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INTUITIONISTIC FUZZY QUASI-METRIC AND PSEUDO-METRIC SPACES

Y. HONG, X. FANG AND B. WANG

Abstract. In this paper, we propose a new definition of intuitionistic fuzzy quasi-metric and pseudo-metric spaces based on intuitionistic fuzzy points. We prove some properties of intuitionistic fuzzy quasi- metric and pseudo-metric spaces, and show that every intuitionistic fuzzy pseudo-metric space is intuitionistic fuzzy regular and intuitionistic fuzzy completely normal and hence intuitionistic fuzzy normal. These are the intuitionistic fuzzy generalization of the corresponding properties of fuzzy quasi-metric and pseudo- metric spaces.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [12]. Fuzzy set theory has been shown to be an useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by considering the degree to which a certain object belongs to a set. However, in fuzzy set theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. In 1983, Antanassov [1, 2] introduces the concept of intuitionistic fuzzy sets, which attrribute both a membership degree and a non-membership degree to every objest. The only constraint on these two degrees is that the sum must smaller than or equal to 1. One of the main problems in the theory of fuzzy topological spaces is to obtain an appropriate and consistent notion of a fuzzy metric space. Many authors have investigated this question and several different notions of a fuzzy metric space have been defined and studied. In [8], Jin Han Park defined the notion of intuitionistic fuzzy metric spaces with the help of continuous $t-$norms and continuous $t-$conorms as a generalization of fuzzy metric space due to George and Veeramani. He also defined a Hausdorff topology on the intuitionistic fuzzy metric space and showed that every metric induces an intuitionistic fuzzy metric space and that some well known results for metric spaces including Baire's theorem and Uniform limit theorem hold for

intuitionistic fuzzy metric spaces. Based on the idea of $\mathscr{L}$-fuzzy sets [4], as a generalization of fuzzy metric space and intuitionistic fuzzy metric space, the authors [9] introduced the notion of $\mathscr{L}$-fuzzy metric spaces and proved a common fixed point theorem for commuting maps in $\mathscr{L}$-fuzzy metric spaces. In [11, 10], R. Saadati generalized some well-known results including the uniform continuity

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theorem, the Ascoli-Arzela theorem and a fuzzy version of Banach and Edelstein contraction theorems to $\mathscr{L}$-fuzzy metric spaces.

V. Gregori etc [5] proved that the topology generated by the intuitionistic fuzzy metric defined by Jin Han Park coincides with the topology generated by the fuzzy metric, and hence, the study of the space of intuitionistic fuzzy metric spaces defined by Jin Han Park reduces to the study of the fuzzy metric space. So some properties of intuitionistic fuzzy metric spaces defined by Jin Han Park follow directly from well-known theorems in fuzzy metric spaces.

In this paper, we propose a new definition of intuitionistic fuzzy quasi-metric and pseudo-metric spaces based on intuitionistic fuzzy points. Our definition of intuitionistic fuzzy metric spaces is different from the definition of intuitionistic fuzzy metric space considered by Jin Han Park, which is based on continuous t-norms and continuous t-conorms. Our definition coincides more formally with the definitions of quasi-metric and pseudo-metric space in real analysis and fuzzy topology, which are based on real numbers and fuzzy points. We also prove some properties of intuitionistic fuzzy quasi-metric and pseudo-metric spaces, which are the generalizations of correponding properties of fuzzy quasi-metric and pseudo- metric spaces [7].

The paper is organized as follows. In section 2, we give some basic definitions needed in the rest of the paper. In section 3, we prove some properties of intuitionistic fuzzy quasi-metric and pseudo-metric spaces.

2. **Preliminaries**

In this section, we give some definitions which are needed in the paper.

Definition 2.1. [2] Let X be a non-empty set. An intuitionistic fuzzy set A, is an object having the form $A = \{\langle x, \mu_A, v_A\rangle | x \in X\}$ where $\mu_A : X \to I$ and $v_A : X \to I$ are functions. $\mu_A(x))$ and $v_A(x))$ respectively denote the degree of membership and the degree of non -membership of $x \in X$ in the set $A and 0 \leq \mu_A(x) + v_A(x) \leq 1$ for each $x \in X$. We write $0_\sim = \{\langle x, 0, 1\rangle | \ x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle | \ x \in X\}$.

Definition 2.2. [2] Let X be a non-empty set, and consider the intuitionistic fuzzy sets

$$A = \{\langle x, \mu_A, v_A\rangle | \ x \in X\}, \quad B = \{\langle x, \mu_B, v_B\rangle | \ x \in X\}.$$

Then

(1)$A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $v_A(x) \geq v_B(x)$ for all $x \in X$
(2)$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
(3)$\bar{A} = \{\langle x, v_A, \mu_A\rangle | \ x \in X\}$.
(4)$A \cap B = \{\langle x, \mu_A \wedge \mu_B, v_A \vee v_B\rangle | \ x \in X\}$.
(5)$A \cup B = \{\langle x, \mu_A \vee \mu_B, v_A \wedge v_B\rangle | \ x \in X\}$.

Definition 2.3. [6] Let X be a non-empty set and $c \in X$. If $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ are two real numbers such that $\alpha + \beta \leq 1$, then the intuitionistic fuzzy set $c(\alpha, \beta) = \{\langle x, c_\alpha, 1 - c_{1-\beta}\rangle | \ x \in X\}$ (where c_α is the fuzzy point in X with support c and value α), is called an intuitionistic fuzzy point in X. The class of all intuitionistic fuzzy points in X is denoted by Ξ.

Let A be an intuitionistic fuzzy set in X. Then $c(\alpha,\ \beta) \in A$ if and only if $\alpha \leq \mu_A(c)$ and $v_A(c) \geq \beta$. In particular, $x(\alpha,\ \beta) \in y(\phi,\ \varphi)$ if and only if $x = y,\ \alpha \leq \phi$ and $\beta \geq \varphi$. The complement of the intuitionistic fuzzy point $c(\alpha,\ \beta)$ is denoted by $c(\beta,\ \alpha)$.

Definition 2.4. [6] Let $\{A_j |\ j \in J\}$ be an arbitrary family of intuitionistic fuzzy sets in X. Then
(1)$\cap A_j = \left\{ \left\langle x,\ \wedge\mu_{A_j},\ \vee v_{A_j} \right\rangle \middle|\ x \in X \right\}$;
(2)$\cup A_j = \left\{ \left\langle x,\ \vee\mu_{A_j},\ \wedge v_{A_j} \right\rangle \middle|\ x \in X \right\}$.

Definition 2.5. [3] An intuitionistic fuzzy topology on a nonempty set X is a family δ of intuitionistic fuzzy sets in X satisfying the following axioms:
(1) $0_\sim,\ 1_\sim \in \delta$;
(2) $G_1 \cap G_2 \in \delta$ for any $G_1, G_2 \in \delta$;
(3) $\cup G_i \in \delta$ for any arbitrary family $\{G_i :\ i \in J\} \subseteq \delta$.

The pair $(X,\ \delta)$ is called an intuitionistic fuzzy topological space and any intuitionistic fuzzy set in δ is called an intuitionistic fuzzy open set in X. The complement $\bar{A}$ of an intuitionistic fuzzy open set A is called an intuitionistic fuzzy closed set in X.

We write:

$$P_0 = \{x(\alpha,\ \beta) :\ x \in X, \alpha \times \beta \in (0,\ 1) \times (0,\ 1),\ \alpha + \beta \leq 1\},$$
$$P_1 = \{x(0,\ \beta) :\ x \in X, \beta \in [0,\ 1)\}, \quad P_* = P_0 \cup P_1 \cup 0_\sim \cup 1_\sim.$$

Definition 2.6. suppose a mapping $d :\ P_* \times P_* \to [0,\ \infty)$ is continuous for membership and non-membership degrees and satisfies the following axioms:
(1) If $y(\alpha_2, \beta_2) \in x(\alpha_1, \beta_1)$, then $d(x(\alpha_1, \beta_1),\ y(\alpha_2, \beta_2)) = 0$;
(2) $d(x(\alpha_1, \beta_1),\ z(\alpha_3, \beta_3)) \leq d(x(\alpha_1, \beta_1),\ y(\alpha_2, \beta_2)) + d(y(\alpha_2, \beta_2),\ z(\alpha_3, \beta_3))$,
for any $x(\alpha_1,\ \beta_1),\ y(\alpha_2,\ \beta_2),\ z(\alpha_3,\ \beta_3) \in P_*$. Then d is called an intuitionistic fuzzy quasi- metric for a set X.

If $d :\ P_* \times P_* \to [0,\ \infty)$ also satisfies the axiom:
(3) $d(x(\alpha_1, \beta_1),\ y(\alpha_2, \beta_2)) = d(x(\beta_1, \alpha_1),\ y(\beta_2, \alpha_2))$,
then d is called an intuitionistic fuzzy pseudo-metric for set X.

Here, we call a mapping continuous for membership and non-membership degrees if and only if for every intuitionistic fuzzy point $x(\alpha,\ \beta) \in P_0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sqrt{(\alpha_1 - \alpha)^2 + (\beta_1 - \beta)^2} < \delta$ and $y(\alpha_1,\ \beta_1) \in P_*$ implies $d(x(\alpha,\ \beta),\ y(\alpha_1,\ \beta_1)) < \varepsilon$.

Example: Let $X = [0, 1]$ and $d :\ P_* \times P_* \to [0,\ \infty)$ be defined as

$$d(x(\alpha, \beta), y(\theta, \vartheta)) = min\{|x - y|, \sqrt{(\alpha - \theta)^2 + (\beta - \vartheta)^2}\},$$

we can prove that d is an intuitionistic fuzzy quasi- metric for set X. In fact

(1) It is obvious that d is continuous for membership and non-membership degrees.

(2) If $x(\alpha,\beta) \in y(\theta,\vartheta)$, then $x = y$, thus $d(x(\alpha,\beta), y(\theta,\vartheta)) = 0$.

(3) For any $x(\alpha_1, \beta_1), y(\alpha_2, \beta_2), z(\alpha_3, \beta_3) \in P_*$, since $|x-z| \leq |x-y|+|y-z|$ and

$$\sqrt{(\alpha_1-\alpha_3)^2+(\beta_1-\beta_3)^2} \leq \sqrt{(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2}+\sqrt{(\alpha_2-\alpha_3)^2+(\beta_2-\beta_3)^2}.$$

So we have

$$\begin{aligned}&d(x(\alpha_1,\beta_1), z(\alpha_3,\beta_3))\\&= min\{|x-z|, \sqrt{(\alpha_1-\alpha_3)^2+(\beta_1-\beta_3)^2}\}\\&\leq min\{|x-y|+|y-z|, \sqrt{(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2}+\sqrt{(\alpha_2-\alpha_3)^2+(\beta_2-\beta_3)^2}\\&= d(x(\alpha_1,\beta_1), y(\alpha_2,\beta_2)) + d(y(\alpha_2,\beta_2), z(\alpha_3,\beta_3)).\end{aligned}$$

Thus d is an intuitionistic fuzzy quasi-metric on the set X.

Let d be an intuitionistic fuzzy quasi-metric on X. Then, for every $x(\alpha, \beta) \in P_*$ and $\varepsilon > 0$, $B_\varepsilon(x(\alpha, \beta)) = \cup\{y(\alpha_1, \beta_1) : d(x(\alpha, \beta), y(\alpha_1, \beta_1)) < \varepsilon\}$ is an intuitionistic fuzzy set, called the intuitionistic fuzzy ε-open ball of $x(\alpha, \beta)$. $B_{\bar{\varepsilon}}(x(\alpha, \beta)) = \cup\{y(\alpha_1, \beta_1) : d(x(\alpha, \beta), y(\alpha_1, \beta_1)) \leq \varepsilon\}$ is called an intuitionistic fuzzy ε-closed ball of $x(\alpha, \beta)$.

Proposition 2.7. *The family of all intuitionistic fuzzy open balls, corresponding to intuitionistic fuzzy quasi-metric (pseudo-metric) d,*

$$B = \{B_\varepsilon(x(\alpha, \beta)) : x(\alpha, \beta) \in P_*, \varepsilon > 0\},$$

forms a base of some intuitionistic fuzzy topology δ_d on X.

Proof. In order to prove that B forms a base of some intuitionistic fuzzy topology δ_d on X, it is sufficient to prove that B satisfies:

(1) If $U, V \in B$ and $x(\alpha, \beta) \subseteq U \cap V$, then there exists $W \in B$ such that $x(\alpha, \beta) \subseteq W \subseteq U \cap V$.

(2) $\cup B = X$.

(3) is obvious. So we only need to prove (1). Let

$$z(\alpha_1, \beta_1) \in B_s(x(\alpha_2,\beta_2)) \cap B_t(y(\alpha_3,\beta_3)),$$

and choose

$$r = \min\{s - d(x(\alpha_2, \beta_2), z(\alpha_1, \beta_1)), t - d(y(\alpha_3, \beta_3), z(\alpha_1, \beta_1))\},$$

then $r > 0$ and $z(\alpha_1, \beta_1) \in B_r(z(\alpha_1, \beta_1)) \subseteq B_s(x(\alpha_2,\beta_2)) \cap B_t(y(\alpha_3,\beta_3))$. In fact, for any $a(\alpha_4, \beta_4) \in B_r(z(\alpha_1, \beta_1))$, by axiom (2) of definition 2.6, we have

$$\begin{aligned}&d(a(\alpha_4, \beta_4), x(\alpha_2, \beta_2))\\&\leq d(a(\alpha_4, \beta_4), z(\alpha_1, \beta_1)) + d(z(\alpha_1, \beta_1), x(\alpha_2, \beta_2))\\&< r + d(z(\alpha_1, \beta_1), x(\alpha_2, \beta_2))\\&\leq \text{s}.\end{aligned}$$

So we have

$$B_r(z(\alpha_1, \beta_1)) \subseteq B_s(x(\alpha_2, \beta_2)).$$

Similarly, we have $B_r(z(\alpha_1, \beta_1)) \subseteq B_s(y(\alpha_3, \beta_3))$. This completes the proof of proposition. □

We shall henceforth call the intuitionistic fuzzy topology δ_d induced by intuitionistic fuzzy quasi-metric (pseudo- metric) d, the intuitionistic fuzzy quasi-metric topology and (X, δ_d) the intuitionistic fuzzy quasi-metric (pseudo-metric) space.

Definition 2.8. An intuitionistic fuzzy set A in an intuitionistic fuzzy topological space $(X,\ \delta)$ is called a neighborhood of $x\,(\alpha,\ \beta)$ if and only if there exists $B \in \delta$ such that $x\,(\alpha,\ \beta) \in B \subseteq A$. It is called an open neighborhood if it is a neighborhood and open.

Definition 2.9. An intuitionistic fuzzy point $c\,(\alpha,\ \beta)$ is said to be quasi-coincident with an intuitionistic fuzzy set A, $(c\,(\alpha,\ \beta)\,qA)$, if and only if $\alpha+\mu_A\,(c) > 1$ and $\beta+v_A\,(c) < 1$. An intuitionistic fuzzy set A is said to be quasi-coincident with an intuitionistic fuzzy set B (AqB), if and only if there exists $x \in X$ such that $\mu_A\,(x) + \mu_B\,(x) > 1$ and $v_A\,(x) + v_B\,(x) < 1$.

3. Some Properties of Intuitionistic Fuzzy Quasi-metric and Pseudo-metric Spaces

In this section, we prove some properties of intuitionistic fuzzy quasi- metric and pseudo-metric spaces.

Theorem 3.1. *Let $(X,\ \delta_d)$ be an intuitionistic fuzzy quasi-metric space, and for any intuitionistic fuzzy point $x\,(\alpha,\ \beta)$ suppose that $x\,(\alpha,\ \beta) \in P_0 \cup P_1$ and $\varepsilon > 0$. Then the intuitionistic fuzzy ε-open ball is an open neighborhood of $x\,(\alpha,\ \beta)$.*

Proof. It is sufficient to show that $x\,(\alpha,\ \beta) \in B_\varepsilon\,(x\,(\alpha,\ \beta))$. If $(\alpha,\ \beta) = (1,\ 0)$, then $d\,(x\,(1,\ 0)\,,\ x\,(1,\ 0)) < \varepsilon$, so we have $x\,(\alpha,\ \beta) \in B_\varepsilon\,(x\,(\alpha,\ \beta))$. If $(\alpha,\ \beta) \in (0,\ 1) \times (0,\ 1)$, since d is continuous for membership and non-membership degrees, there exist are $\alpha_1 > \alpha$ and $\beta_1 < \beta$ such that $d\,(x\,(\alpha,\ \beta)\,,\ x\,(\alpha_1,\ \beta_1)) < \varepsilon$. Thus we have $x\,(\alpha_1,\ \beta_1) \in B_\varepsilon\,(x\,(\alpha,\ \beta))$ and hence $x\,(\alpha,\ \beta) \in B_\varepsilon\,(x\,(\alpha,\ \beta))$. □

Theorem 3.2. *Let $(X,\ \delta_d)$ be an intuitionistic fuzzy pseudo-metric space. If $0 < \alpha_0 = \mu_{B_\varepsilon(x(\alpha,\ \beta))}\,(y) < 1$ and $0 < \beta_0 = v_{B_\varepsilon(x(\alpha,\ \beta))}\,(y) < 1$, then*

$$d\,(x\,(\alpha,\ \beta)\,,\ y\,(\alpha_0,\ \beta_0)) = \varepsilon.$$

Proof. Let $\alpha_n,\ n = 1, 2, \cdots,$ be a strictly increasing sequence, convergent to α_0 and $\beta_n,\ n = 1, 2, \cdots,$ be a strictly decreasing Sequence, convergent to β_0. For any $n_1 <\ n_2$, by the axiom 1 and 2 of definition 2.6, we have

$$d\,(x\,(\alpha,\ \beta)\,, y\,(\alpha_{n_2},\ \beta_{n_2})) \le d\,(x\,(\alpha,\ \beta)\,, y\,(\alpha_{n_1},\ \beta_{n_1})) < \varepsilon.$$

Therefore

$$\lim_{n\to\infty} d\,(x\,(\alpha,\ \beta)\,, y\,(\alpha_n,\ \beta_n)) = \varepsilon^* \le \varepsilon.$$

Since d is continuous for membership and non-membership degrees, we have

$$\lim_{n\to\infty} d\,(y\,(\alpha_n,\ \beta_n)\,, y\,(\alpha,\ \beta)) = 0.$$

By axiom 3 of definition 2.6, we have

$$\begin{aligned} &d\,(x\,(\alpha,\ \beta)\,, y\,(\alpha_{n_2},\ \beta_{n_2}))\\ &\le d\,(x\,(\alpha,\ \beta)\,, y\,(\alpha_0,\ \beta_0))\\ &\le d\,(x\,(\alpha,\ \beta)\,, y\,(\alpha_{n_2},\ \beta_{n_2})) + d\,(y\,(\alpha_{n_2},\ \beta_{n_2})\,, y\,(\alpha_0,\ \beta_0))\,, \end{aligned}$$

for any n_2. Hence $d\,(x\,(\alpha,\ \beta)\,, y\,(\alpha_0,\ \beta_0)) = \varepsilon^*$.

Next we show that $\varepsilon = \varepsilon^*$. Suppose that $\varepsilon > \varepsilon^*$, since d is continuous for membership and non-membership degrees, there exists $\delta > 0$ such that $\sqrt{(\alpha_* - \alpha)^2 + (\beta_* - \beta)^2} < \delta$ implies $d\,(y\,(\alpha,\ \beta)\,,\ y\,(\alpha_*,\ \beta_*)) < \varepsilon - \varepsilon^*$ for intuitionistic fuzzy point $y\,(\alpha_0,\ \beta_0)$. Choose $\alpha_0 < \alpha^1 < \alpha_0 + \frac{\delta}{2},\ \beta_0 - \frac{\delta}{2} < \beta^1 < \beta_0$. Then

$$\begin{aligned}&d\left(x\left(\alpha,\ \beta\right),y\left(\alpha^1,\ \beta^1\right)\right)\\&\leq d\left(x\left(\alpha,\ \beta\right),y\left(\alpha_0,\ \beta_0\right)\right)+d\left(y\left(\alpha_0,\ \beta_0\right),y\left(\alpha^1,\ \beta^1\right)\right)\\&<\varepsilon,\end{aligned}$$

and so $y\left(\alpha^1,\ \beta^1\right)\in B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)$. This contradicts the definition of $\left(\alpha^1,\ \beta^1\right)$. Therefore, we have $\varepsilon=\varepsilon^*$, and $d\left(x\left(\alpha,\ \beta\right),\ y\left(\alpha_0,\ \beta_0\right)\right)=\varepsilon$. □

Theorem 3.3. *Let $(X,\ \delta_d)$ be an intuitionistic fuzzy pseudo-metric space. Then, for any intuitionistic fuzzy point $x\left(\alpha,\ \beta\right)$, and $\varepsilon^*>\varepsilon>0$, we have $B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)\subseteq B_{\varepsilon^*}\left(x\left(\alpha,\ \beta\right)\right)$ and $B_{\varepsilon^*}\left(x\left(\alpha,\ \beta\right)\right)$ is a neighborhood of $\overline{B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)}$.*

Proof. Given any $z\left(\alpha_1,\ \beta_1\right)\in\overline{B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)}$, each neighborhood of $z\left(\beta_1,\ \alpha_1\right)$ and $B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)$ are quasi-coincident. If $\left(\alpha_1,\ \beta_1\right)\in(0,\ 1)\times(0,\ 1)$, then $B_\delta\left(z\left(\beta_1,\ \alpha_1\right)\right)$ is quasi-coincident with $B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)$ for any $\delta>0$. That is, there is an intuitionistic fuzzy point $y\left(\alpha_2,\ \beta_2\right)\in B_\delta\left(z\left(\beta_1,\ \alpha_1\right)\right)$ and $y\left(\alpha_2,\ \beta_2\right)\not\subset B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)$, such that

$$d\left(z\left(\beta_1,\ \alpha_1\right),\ y\left(\alpha_2,\ \beta_2\right)\right)\leq\delta,$$

and

$$d\left(y\left(\alpha_2,\ \beta_2\right),\ x\left(\beta,\ \alpha\right)\right)=d\left(x\left(\alpha,\ \beta\right),\ y\left(\beta_2,\ \alpha_2\right)\right)<\varepsilon.$$

Hence

$$\begin{aligned}&d\left(x\left(\alpha,\ \beta\right),\ z\left(\alpha_1,\ \beta_1\right)\right)\\&\leq d\left(z\left(\beta_1,\ \alpha_1\right),\ y\left(\alpha_1,\ \beta_1\right)\right)+d\left(y\left(\alpha_1,\ \beta_1\right),\ x\left(\beta,\ \alpha\right)\right)\\&<\varepsilon+\delta.\end{aligned}$$

Since $\delta>0$ is arbitrary, we have $d\left(x\left(\alpha,\ \beta\right),\ z\left(\alpha_1,\ \beta_1\right)\right)\leq\varepsilon$. □

Theorem 3.4. *Let $(X,\ \delta_d)$ be an intuitionistic fuzzy pseudo-metric space, then any intuitionistic fuzzy ε-closed ball*

$$B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)=\cup\left\{y\left(\alpha_1,\ \beta_1\right):\ d\left(x\left(\alpha,\ \beta\right),\ y\left(\alpha_1,\ \beta_1\right)\right)\leq\varepsilon\right\},$$

is a closed intuitionistic fuzzy set and

$$B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)=\bigcap_{\varepsilon^*>\varepsilon}B_{\varepsilon^*}\left(x\left(\alpha,\ \beta\right)\right).$$

Proof. Given any intuitionistic fuzzy point $z\left(\alpha_1,\ \beta_1\right)\in\overline{B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)}$, by the proof of Theorem 3.3 we have $d\left(x\left(\alpha,\ \beta\right),\ z\left(\alpha_1,\ \beta_1\right)\right)\leq\varepsilon$. Hence $\overline{B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)}\subseteq B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)$ and so $B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)$ is a closed intuitionistic fuzzy set.

If $y\left(\alpha_2,\ \beta_2\right)\in\bigcap_{\varepsilon^*>\varepsilon}B_{\varepsilon^*}\left(x\left(\alpha,\ \beta\right)\right)$, then

$$d\left(x\left(\alpha,\ \beta\right),\ y\left(\alpha_2,\ \beta_2\right)\right)\leq\varepsilon^*$$

for any $\varepsilon^*>\varepsilon$. Hence $d\left(x\left(\alpha,\ \beta\right),\ y\left(\alpha_2,\ \beta_2\right)\right)\leq\varepsilon$ and we have $z\left(\alpha_1,\ \beta_1\right)\in B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)$. Then $\bigcap_{\varepsilon^*>\varepsilon}B_{\varepsilon^*}\left(x\left(\alpha,\ \beta\right)\right)\subseteq B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)$. Thus we have $B_{\bar\varepsilon}\left(x\left(\alpha,\ \beta\right)\right)=\bigcap_{\varepsilon^*>\varepsilon}B_{\varepsilon^*}\left(x\left(\alpha,\ \beta\right)\right)$. □

Theorem 3.5. *Every intuitionistic fuzzy pseudo-metric space $(X,\ \delta_d)$ is intuitionistic fuzzy regular.*

Proof. Given any open intuitionistic fuzzy set A and an intuitionistic fuzzy point $x\left(\alpha,\ \beta\right)\in A$, by Theorem 3.1 and Theorem 3.3, we have

$$x\left(\alpha,\ \beta\right)\in B_{\frac{\varepsilon}{2}}\left(x\left(\alpha,\ \beta\right)\right)\subseteq\overline{B_{\frac{\varepsilon}{2}}\left(x\left(\alpha,\ \beta\right)\right)}\subseteq B_\varepsilon\left(x\left(\alpha,\ \beta\right)\right)\subseteq A.$$

Moreover, $B_{\frac{\varepsilon}{2}}(x(\alpha,\ \beta))$ and A are neighborhoods of $x(\alpha,\ \beta) \in A$ and $\overline{B_{\frac{\varepsilon}{2}}(x(\alpha,\ \beta))}$, respectively. Hence $(X,\ \delta_d)$ is intuitionistic fuzzy regular. □

Theorem 3.6. *Every intuitionistic fuzzy pseudo-metric space $(X,\ \delta_d)$ is intuitionistic fuzzy completely normal and hence intuitionistic fuzzy normal.*

Proof. Let A be an intuitionistic fuzzy set in intuitionistic fuzzy pseudo-metric space $(X,\ \delta_d)$ and B be its neighborhood satisfying the condition $P(x, A(x)) \in B,\ x \in X$. We prove that there is an open intuitionistic fuzzy set O such that $A \subseteq O \subseteq \bar{O} \subseteq B$. Moreover, O is a neighborhood of A and $P\left(x, \bar{O}(x)\right) \in B,\ x \in X$.

Since B is a neighborhood of A, there is an $\varepsilon = \varepsilon(x(\alpha,\ \beta))$ such that $B_{\varepsilon}(x(\alpha,\ \beta)) \subseteq B^{\circ}$ for any point $x(\alpha,\ \beta) \in A$. Let

$$O_A = \cup \left\{ B_{\frac{\varepsilon}{2}}(x(\alpha,\ \beta)) :\ x(\alpha,\ \beta) \in A,\ \varepsilon = \varepsilon(x(\alpha,\ \beta)) \right\},$$

then O_A is an open neighborhood of A. Since $P\left(x, \overline{A}(x)\right) \in B,\ x \in X$, $\overline{A'}$ is an open neighborhood of B', and so we can define an open neighborhood $O_{B'}$ of B' similarly. Now we prove that the intuitionistic fuzzy sets O_A and $O_{B'}$ are quasi-discoincident. In fact, if O_A and $O_{B'}$ are quasi-coincident, then there exists an intuitionistic fuzzy point $z(\alpha_1,\ \beta_1) \in A$ quasi-coincident with $O_{B'}$ i.e. $z(\beta_1,\ \alpha_1) \in O_{B'}$. That is, there are intuitionistic fuzzy points $x(\alpha,\ \beta) \in A$ and $y(\alpha_2,\ \beta_2) \not\subset B$ such that $d(x(\alpha,\beta), z(\alpha_1,\beta_1)) < \epsilon(x(\alpha,\beta))/2$ and $d(y(\beta_1,\alpha_1), z(\beta_1,\alpha_1)) < \epsilon(x(\alpha,\beta))/2$. By axioms (2) and (3) of definition 2.6, we have

$$\begin{aligned} & d(x(\alpha,\beta), y(\alpha_2,\beta_2)) \\ & \leq d(x(\alpha,\beta), z(\alpha_1,\beta_1)) + d(z(\alpha_1,\beta_1), y(\alpha_2,\beta_2)) \\ & < max\{\epsilon(\alpha,\beta), \epsilon y(\beta_2,\alpha_2)\}. \end{aligned}$$

On the other hand, since $x(\alpha,\ \beta) \in A$ and $y(\alpha_2,\ \beta_2) \not\subset B$, we have

$$d(x(\alpha,\beta), y(\alpha_2,\beta_2)) \geq \epsilon(x(\alpha,\beta)),$$

$$d(x(\beta_2,\alpha_2), x(\beta,\alpha)) \geq \epsilon(y(\beta_2,\alpha_2)),$$

Hence

$$d(x(\alpha,\beta), y(\alpha_2,\beta_2)) \geq max\{\epsilon x(\alpha,\beta), \epsilon y(\beta_2,\alpha_2)\}.$$

This is a contradiction. Thus O_A and $O_{B'}$ are quasi-discoincident, so we have $A \subseteq O_A \subseteq O'_{B'} \subseteq B$ and $P(x, \overline{O'}_{B'}(x)) \in B, x \in X$. Since $O'_{B'}$ is a closed intuitionistic fuzzy set, we have $A \subseteq O_A \subseteq \overline{O}_A \subseteq B$. Moreover, O_A is a neighborhood of A and $P(x, \overline{O}_A(x)) \in B, x \in X$. □

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Yongfa Hong*, College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao, Shandong, 266510, P. R. China
E-mail address: `hzycfl@ 126.com`

Xianwen Fang, Department of Mathematics and Physics, Anhui University of Science and Technology, Huainan,Anhui, 232001, P. R. China

Binguo Wang, College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao, Shandong, 266510, P. R. China

∗Corresponding author