ON $(\in, \in \vee q)$-FUZZY IDEALS OF BCI-ALGEBRAS

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Abstract. The aim of this paper is to introduce the notions of $(\in, \in \vee q)$-fuzzy p-ideals, $(\in, \in \vee q)$-fuzzy q-ideals and $(\in, \in \vee q)$-fuzzy a-ideals in BCI-algebras and to investigate some of their properties. Several characterization theorems for these generalized fuzzy ideals are proved and the relationship among these generalized fuzzy ideals of BCI-algebras is discussed. It is shown that a fuzzy set of a BCI-algebra is an $(\in, \in \vee q)$-fuzzy a-ideal if and only if it is both an $(\in, \in \vee q)$-fuzzy p-ideal and an $(\in, \in \vee q)$-fuzzy q-ideal. Finally, the concept of implication-based fuzzy a-ideals in BCI-algebras is introduced and, in particular, the implication operators in Lukasiewicz system of continuous-valued logic are discussed.

1. Introduction

Logic appears in a 'sacred' form (resp., a 'profane') which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold. It is a tool for applications, as well as a technique for laying the foundation. Non-classical logic including many-valued logic, fuzzy logic, etc., uses classical logic to handle information with various facets of uncertainty ([34] for generalized theory of uncertainty), such as fuzziness and randomness and has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Non-comparability is another important kind of uncertainty and is often encountered in real life.

In recent years, the study of t-norm-based logic systems and the corresponding pseudo-logic systems has been become an important topic in the field of logic. t-norm-based algebraic investigations were first of the algebraic investigations, and in the case of pseudo-logic systems, algebraic development preceded the corresponding logical development. For more details, the reader is referred to [26]. As it is well known, BCK and BCI-algebras are two classes of logical algebras which were introduced by Imai and Iseki [8,11,12]. BCI-algebras are generalizations of BCK-algebras and both these logical algebras have been extensively investigated (, [13-30,35-38]). Jun [13-18] investigated several kinds of (fuzzy) ideals of BCI/BCK-algebras and Liu etc. studied certain types of (fuzzy) ideals of BCI-algebras [21-27]. Zhan etc. [35] obtained results on f-derivations in BCI-algebras. Iorgulescu

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[9,10] showed that under condition (S) pocrims and BCK-algebras are categorically isomorphic, and residuated lattices and bounded BCK lattices are categorically isomorphic. Iseki and Tanaka [12] proved that Boolean algebras are equivalent to bounded implicative BCK-algebras and Mundici [30] proved that MV-algebras are equivalent to bounded commutative BCK-algebras. Hence, most of the algebras related to the t-norm based logic, such as MTL [6], BL[7], hoop, MV[3](i.e., lattice implication algebra) and Boolean algebras etc., are extensions of BCK-algebras (i.e., they are subclasses of BCK-algebras). This shows that BCK/BCI-algebras are quite general structures.

After the introduction of fuzzy sets by Zadeh [33], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, the $(\in, \in\vee\mathrm{q})$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [2] using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets introduced by Pu and Liu [31]. In fact, the $(\in, \in\vee\mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems of other algebraic structures. Jun ([14,15]) introduced the concept of (α, β)-fuzzy subalgebras (ideals) of a BCK/BCI-algebra and investigated their properties. Recently, Davvaz [4] applied this theory to near-rings and obtained some useful results. Furthermore, Davvaz and Corsini [5] redefined fuzzy H_v-submodules and many valued implications. For more details, the reader is referred to [4,5,14,15,19].

This paper is a continuation of [14,15]. In section 2, we recall some basic definitions and results of BCI-algebras. In section 3, we introduce the notion of $(\in, \in\vee\mathrm{q})$-fuzzy p-ideals in BCI-algebras and investigate some of their properties. In section 4, we introduce the notion of $(\in, \in\vee\mathrm{q})$-fuzzy a-ideals of BCI-algebras and study the relationship among these generalized fuzzy ideals of BCI-algebras Finally, in section 5, we study the concept of implication-based fuzzy a-ideals in BCI-algebras, and in particular, discuss the implication operators in Lukasiewicz system of continuous-valued logic.

2. **Preliminaries**

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the axioms:

(i) $((x*y)*(x*z))*(z*y) = 0$;
(ii) $(x*(x*y))*y = 0$;
(iii) $x*x = 0$;
(iv) $x*y = 0$ and $y*x = 0$ imply $x = y$.

We can define a partial ordering " $\leq$ " by $x \leq y$ if and only if $x*y = 0$.

If a BCI-algebra X satisfies $0*x = 0$ for all $x \in X$, then we say that X is a BCK-algebra. In what follows, X will denote a BCI-algebra unless otherwise specified.

Lemma 2.1. *([16]) For any BCI-algebra X, we have:*
(i) $0*(0*((x*z)*(y*z))) = (0*y)*(0*x)$;

(ii) $0*(0*(x*y)) = (0*y)*(0*x)$.

A non-empty subset I of X is called an *ideal* of X if (I1) $0 \in I$; (I2) $x*y \in I$ and $y \in I$ imply $x \in I$. A non-empty subset I of X is called a *p-ideal* if it satisfies (I1) and (I3) $(x*z)*(y*z) \in I$ and $y \in I$ imply $x \in I$. A non-empty subset I of X is called a *q-ideal* if it satisfies (I1) and (I4) $x*(y*z) \in I$ and $y \in I$ imply $x*z \in I$. A non-empty subset I of X is called an *a-ideal* if it satisfies (I1) and (I5) $(x*z)*(0*y) \in I$ and $z \in I$ imply $y*x \in I$([25,36,37,38]).

Definition 2.2. ([16]) A fuzzy set μ in X is called a fuzzy ideal of X if it satisfies the following conditions:
(F1) $\mu(0) \geq \mu(x), \forall x \in X$,
(F2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}, \forall x, y \in X$.

Definition 2.3. ([16,23])
(i) A fuzzy set μ in X is called a fuzzy p-ideal of X if it satisfies (F1) and (F3) $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y)\}$, for all $x, y, z \in X$.

(ii) A fuzzy set μ in X is called a fuzzy q-ideal of X if it satisfies (F1) and
(F4) $\mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\}$, for all $x, y, z \in X$.

Definition 2.4. ([27]) A fuzzy set μ in X is called a fuzzy a-ideal of X if it satisfies (F1) and
(F5) $\mu(y*x) \geq \min\{\mu((x*z)*(0*y)), \mu(z)\}$, for all $x, y, z \in X$.

For any fuzzy set μ of X and $t \in (0,1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a *level subset* of μ.

Theorem 2.5. *([16,23,27]) A fuzzy set μ in X is a fuzzy p-ideal(resp., q-ideal (a-ideal)) of X if and only if for all $t \in (0,1]$, each non-empty level subset μ_t is a p-ideal(resp.,q-ideal (a-ideal)) of X.*

A fuzzy set μ of a BCI-algebra X of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t and is denoted by $U(x;t)$.* A fuzzy point $U(x;t)$ is said to *belong to* (resp. be *quasi-coincident with*) a fuzzy set μ, written as $U(x;t) \in \mu$ (resp. $U(x;t)q\mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $U(x;t) \in \mu$ or (resp. and) $U(x;t)q\mu$, then we write $U(x;t) \in\vee q$(resp. $\in\wedge q$) μ . The symbol $\overline{\in\vee q}$ means $\in\vee q$ does not hold. The concept of (α, β)-fuzzy subsemigroup, where α and β are any two of $\{\in, q, \in\vee q, \in\wedge q\}$ with $\alpha \neq \in\wedge q$, was introduced in [2] using the notion of " belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy subsets. Indeed, the most viable generalization of Rosenfeld's fuzzy subgroup is the $(\in, \in\vee q)$-fuzzy subgroup. For more information about $(\in, \in\vee q)$-fuzzy subgroups, the reader is referred to [1] .

In [14], Jun introduced the concept of (α, β)-fuzzy ideals of a BCK/BCI-algebra and investigated related results.

Definition 2.6. ([14]) A fuzzy set μ of X is called an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X if for all $t, r \in (0, 1]$ and $x, y \in X$,
(F6) $U(x;t) \in \mu$ implies $U(0;t) \in\vee\mathrm{q}\mu$,
(F7) $U(x*y;t) \in \mu$ and $U(y;r) \in \mu$ imply $U(x;\min\{t,r\}) \in\vee\mathrm{q}\mu$.

Lemma 2.7. *([14]) The conditions (F6) and (F7) in Definition 2.6, are respectively equivalent to the following:*
(F8) $\mu(0) \geq \min\{\mu(x), 0.5\}$, *for all* $x \in X$,
(F9) $\forall x, y \in X, \mu(x) \geq \min\{\mu(x*y), \mu(y), 0.5\}$.

Lemma 2.8. *([14]) A fuzzy set μ in X is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X if and only if the set μ_t is an ideal of X for all $0 < t \leq 0.5$.*

Lemma 2.9. *([14]) Let μ be a fuzzy set in X. Then μ_t is an ideal of X for all $0.5 < t \leq 1$ if and only if it satisfies:*
(F10) $\forall x \in X, \max\{\mu(0), 0.5\} \geq \mu(x)$,
(F11) $\forall x, y \in X, \max\{\mu(x), 0.5\} \geq \min\{\mu(x*y), \mu(y)\}$.

Lemma 2.10. *Let μ be an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X. Then $x*y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z), 0.5\}$.*

Proof. Since $x*y \leq z$, then $(x*y)*z = 0$. By Lemma 2.7, $\mu(x*y) \geq \min\{\mu((x*y)*z), \mu(z), 0.5\} = \min\{\mu(0), \mu(z), 0.5\} \geq \min\{\mu(z), 0.5\}$. Thus, $\mu(x) \geq \min\{\mu(x*y), \mu(y), 0.5\} \geq \min\{\mu(z), \mu(y), 0.5\}$. □

Lemma 2.11. *([14]) Every fuzzy ideal of X is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal.*

3. $(\in, \in\vee\mathrm{q})$-fuzzy p-ideals

In this section, we introduce the concept of $(\in, \in\vee\mathrm{q})$-fuzzy p-ideals and discuss their properties.

Definition 3.1. An $(\in, \in\vee\mathrm{q})$-fuzzy ideal μ of X is called an $(\in, \in\vee\mathrm{q})$-fuzzy p-ideal of X if it satisfies:
(F12) $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y), 0.5\}$, for all $x, y, z \in X$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a BCI-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a fuzzy set μ in X by $\mu(0) = 0.8, \mu(1) = 0.7$, and $\mu(2) = \mu(3) = 0.3$. It is easy to verify that μ is an $(\in, \in\vee q)$-fuzzy p-ideal of X.

Theorem 3.3. *Every fuzzy p-ideal of X is an $(\in, \in\vee q)$-fuzzy p-ideal, but the converse may not be true.*

Proof. Let μ be a fuzzy p-ideal of X, then it is also a fuzzy ideal of X. By Lemma 2.11, we know that μ is an $(\in, \in\vee q)$-fuzzy ideal of X.

Then by (F3), for any $x, y, z \in X$ we have
$\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$.

(i) If $\min\{\mu((x*z)*(y*z)), \mu(y)\} \geq 0.5$, then $\mu((x*z)*(y*z)) \geq 0.5$ and $\mu(y) \geq 0.5$, which implies that $\mu(x) \geq 0.5$. Thus, $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y), 0.5\}$.

(ii) If $\min\{\mu((x*z)*(y*z)), \mu(y)\} < 0.5$, then $\min\{\mu((x*z)*(y*z)), \mu(y)\} = \min\{\mu((x*z)*(y*z)), \mu(y), 0.5\}$.

Thus, $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y), 0.5\}$.

This proves that μ satisfies (F12), and so μ is an $(\in, \in\vee q)$-fuzzy p-ideal of X.

The following example showa that the converse is not generally true.

Define a fuzzy set ν in X in Example 3.2 by $\nu(0) = 0.7, \nu(1) = \nu(2) = 0.8$ and $\nu(3) = 0.6$. It is now easy to verify that ν is an $(\in, \in\vee q)$-fuzzy p-ideal of X, but it is not a fuzzy p-ideal of X. □

Lemma 3.4. *Let μ be an $(\in, \in\vee q)$-fuzzy ideal of X, then $\mu(0 * (0 * x)) \geq \min\{\mu(x), 0.5\}$, for all $x \in X$.*

Proof. For any $x \in X$, we have
$\mu(0 * (0 * x)) \geq \min\{\mu((0 * (0 * x)) * x), \mu(x), 0.5\}$
$= \min\{\mu(0), \mu(x), 0.5\} \geq \min\{\mu(x), 0.5\}$. □

Proposition 3.5. *Let μ be an $(\in, \in\vee q)$-fuzzy ideal of X. Then the following are equivalent:*

(i) μ is an $(\in, \in\vee q)$-fuzzy p-ideal,

(ii) $\mu(x) \geq \min\{\mu(0 * (0 * x)), 0.5\}$, for all $x \in X$.

Proof. (i)⇒ (ii) Let μ be an $(\in, \in\vee q)$-fuzzy p-ideal of X. Putting $z = x$ and $y = 0$ in (F12), we have
$\mu(x) \geq \min\{\mu((x * x) * (0 * x)), \mu(0), 0.5\} = \min\{\mu(0 * (0 * x)), 0.5\}$.

(ii)⇒ (i) For any $x, y, z \in X$, we have
$\mu(x) \geq \min\{\mu(x * y), \mu(y), 0.5\}$ (by (F9))
$\geq \min\{\mu(0 * (0 * (x * y))), \mu(y), 0.5\}$ (by (ii))
$= \min\{\mu((0*y)*(0*x)), \mu(y), 0.5\}$ (by Lemma 2.1(ii))
$= \min\{\mu(0 * (0 * ((x * z) * (y * z)))), \mu(y), 0.5\}$ (by Lemma 2.1(i))
$\geq \min\{\mu((x * z) * (y * z)), \mu(y), 0.5\}$. (by Lemma 3.4)
Hence, μ is an $(\in, \in\vee q)$-fuzzy p-ideal of X. □

In what follows, we characterize the $(\in, \in\vee q)$-fuzzy p-ideals using their level p-ideals.

Theorem 3.6. *Let μ be an $(\in, \in\vee\mathrm{q})$-fuzzy p-ideal of X. Then for all $0 < t \leq 0.5$, μ_t is an empty set or a p-ideal of X. Conversely, if μ is a fuzzy set of X such that $\mu_t(\neq \emptyset)$ is a p-ideal of X for all $0 < t \leq 0.5$, then μ is an $(\in, \in\vee\mathrm{q})$-fuzzy p-ideal of X.*

Proof. Let μ be an $(\in, \in\vee\mathrm{q})$-fuzzy p-ideal of X and $0 < t \leq 0.5$. Then, by Lemma 2.8, we know that μ_t is an ideal of X. Let $(x * z) * (y * z) \in \mu_t$ and $y \in \mu_t$, then $\mu((x * z) * (y * z)) \geq t$ and $\mu(y) \geq t$. It follows that
$\mu(x)$
$\geq \min\{\mu((x * z) * (y * z)), \mu(y), 0.5\}$
$\geq \min\{t, 0.5\}$
$= t$,
which implies $x \in \mu_t$. Thus, μ_t is a p-ideal of X.

Conversely, let μ be a fuzzy set of X such that $\mu_t(\neq \emptyset)$ is a p-ideal of X for all $0 < t \leq 0.5$. Then, by Lemma 2.8, μ is an $(\in, \in\vee\mathrm{q})$-fuzzy ideal of X. We can write
$\mu((x * z) * (y * z)) \geq \min\{\mu((x * z) * (y * z)), \mu(y), 0.5\} = t_0$,
$\mu(y) \geq \min\{\mu((x * z) * (y * z)), \mu(y), 0.5\} = t_0$.
Thus, $(x * z) * (y * z), y \in \mu_{t_0}$, which implies $x \in \mu_{t_0}$, and so,
$\mu(x) \geq t_0 = \min\{\mu((x*z)*(y*z)), \mu(y), 0.5\}$. Therefore, μ is an $(\in, \in\vee\mathrm{q})$-fuzzy p-ideal of X. □

We can prove a similar result for the case when μ_t is a p-ideal of X for $0.5 < t \leq 1$.

Theorem 3.7. *Let μ be a fuzzy set of X. Then $\mu_t(\neq \emptyset)$ is a p-ideal of X for all $0.5 < t \leq 1$ if and only if it satisfies (F10), (F11) and*
(F13) $\max\{\mu(x), 0.5\} \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$, *for all $x, y, z \in X$.*

Proof. Assume that $\mu_t(\neq \emptyset)$ is a p-ideal of X. Then, it follows from Lemma 2.9 that(F10) and (F11) hold.

If there exist $x, y, z \in X$ such that $\max\{\mu(x), 0.5\} < \min\{\mu((x * z) * (y * z)), \mu(y)\} = t$, then $0.5 < t \leq 1$, $\mu(x) < t$, and $(x * z) * (y * z), y \in \mu_t$. Since μ_t is a p-ideal of X, we have $x \in \mu_t$. This leads to a contradiction. Hence (F13) holds.

Conversely, suppose that the conditions (F10),(F11) and (F13) hold. By Lemma 2.9, we know that μ_t is an ideal of X. Assume that $0.5 < t \leq 1, (x*z)*(y*z), y \in \mu_t$. Then $0.5 < t \leq \min\{\mu((x * z) * (y * z)), \mu(y)\} \leq \max\{\mu(x), 0.5\} = \mu(x)$. Therefore μ_t is a p-ideal of X. □

Let μ be a fuzzy set of X and $J = \{t | t \in (0, 1]$ and μ_t is an empty set or a p-ideal of $X\}$. In particular, if $J = (0, 1]$, then μ is an ordinary fuzzy p-ideal of X (Theorem 2.5); if $J = (0, 0.5]$, μ is an $(\in, \in\vee\mathrm{q})$-fuzzy p-ideal of X(Theorem 3.6).

In [32], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization the fuzzy subgroups of Rosenfeld as well as Bhakat and Das. Based on the results of [32], we can extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy p-ideal with thresholds as follows:

Definition 3.8. Let $\alpha, \beta \in [0,1]$ and $\alpha < \beta$. Then a fuzzy set μ of X is called a fuzzy p-ideal with thresholds $(\alpha, \beta]$ of X if it satisfies,
(F14) $\forall x \in X, \max\{\mu(0), \alpha\} \geq \min\{\mu(x), \beta\}$ and
(F15) $\forall x, y, z \in X, \max\{\mu(x), \alpha\} \geq \min\{\mu((x*z)*(y*z)), \mu(y), \beta\}$.

We now characterize fuzzy p-ideals with thresholds using their level p-ideals.

Theorem 3.9. *A fuzzy set μ of X is a fuzzy p-ideal with thresholds $(\alpha, \beta]$ of X if and only if $\mu_t (\neq \emptyset)$ is a p-ideal of X for all $\alpha < t \leq \beta$.*

Proof. The proof is similar to the proofs of Theorems 3.6 and 3.7. □

4. $(\in, \in\vee q)$-fuzzy a-ideals

In this section, we introduce the concepts of $(\in, \in\vee q)$-fuzzy a-(q-) ideals in BCI-algebras and investigate some of their properties.

Definition 4.1. An $(\in, \in\vee q)$-fuzzy ideal of X is called an $(\in, \in\vee q)$-fuzzy a-ideal of X if it satisfies the following additional condition:
(F16) $\mu(y*x) \geq \min\{\mu((x*z)*(0*y)), \mu(z), 0.5\}$, for all $x, y, z \in X$.

Example 4.2. Consider the BCI-algebra X in Example 3.2.
Define a fuzzy set ω in X by $\omega(0) = \omega(1) = 0.8, \omega(2) = \omega(3) = 0.3$. It is now easy to verify that ω is an $(\in, \in\vee q)$-fuzzy a-ideal of X.

A proof similar to that of Theorem 3.3, shows that every fuzzy a-ideal of X is an $(\in, \in\vee q)$-fuzzy a-ideal. However, as the following example shows, the converse is not true in general.

Example 4.3. Let $X = \{0, 1, 2\}$ be a BCI-algebra with Cayley table as follows:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Define a fuzzy set μ in X by $\mu(0) = 0.6, \mu(1) = \mu(2) = 0.7$. It is now easy to verify that μ is an $(\in, \in\vee q)$-fuzzy a-ideal of X, but μ is not a fuzzy a-ideal of X.

Characterizations of $(\in, \in\vee q)$-fuzzy a-ideals are given by the following proposition.

Proposition 4.4. *Let μ be an $(\in, \in\vee q)$-fuzzy ideal of X. Then the following are equivalent:*
(i) μ is an $(\in, \in\vee q)$-fuzzy a-ideal,
(ii) $\mu(y*(x*z)) \geq \min\{\mu((x*z)*(0*y)), 0.5\}$, for all $x, y, z \in X$,
(iii) $\mu(y*x) \geq \min\{\mu(x*(0*y)), 0.5\}$, for all $x, y \in X$.

Proof. (i) $\Rightarrow$ (ii) By (F16), we have
$\mu(y*(x*z)) \geq \min\{\mu(((x*z)*0)*(0*y)), \mu(0), 0.5\}$
$= \min\{\mu((x*z)*(0*y)), \mu(0), 0.5\}$
$\geq \min\{\mu((x*z)*(0*y)), 0.5\}$.
(ii) $\Rightarrow$ (iii) Putting $z = 0$ in (ii), we get (iii).
(iii)$\Rightarrow$(i) Since $(x*(0*y))*((x*z)*(0*y)) \leq x*(x*z) \leq z$,
it follows by Lemma 2.10 that
$\mu(x*(0*y)) \geq \min\{\mu((x*z)*(0*y)), \mu(z), 0.5\}$.
From (iii), we have $\mu(y*x) \geq \min\{\mu(x*(0*y)), 0.5\}$
$\geq \min\{\mu((x*z)*(0*y)), \mu(z), 0.5\}$.
Hence μ satisfies (F16), and so μ is an $(\in, \in \vee \mathrm{q})$-fuzzy a-ideal of X. □

Using the level a-ideals of BCI-algebras, we can characterize the $(\in, \in \vee \mathrm{q})$-fuzzy a-ideals as follows:

Theorem 4.5. *Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy a-ideal of X. Then for all $0 < t \leq 0.5$, μ_t is an empty set or an a-ideal of X. Conversely, if μ is a fuzzy set of X such that $\mu_t(\neq \emptyset)$ is an a-ideal of X for all $0 < t \leq 0.5$, then μ is an $(\in, \in \vee \mathrm{q})$-fuzzy a-ideal of X.*

Proof. The proof is similar to that of Theorem 3.6. □

We can prove a similar result for the case when μ_t is an a-ideal of X, for all $0.5 < t \leq 1$.

Theorem 4.6. *Let μ be a fuzzy set of X. Then $\mu_t(\neq \emptyset)$ is an a-ideal of X for all $0.5 < t \leq 1$ if and only if it satisfies (F10), (F11) and*
(F17) $\max\{\mu(y*x), 0.5\} \geq \min\{\mu((x*z)*(0*y)), \mu(z)\}$, *for all $x, y, z \in X$.*

Proof. It is similar to the proof of Theorem 3.7. □

Definition 4.7. An $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X is called an $(\in, \in \vee \mathrm{q})$-fuzzy q-ideal of X if it satisfies the following additional condition:
(F18) $\mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y), 0.5\}$, for all $x, y, z \in X$.

Example 4.8. Let $X = \{0, 1, 2\}$ be a BCI-algebra with Cayley table as follows:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a fuzzy set μ in X by $\mu(0) = 0.8, \mu(1) = \mu(2) = 0.3$. It is now easy to verify that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy q-ideal of X.

Proposition 4.9. *An $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X is an $(\in, \in \vee \mathrm{q})$-fuzzy q-ideal of X if and only if it satisfies:*
(F19) $\mu(x*y) \geq \min\{\mu(x*(0*y)), 0.5\}$, *for all $x, y \in X$.*

Proof. Let μ be an $(\in, \in\vee q)$-fuzzy q-ideal of X. Putting $z = y$ and $y = 0$ in (F18), we have

$\mu(x * y) \geq \min\{\mu(x * (0 * y)), \mu(0), 0.5\}$
$\geq \min\{\mu(x * (0 * y)), 0.5\}$.
Thus, μ satisfies (F19).
Conversely, assume that μ satisfies (F19), then we have
$\mu((x * y) * z) \geq \min\{\mu((x * y) * (0 * z)), 0.5\}$. (*)
Since $((x * y) * (0 * z)) * (x * (y * z))$
$= ((x * y) * (x * (y * z))) * (0 * z)$
$\leq ((y * z) * y) * (0 * z)$
$= (0 * z) * (0 * z) = 0$,
that is, $((x * y) * (0 * z)) * (x * (y * z)) = 0$.
By Lemma 2.10, we have
$\mu((x * y) * (0 * z))$
$\geq \min\{\mu(x * (y * z)), \mu(0), 0.5\}$
$\geq \min\{\mu(x * (y * z)), 0.5\}$. (**)
It follows from (*) and (**) that
$\mu((x * y) * z) \geq \min\{\mu((x * y) * (0 * z)), 0.5\} \geq \min\{\mu(x * (y * z)), 0.5\}$.
Hence, $\mu(x * z) \geq \min\{\mu((x * z) * y), \mu(y), 0.5\}$
$= \min\{\mu((x * y) * z), \mu(y), 0.5\}$
$\geq \min\{\mu(x * (y * z)), \mu(y), 0.5\}$.
This proves that μ satisfies (F18), and so μ is an $(\in, \in\vee q)$-fuzzy q-ideal of X. □

We now discuss the relations among $(\in, \in\vee q)$-fuzzy p-ideals, $(\in, \in\vee q)$-fuzzy q-ideals and $(\in, \in\vee q)$-fuzzy a-ideals in BCI-algebras.

Theorem 4.10. *Every $(\in, \in\vee q)$-fuzzy a-ideal of X is an $(\in, \in\vee q)$-fuzzy p-ideal, but the converse may not be true.*

Proof. Putting $x = 0$ in Proposition 4.4, we get $\mu(y) \geq \min\{\mu(0 * (0 * y)), 0.5\}$. By Proposition 3.5, we know that μ is an $(\in, \in\vee q)$-fuzzy p-ideal of X.

To show that the converse is not generally true, define a fuzzy set ν in X of Example 4.3 by $\nu(0) = 0.8, \nu(1) = \nu(2) = 0.3$. It is easy to check that ν is an $(\in, \in\vee q)$-fuzzy p-ideal of X, but ν is not an $(\in, \in\vee q)$-fuzzy a-ideal of X, because
$\nu(2 * 1) = \nu(1) = 0.3 \ngeq 0.5 = \min\{\nu((1 * 0) * (0 * 2)), \nu(0), 0.5\}$. □

Theorem 4.11. *Every $(\in, \in\vee q)$-fuzzy a-ideal of X is an $(\in, \in\vee q)$-fuzzy q-ideal, but the converse may not be true.*

Proof. Since $(0 * (0 * (y * (0 * x)))) * (x * (0 * y))$
$= ((0 * (0 * y)) * (0 * (0 * (0 * x))) * (x * (0 * y))$
$\leq (x * (0 * y)) * (x * (0 * y)) = 0$.
Thus, by Lemma 2.10, we get,
$\mu(0 * (0 * (y * (0 * x)))) \geq \min\{\mu(x * (0 * y)), \mu(0), 0.5\}$.
By Theorem 4.10, we know that μ is an $(\in, \in\vee q)$-fuzzy p-ideal of X. Then, by Proposition 3.5, we have
$\mu(y * (0 * x))$

$\geq \min\{\mu(0*(0*(y*(0*x)))), 0.5\}$
$\geq \min\{\mu(x*(0*y)), 0.5\}$.
Thus, by Proposition 4.4(ii), we have
$\mu(x*y)$
$\geq \min\{\mu(y*(0*x)), 0.5\}$
$\geq \min\{\mu(x*(0*y)), 0.5\}$.
It follows from Proposition 4.9 that μ is an $(\in, \in\vee q)$-fuzzy q-ideal of X.

To show that the converse is not generally true, we once again consider Example 4.8. We know that μ is an $(\in, \in\vee q)$-fuzzy q-ideal of X, but μ is not an $(\in, \in\vee q)$-fuzzy a-ideal of X, because: $\mu(1*0) = \mu(1) = 0.3 \ngeq 0.5 = \min\{\mu((0*0)*(0*1)), \mu(0), 0.5\}$. □

Lemma 4.12. *If μ is an $(\in, \in\vee q)$-fuzzy q-ideal of X, then for all $x \in X$, $\mu(0*x) \geq \min\{\mu(x), 0.5\}$.*

Proof. Putting $x = 0, y = x$ and $z = x$ in (F18), we have $\mu(0*x) \geq \min\{\mu(0*(x*x)), \mu(x), 0.5\} = \min\{\mu(0), \mu(x), 0.5\} \geq \min\{\mu(x), 0.5\}$. □

Theorem 4.13. *A fuzzy set μ in X is an $(\in, \in\vee q)$-fuzzy a-ideal of X if and only if it is both an $(\in, \in\vee q)$-fuzzy p-ideal and an $(\in, \in\vee q)$-fuzzy q-ideal.*

Proof. Necessity: By Theorem 4.10 and 4.11.

Sufficiency: Let μ be both an $(\in, \in\vee q)$-fuzzy p-ideal and an $(\in, \in\vee q)$-fuzzy q-ideal of X. By Proposition 4.9, we have $\mu(x*y) \geq \min\{\mu(x*(0*y)), 0.5\}$. Since $0*(y*x) \leq x*y$, we have $\mu(0*(y*x)) \geq \min\{\mu(x*y), 0.5\} \geq \min\{\mu(x*(0*y)), 0.5\}$. It follows from Lemma 4.12 that $\mu(0*(0*(y*x))) \geq \min\{\mu(0*(y*x)), 0.5\}$ $\geq \min\{\mu(x*(0*y)), 0.5\}$.

Applying Proposition 3.5, we have
$\mu(y*x) \geq \min\{\mu(0*(0*(y*x))), 0.5\} \geq \min\{\mu(x*(0*y)), 0.5\}$.
It follows from Proposition 4.4 that μ is an $(\in, \in\vee q)$-fuzzy a-ideal of X. □

5. Implication-Based Fuzzy a-ideals

Fuzzy logic is an extension of set theoretic variables in terms of the linguistic variable truth. Some operators, like $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic can also be defined by using the truth tables. Also, the extension principle can be used to derive definitions of the operators.

In fuzzy logic, we denote the truth value of fuzzy proposition P by $[P]$. In what follows, we display the fuzzy logical and corresponding set-theoretical notions:

$[x \in \mu] = \mu(x)$;
$[x \notin \mu] = 1 - \mu(x)$;
$[P \wedge Q] = \min\{[P], [Q]\}$;
$[P \vee Q] = \max\{[P], [Q]\}$;
$[P \rightarrow Q] = \min\{1, 1 - [P] + [Q]\}$;

$[\forall x P(x)] = \inf[P(x)]$;
$\models P$ if and only if $[P] = 1$ for all valuations.

Of course, various implication operators may be defined similarly. We only show a selection in the following table, where α denotes the degree of truth (or degree of membership) of the premise and β denotes the respective values for the consequence, and I the resulting degree of truth for the implication:

Name	Definition of Implication Operators
Early Zadeh	$I_m(\alpha,\beta) = \max\{1-\alpha, \min\{\alpha,\beta\}\}$
Lukasiewicz	$I_a(\alpha,\beta) = \min\{1, 1-\alpha+\beta\}$
Standard Star(Godel)	$I_g(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta \\ \beta & \text{if} \quad \alpha > \beta \end{cases}$
Contraposition of Godel	$I_{cg}(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta \\ 1-\alpha & \text{if} \quad \alpha > \beta \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta \\ 0 & \text{if} \quad \alpha > \beta \end{cases}$
Kleene-Dienes	$I_b(\alpha,\beta) = \max\{1-\alpha, \beta\}$

The " quality" of these implication operators may be evaluated by either empirical or axiomatic methods.

In the following definition, we consider the implication operators in the Lukasiewicz system of continuous-valued logic.

Definition 5.1. A fuzzy set μ of X is called a fuzzifying a-ideal of X if it satisfies the following conditions::
(F20) for any $x \in X, \models [x \in \mu] \rightarrow [0 \in \mu]$,
(F21) for any $x, y, z \in X$, $\models [(x * z) * (0 * y) \in \mu] \wedge [z \in \mu] \rightarrow [y * x \in \mu]$.

Clearly, Definition 5.1 is equivalent to Definition 2.4. Therefore a fuzzifying a-ideal is an ordinary fuzzy a-ideal.

Now, we introduce the concept of t-tautology, i.e.,
$\models_t P$ if and only if $[P] \geq t$, for all valuations.

Using the results in [32], we can extend the concept of implication-based fuzzy implicative ideals in as follows:

Definition 5.2. Let μ be a fuzzy set of X and $t \in (0, 1]$ is a fixed number. Then μ is called a *t-implication-based fuzzy a-ideal* of X if the following conditions hold:
(F22) for any $x \in X, \models_t [x \in \mu] \rightarrow [0 \in \mu]$,
(F23) for any $x, y, z \in X$, $\models_t [(x * z) * (0 * y) \in \mu] \wedge [z \in \mu] \rightarrow [y * x \in \mu]$.

Now, if I is an implication operator then we have the following corollary:

Corollary 5.3. *A fuzzy set μ of X is a t-implication-based fuzzy a-ideal of X if and only if it satisfies:*
(F24) *for any $x \in X, I(\mu(x), \mu(0)) \geq t$,*

(F25) *for any* $x, y, z \in X$, $I(\mu((x * z) * (0 * y)) \wedge \mu(z), \mu(y * x)) \geq t$.

Let μ be a fuzzy set of X. Then we have the following theorem:

Theorem 5.4. *(i) Let* $I = I_{gr}$. *Then* μ *is an 0.5-implication-based fuzzy a-ideal of* X *if and only if* μ *is a fuzzy a-ideal with thresholds* $(r = 0, s = 1)$ *of* X*;*

(ii) Let $I = I_g$. *Then* μ *is an 0.5-implication-based fuzzy a-ideal of* X *if and only if* μ *is a fuzzy a-ideal with thresholds* $(r = 0, s = 0.5)$ *of* X*;*

(iii) Let $I = I_{cg}$. *Then* μ *is an 0.5-implication-based fuzzy a-ideal of* X *if and only if* μ *is a fuzzy a-ideal with thresholds* $(r = 0.5, s = 1)$ *of* X.

6. Conclusions

It is clear that ideals with special properties play an important role in the study of the structure of an algebraic system. In this paper, we studied the notions of $(\in, \in \vee \mathrm{q})$-fuzzy *a*-(*p*- and *q*-) ideals in BCI-algebras and investigated the relationship among these. Finally, we investigated the concept of implication-based fuzzy *a*-ideals in BCI-algebras. The results can be applied to other algebraic structures. It is our hope that this work will serve as a foundation for further study of the theory of BCK/BCI-algebras.

In the future, we plan to study (α, β)-fuzzy *a*-(*p*- and *q*-) ideals in BCI-algebras, where α, β is any one of $\in, q, \in \vee \mathrm{q}$ or $\in \wedge \mathrm{q}$. For an (α, β)-fuzzy *a*-(*p*- and *q*-) ideal in BCI-algebras, we can consider twelve different types of such structures resulting from three choices of α and four choices of β. But, in this report, we have only discussed the $(\in, \in \vee \mathrm{q})$-type. In the future we shall focus on other types of structures and the relationships between them. We shall also consider quotient BCI-algebras via (α, β)-fuzzy *a*-(*p*- and *q*-) ideals.

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References

[1] S. K. Bhakat, $(\in, \in \vee \mathrm{q})$-*fuzzy normal, quasinormal and maximal subgroups*, Fuzzy Sets and Systems, **112** (2000), 299-312.

[2] S. K. Bhakat and P. Das, $(\in, \in \vee \mathrm{q})$-*fuzzy subgroups*, Fuzzy Sets and Systems, **80** (1996), 359-368.

[3] C. C. Chang, *Algebraic analysis of many valued logic*, Trans. Amer. Math. Soc., **88** (1958), 467-490.

[4] B. Davvaz, $(\in, \in \vee q)$-*fuzzy subnear-rings and ideals*, Soft Computing, **10** (2006), 206-211.

[5] B. Davvaz and P. Corsini, *Redefined fuzzy* H_v*-submodules and many valued implications*, Inform. Sci., **177** (2007), 865-875.

[6] F. Esteva and L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems, **124** (2001), 271-288.

[7] P. Hájek, *Metamathematics of fuzzy logic*, Kluwer Academic Press, Dordrecht, 1998.

[8] Y. Imai and K. Iseki, *On axiom system of propositional calculus*, Proc. Japan Acad., **42** (1966), 19-22.

[9] A. Iorgulescu, *Some direct ascendents of wajsberg and MV algebras*, Sci. Math. Japon., **57** (2003), 583-647.

[10] A. Iorgulescu, *Pseudo-Iseki algebras. connection with pseudo-BL algebras*, Multiple-Valued Logic and Soft Computing, **11** (2005), 263-308.
[11] K. Iseki, *An algebra related with a propositional calculus*, Proc. Japan Acad., **42** (1966), 26-29.
[12] K. Iseki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japon., **21** (1966), 351-366.
[13] Y. B. Jun, *Closed fuzzy ideals in BCI-algebras*, Math. Japon., **38** (1993), 199-202.
[14] Y. B. Jun, *On (α, β)-fuzzy ideals of BCK/BCI-algebras*, Sci. Math. Japon., **60** (2004), 613-617.
[15] Y. B. Jun, *On (α, β)-fuzzy subalgebras of BCK/BCI-algebras*, Bull. Korean Math. Soc., **42** (2005), 703-711.
[16] Y. B. Jun and J. Meng, *Fuzzy p-ideals in BCI-algebras*, Math. Japon, **40** (1994), 271-282.
[17] Y. B. Jun and J. Meng, *Fuzzy commutative ideals in BCI-algebras*, Comm. Korean Math. Soc., **9** (1994), 19-25.
[18] Y. B. Jun and W. H. Shim, *Fuzzy strong implicative hyper BCK-ideals of hyper BCK-algebras*, Inform. Sci., **170** (2005), 351-361.
[19] Y. B. Jun, Y. Xu and J. Ma, *Redefined fuzzy implicative filters*, Inform. Sci., **177** (2007), 1422-1429.
[20] T. D. Lei and C. C. Xi, *p-radical in BCI-algebras*, Math. Japon., **30** (1995), 511-517.
[21] Y. L. Liu, *Some results on p-semisimple BCI-algebras*, Math. Japon, **30** (1985), 511-517.
[22] Y. L. Liu, S. Y. Liu and J. Meng, *FSI-ideals and FSC-ideals of BCI-algebras*, Bull. Korean Math. Soc., **41** (2004), 167-179.
[23] Y. L. Liu and J. Meng, *Fuzzy q-ideals of BCI-algebras*, J. Fuzzy Math., **8** (2000), 873-881.
[24] Y. L. Liu and J. Meng, *Fuzzy ideals in BCI-algebras*, Fuzzy Sets and Systems, **123** (2001), 227-237.
[25] Y. L. Liu, J. Meng, X. H. Zhang and Z. C. Yue, *q-ideals and a-ideals in BCI-algebras*, SEA Bull. Math., **24** (2000), 243-253.
[26] Y. L. Liu, Y. Xu and J. Meng, *BCI-implicative ideals of BCI-algebras*, Inform. Sci., **177** (2007), 4987-4996.
[27] Y. L. Liu and X. H. Zhang, *Fuzzy a-ideals in BCI-algebras*, Adv. in Math.(China), **31** (2002), 65-73.
[28] J. Meng and X. Guo, *On fuzzy ideals in BCK-algebras*, Fuzzy Sets and Systems, **149** (2005), 509-525.
[29] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co., Seoul, Korean, 1994.
[30] D. Mundici, *MV algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Japon., **31** (1986), 889-894.
[31] P. M. Pu and Y. M. Liu, *Fuzzy topology I: Neighourhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., **76** (1980), 571-599.
[32] X. H. Yuan, C. Zhang and Y. H. Ren, *Generalized fuzzy groups and many valued applications*, Fuzzy Sets and Systems, **138** (2003), 205-211.
[33] L. A. Zadeh, *Fuzzy sets*, Inform. Control, **8** (1965), 338-353.
[34] L. A. Zadeh, *Toward a generalized theory of uncertainty (GTU)-an outine*, Inform. Sci., **172** (2005), 1-40.
[35] J. Zhan and Y. L. Liu, *On f-derivation of BCI-algebras*, Int. J. Math. Math. Sci., 2005, 1675-1684.
[36] J. Zhan and Z. Tan, *Fuzzy a-ideals of IS-algebras*, Sci. Math. Japon, **58** (2003), 85-87.
[37] J. Zhan and Z. Tan, *Intuitionistic fuzzy a-ideals in BCI-algebras*, Soochow Math. J., **30** (2004), 207-216.
[38] X. H. Zhang, H. Jiang and S. A. Bhatti, *On p-ideals of BCI-algebras*, Punjab Univ. J. Math., **27** (1994), 121-128.

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