t-norm and N is a fuzzy set on  $X \times (0, \infty)$  satisfying the following conditions for every  $x, y \in X$  and t, s > 0,

 $\begin{array}{l} (i) \ N(x,t) > 0, \\ (ii) \ N(x,t) = 1 \Leftrightarrow x = 0, \\ (iii) \ N(\alpha x,t) = N(x,t/\mid \alpha \mid), \mbox{ for all } \alpha \neq 0, \\ (iv) \ N(x,t) * N(y,s) \leq N(x+y,t+s), \\ (v) \ N(x,.) : (0,\infty) \longrightarrow [0,1] \mbox{ is continuous}, \\ (vi) \ \lim_{t \to \infty} N(x,t) = 1. \end{array}$ 

**Lemma 1.3.** [6] Let N be a fuzzy norm. Then: (i) N(x,t) is nondecreasing with respect to t for each  $x \in X$ , (ii) N(x-y,t) = N(y-x,t).

**Remark 1.4.** As was shown in [6], every fuzzy normed space induces a fuzzy metric space on it and is therefore a topological space.

**Definition 1.5.** [6] Let (X, N, \*) be a fuzzy normed space. The open ball B(x, r, t) and the closed ball B[x,r,t] with the center  $x \in X$  and radius 0 < r < 1, t > 0 are

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defined as follows:

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$$B(x, r, t) = \{ y \in X : N(x - y, t) > 1 - r \},\$$
  
$$B[x, r, t] = \{ y \in X : N(x - y, t) \ge 1 - r \}.$$

**Lemma 1.6.** [6] If (X, N, \*) is a fuzzy normed space. Then: (i) the function  $(x, y) \longrightarrow x + y$  is continuous, (ii) the function  $(\alpha, x) \longrightarrow \alpha x$  is continuous.

## 2. t-best Approximation

**Definition 2.1.** [8] Let A be a nonempty subset of a fuzzy normed space (X, N, \*). For  $x \in X$ , t > 0, let

$$d(A, x, t) = \sup\{N(y - x, t) : y \in A\}.$$

An element  $y_0 \in A$  is said to be a t-best approximation of x from A if

$$N(y_0 - x, t) = d(A, x, t)$$

**Definition 2.2.** Let A be a nonempty set of a fuzzy normed space (X, N, \*). For  $x \in X, t > 0$ , we shall denote the set of all elements of t-best approximation of x from A by  $P_A^t(x)$ ; i.e.,

$$P_A^t(x) = \{ y \in A : d(A, x, t) = N(y - x, t) \}.$$

If each  $x \in X$  has at least (respectively exactly) one t-best approximation in A, then A is called a t-proximinal (respectively t-Chebyshev) set.

**Definition 2.3.** For t > 0, a nonempty closed subset A of a fuzzy normed space (X, N, \*) is said to be t-boundedly compact if for each x in X and 0 < r < 1 $1, B[x, r, t] \cap A$  is a compact subset of X.

**Theorem 2.4.** (Invariance by translation and scalar multiplication)

Let A be a nonempty subset of a fuzzy normed space (X, N, \*). Then:

 $(i) \ d(A+y,x+y,t)=d(A,x,t), \ for \ every \ x,y\in X \ and \ t>0,$ 

 $\begin{array}{l} (ii) \ P_A^t(x+y) = P_A^t(x) + y, \ for \ every \ x, y \in X \ and \ t > 0, \\ (iii) \ d(\alpha A, \alpha x, t) = d(A, x, t/ \mid \alpha \mid) \ for \ every \ x \in X, \ t > 0 \ and \ \alpha \in \mathbb{R} \setminus \{0\}, \end{array}$ 

(iv)  $P_{\alpha A}^{|\alpha|t}(\alpha x) = \alpha P_A^t(x)$ , for every  $x \in X$ , t > 0 and  $\alpha \in \mathbb{R} \setminus \{0\}$ ,

(v) A is t-proximinal (respectively t-chebyshev) if and only if A+y is t-proximinal (respectively t-chebyshev), for any given  $y \in X$ ,

(vi) A is t-proximinal (respectively t-chebyshev) if and only if  $\alpha A$  is  $|\alpha|$  tproximinal (respectively  $| \alpha | t$ -chebyshev), for any given  $\alpha \in \mathbb{R} \setminus \{0\}$ .

*Proof.* (i) For any  $x, y \in X$  and t > 0,

$$\begin{split} d(A+y,x+y,t) &= \sup\{N((z+y)-(x+y),t): z \in A\} \\ &= \sup\{N(z-x,t): z \in A\} \\ &= d(A,x,t). \end{split}$$

(ii) Using (i),  $y_0 \in P_{A+y}^t(x+y)$  if and only if  $y_0 \in A+y$  and d(A+y, x+y, t) = $N(x+y-y_0,t)$  if and only if  $y_0-y \in A$  and  $d(A,x,t) = N(x-(y_0-y),t)$  if and only if  $y_0 - y \in P_A^t(x)$ ; i.e.,  $y_0 \in P_A^t(x) + y$ . (*iii*) We have,

$$d(\alpha A, \alpha x, t) = \sup\{N(\alpha x - \alpha z, t) : z \in A\}$$
  
= sup{N(\alpha(x - z), t) : z \in A}  
= sup{N(x - z, t/ | \alpha |) : z \in A}  
= d(A, x, t/ | \alpha |).

(*iv*) From (*iii*) it follows that  $y_0 \in P_{\alpha A}^{|\alpha|t}(\alpha x)$  if and only if  $y_0 \in \alpha A$  and  $d(\alpha A, \alpha x, |\alpha A, \beta A)$  $\alpha \mid t) = N(\alpha x - y_0, \mid \alpha \mid t)$  if and only if  $y_0/\alpha \in A$  and  $N(x - y_0/\alpha, t) = d(A, x, t)$ . However, this is equivalent to  $y_0/\alpha \in P_A^t(x)$ ; i.e.,  $y_0 \in \alpha P_A^t(x)$ . (v) is an immediate consequence of (ii), and (vi) follows from (iv). 

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**Corollary 2.5.** Let M be a nonempty subspace of x. Then: (i) d(M, x + y, t) = d(M, x, t), for every t > 0,  $x \in X$  and  $y \in M$ , (ii)  $P_M^t(x+y) = P_M^t(x) + y$ , for every t > 0,  $x \in X$  and  $y \in M$ , (iii)  $d(M, \alpha x, |\alpha t|) = d(M, x, t)$ , for every t > 0,  $x \in X$  and  $\alpha \in \mathbb{R} \setminus \{0\}$ , (iv)  $P_M^{|\alpha|t}(\alpha x) = \alpha P_M^t(x)$ , for every  $t > 0, x \in X$  and  $\alpha \in \mathbb{R} \setminus \{0\}$ .

*Proof.* (i) and (ii) follow from theorem (2.4)(i) and (2.4)(ii), and the fact that if M is a subspace and  $y \in M$ , then M + y = M.

(iii) and (iv) follow from Theorem 2.4 ((iii) and (iv)), and the fact that if M is a subspace and  $\alpha \neq 0$ , then  $\alpha M = M$ .

**Definition 2.6.** For  $x \in X$ , 0 < r < 1, t > 0,

$$S[x, r, t] = \{ y \in X : N(x - y, t) = 1 - r \},\$$
  
$$e_A^t(x) = 1 - d(A, x, t).$$

**Theorem 2.7.** Let (X, N, \*) be a fuzzy normed space, A be a subset of  $X, x \in X \setminus \overline{A}$ and t > 0. Then we have,

$$P_A^t(x) = A \cap B[x, e_A^t(x), t]$$
  
=  $A \cap S[x, e_A^t(x), t].$  (1)

*Proof.* The inclusions

$$P_A^t(x) \subseteq A \cap S[x, e_A^t(x), t] \subseteq A \cap B[x, e_A^t(x), t]$$
<sup>(2)</sup>

are obvious by the definitions of  $P_A^t(x)$  and  $e_A^t(x)$ . Conversely, let  $y \in A \cap B[x, e_A^t(x), t]$ , then we have,  $y \in A$  and

$$N(y - x, t) \ge 1 - e_A^t(x) = d(A, x, t) \ge N(y - x, t).$$

Therefore  $y \in A$  and

$$N(y - x, t) = d(A, x, t),$$

which implies that  $y \in P_A^t(x)$ . So,  $A \cap B[x, e_A^t(x), t] \subset P_A^t(x)$ , whence, by (2), we have (1), which completes the proof.

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**Remark 2.8.** Let (X, N, \*) be a fuzzy normed space, A be a subset of  $X, x \in X \setminus \overline{A}$  and t > 0. Then we have

$$A \cap B(x, e_A^t(x), t) = \emptyset, \tag{3}$$

because, if  $y_0 \in A \cap B(x, e_A^t(x), t)$  then  $d(A, x, t) \ge N(x - y_0, t) > d(A, x, t)$ , which is absurd.

**Corollary 2.9.** Let (X, N, \*) be a fuzzy normed space, A be a subset of  $X, x \in X \setminus \overline{A}$  with  $P_A^t(x) \neq \emptyset$  and 0 < r < 1 such that :

$$\emptyset \neq A \cap B[x, r, t] \subseteq S[x, r, t].$$
(4)

Then we have

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$$r = e_A^t(x),$$

and consequently  $A \cap B[x, r, t] = P_A^t(x)$ .

*Proof.* If  $r < e_A^t(x)$ , then by the definition of  $e_A^t(x)$  we have  $A \cap B[x, r, t] = \emptyset$ , which contradicts (4). If  $r > e_A^t(x)$ , since  $P_A^t(x) \neq \emptyset$ , then by (1) we have

$$\emptyset \neq P_A^t(x) = A \cap B[x, e_A^t(x), t] \subseteq A \cap B(x, r, t),$$

which contradicts (4), and this completes the proof.

**Definition 2.10.** Let (X, N, \*) be a fuzzy normed space, 0 < r < 1 and t > 0. We shall say that a set  $A \subset X$  supports the cell B[x, r, t], or that A is a support set of the cell B[x, r, t], if we have d(A, B[x, r, t], t) = 1 and  $A \cap B(x, r, t) = \emptyset$ .

**Theorem 2.11.** Let (X, N, \*) be a fuzzy normed space, A a non-void set in X,  $x \in X \setminus \overline{A}$ ,  $a_0 \in A$  and t > 0. We have  $a_0 \in P_A^t(x)$  if and only if the set A supports the cell  $B = B[x, 1 - N(a_0 - x, t), t]$ .

*Proof.* Assume that  $a_0 \in P_A^t(x)$ . Hence  $N(a_0 - x, t) = d(A, x, t)$ . Then by (3), we have  $A \cap B(x, 1 - N(a_0 - x, t), t) = \emptyset$ , on the other hand, since  $a_0 \in A \cap B[x, 1 - N(a_0 - x, t), t]$ , we have d(A, B, t) = 1. Consequently, the set A supports the cell B. Conversely, suppose  $a_0 \notin P_A^t(x)$ , hence  $N(a_0 - x, t) < d(A, x, t)$ , and let  $0 < \varepsilon < 1$  be such that  $N(a_0 - x, t) < d(A, x, t) - \varepsilon$ . Then there exists an  $a \in A$  such that  $N(a_0 - x, t) < d(A, x, t) - \varepsilon < N(a - x, t)$ , hence  $a \in B(x, 1 - N(a_0 - x, t), t)$ . Consequently, A does not support the cell B, which completes the proof.

**Remark 2.12.** We recall that a set A in a topological space  $\tau$  is said to be countably compact, if every countable open cover of A has a finite subcover, or, what is equivalent, if for every decreasing sequence  $A_1 \supset A_2 \supset \cdots$  of non-void closed subsets of A we have  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ .

**Theorem 2.13.** Let (X, N, \*) be a fuzzy normed space,  $\tau$  be an arbitrary topology on X and t > 0. If A is a non-void set of X such that for  $A \cap B[x, r, t]$  is  $\tau$ -countably compact, then A is t-proximinal. t-best Approximation in Fuzzy Normed Spaces

Proof. For every  $n \in \mathbb{N}, 0 < 1 - d(A, x, t) + \frac{d(A, x, t)}{n+1} < 1.$  Put

$$A_n^t = A \cap B[x, 1 - d(A, x, t) + \frac{d(A, x, t)}{n+1}, t] \ (n = 1, 2, \ldots).$$

Since for every  $n \in \mathbb{N}$ ,  $d(A, x, t)(1 - \frac{1}{n+1}) < d(A, x, t)$ , obviously  $A_1^t \supset A_2^t \supset \cdots$ and each  $A_n^t$  is non-void. Hence there exists  $a_n^t \in A$  such that

$$d(A, x, t)(1 - \frac{1}{n+1}) < N(a_n^t - x, t)$$

It follows that  $a_n^t \in A_n^t$ . Now, since each  $A_n^t$  is  $\tau$ -countably compact and  $\tau$ -closed, we conclude that there exists an  $a_0 \in \bigcap_{n=1}^{\infty} A_n^t$ . Then we have

$$d(A, x, t) \ge N(a_0 - x, t) \ge d(A, x, t)(1 - \frac{1}{n+1})(n = 1, 2, \ldots),$$

whence  $a_0 \in P_A^t(x)$  which completes the proof.

**Definition 2.14.** Let A be a nonempty subset of a fuzzy normed space (X, N, \*). An element  $y_0 \in A$  is said to be an F-best approximation of  $x \in X$  from A if it is a t-best approximation of x from A, for every t > 0, i.e.,

$$y_0 \in \bigcap_{t \in (0,\infty)} P_A^t(x).$$

The set of all elements of F-best approximations of X from A is denoted by  $FP_A(x)$ , i.e.,

$$FP_A(x) = \bigcap_{t \in (0,\infty)} P_A^t(x).$$

If each  $x \in X$  has at least (respectively exactly) one F-best approximation in A, then A is called a F-proximinal (respectively F-chebyshev) set.

**Example 2.15.** Let  $X = \mathbb{R}^2$ . For  $a, b \in [0, 1]$ , let a \* b = ab. Define  $N : \mathbb{R}^2 \times (0, \infty) \to [0, 1]$  by

$$N((x_1, x_2), t) = (\exp \frac{\sqrt{x_1^2 + x_2^2}}{t})^{-1}$$

Then (X, N, \*) is a fuzzy normed space. Let  $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, 0 \le x_2 \le x_1^2\}$  and x = (0, 2). Then for every t > 0,

$$N((-1,1) - (0,3), t) = N((1,1) - (0,3), t) = (\exp \frac{\sqrt{5}}{t})^{-1}.$$

On the other hand,

$$d(A, (0,3), t) = \sup\{N((x_1, x_2) - (0,3), t) \mid -1 \le x_1 \le 1, 0 \le x_2 \le x_1^2\}$$
  
= 
$$\sup\{(\exp\frac{\sqrt{x_1^2 + (x_2 - 3)^2}}{t})^{-1} \mid -1 \le x_1 \le 1, 0 \le x_2 \le x_1^2\}$$
  
= 
$$(\exp\frac{\sqrt{5}}{t})^{-1}.$$

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So, for every t > 0,  $y_0 = (-1, 1)$  and  $y_1 = (1, -1)$  are t-best approximations of (0, 3) from A. Therefore  $y_0 = (-1, 1)$  and  $y_1 = (1, -1)$  are *F*-best approximations of x = (0, 3) from A. Therefore A is not an *F*-Chebyshev set.

**Example 2.16.** Let  $X = \mathbb{R}$ . For  $a, b \in [0, 1]$ , let a \* b = ab. Define

 $N: \mathbb{R} \times (0, \infty) \to [0, 1],$ 

by

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$$N(x,t) = \frac{t}{t+|x|}.$$

Then (X, N, \*) is a fuzzy normed space. Let A = [0, 1]. Then, for every x > 1, 1 is an F-best approximation of x from A and for every x < 0, 0 is an F-best approximation of x from A. So A is an F-proximinal set.

**Remark 2.17.** For an arbitrary set  $A \subset X$  we shall denote by  $\partial A$  the boundary of A, by *IntA* the interior of A (hence  $\partial A = \overline{A} \setminus IntA$ ), and by  $\mathcal{M}_A$  the set of all elements of the F-best approximation of the elements  $x \in X$  from A, i.e.

$$\mathcal{M}_A = \bigcup_{x \in X} FP_A(x).$$

**Theorem 2.18.** Let (X, N, \*) be a fuzzy normed space and A be a F-best proximinal set in X. Then

$$\partial A \subset \overline{\mathcal{M}}_A.$$

*Proof.* If  $\partial A = \emptyset$ , the statement is obvious. If  $\partial A \neq \emptyset$ , let  $a_0 \in \partial A$ ,  $0 < \varepsilon < 1$  and t > 0 be arbitrary. Then there exists  $0 < \varepsilon' < 1$  such that  $(1-\varepsilon')*(1-\varepsilon') > 1-\varepsilon$  and the cell  $B(a_0, \varepsilon', t/2)$  contains at least one element  $x \in X \setminus A$ . Let  $\pi_A(x) \in FP_A(x)$  (it exists, since by hypothesis, A is F-proximinal). Then we have,

$$N(a_0 - \pi_A(x), t) \ge N(a_0 - x, t/2) * N(x - \pi_A(x), t/2)$$
  
=  $N(a_0 - x, t/2) * N(A, x, t/2)$   
 $\ge N(a_0 - x, t/2) * N(a_0 - x, t/2)$   
 $\ge (1 - \varepsilon') * (1 - \varepsilon')$   
 $> 1 - \varepsilon.$ 

So,  $B(a_0, \varepsilon, t) \cap \mathfrak{M}_A \neq \emptyset$  and since  $\varepsilon > 0$  is arbitrary, we obtain  $a_0 \in \overline{\mathfrak{M}}_A$  which completes the proof.

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