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ON L-FUZZIFYING CONVERGENCE SPACES

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Abstract. Based on a complete Heyting algebra L, the relations between L-fuzzifying convergence spaces and L-fuzzifying topological spaces are studied. It is shown that, as a reflective subcategory, the category of L-fuzzifying topological spaces could be embedded in the category of L-fuzzifying convergence spaces and the latter is cartesian closed. Also the specialization L-preorder of L-fuzzifying convergence spaces and that of L-fuzzifying topological spaces are investigated.

1. **Introduction**

Since Chang [2] introduced fuzzy set theory to topology, many researchers have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. In Chang's I-topology on a set X, each open set was fuzzy, while the topology itself was a crisp subset of the family of all fuzzy subsets of X. From a different direction, the fundamental idea of a topology itself being fuzzy was first defined by Höhle [7] in 1980, then was independently generalized by each of Kubiak [13] and Šostak [19] in 1985, and then independently rediscovered by Ying [21] in Höhle's original setting in 1991: in Höhle's approach a topology was an L-subset of a traditional powerset. In 1999, the axioms of many valued L-fuzzy topological spaces and L-fuzzy continuous maps are given a lattice-theoretical foundation by Höhle and Šostak [9] and a categorical foundation by Rodabaugh [18].

Convergence theory of filters or nets provides a good tool for interpreting topological structures and plays an important role in fuzzy topology. In crisp situation, there are close relations between topological spaces and convergence spaces. For a nonempty set X, let $\mathbb{F}(X)$ denote the set of all filters (which are equivalent to proper lattice-theoretical filters of $(2^X, \subseteq)$) on X and for each $x \in X$, let $[x]$ denote the principal filter generated by $\{x\}$. A convergence structure [22] on X is a subset $R \subseteq \mathbb{F}(X) \times X$ satisfying the following conditions:

(Conv1) $([x],\ x) \in R$ for all $x \in X$;

(Conv2) $(F,\ x) \in R$ and $F \subseteq G \in \mathbb{F}(X)$ imply $(G,\ x) \in R$.

A convergence structure R on X is called a limit structure if R satisfies the additional condition:

(Lim) $(F,\ x),\ (G,\ x) \in R$ implies $(F \cap G,\ x) \in R$.

For a convergence structure (resp., limit structure) R on X, the pair $(X,\ R)$ is

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called a convergence space (resp., limit space). A map $f : (X,\ R_X) \longrightarrow (Y,\ R_Y)$ between two convergence spaces is called continuous if $\forall F \in \mathbb{F}(X),\ (F,\ x) \in R_X$ implies $(f(F),\ f(x)) \in R_Y$, where $f(F)$ is a filter on Y generated by the filter base $\{f(A) |\ A \in F\}$. **Conv** denotes the category of convergence spaces with continuous maps and **Lim** the full subcategory of **Conv** formed by all limit spaces. It is well-known that both **Conv** and **Lim** are cartesian closed and the category of topological spaces **Top** can be embedded in them as a reflective subcategory [22].

Many researchers extended convergence structures and limit structures to fuzzy setting. In the framework of I-topology, Lowen [16] defined the concept of a prefilter as a subset of I^X (a lattice-theoretical filter of I^X under pointwise order) in order to study the theory of fuzzy topological spaces. Also, K.C. Min [17] introduced fuzzy limit spaces using prefilters. In the framework of fuzzy topology, Xu [20] introduced fuzzy topological limit structures and characterized fuzzy topologies by filter convergence structures. In L-fuzzy setting, Lowen et al. [15] used prefilters to define the notion of an I-fuzzy convergence space, and showed that the category of all such objects had several desirable properties, such as being cartesian closed. Höhle and Šostak [9] introduced the idea of an (resp., a stratified) L-filter as a map from L^X to L and showed that stratified L-filters provided a fruitful tool employed in the development of general lattice-valued topological spaces. Later, for a complete Heyting algebra L, Jäger [11] defined stratified L-fuzzy convergence spaces (which are called L-generalized convergence spaces in [12]) and proved that the resulting category is a cartesian closed topological category and the category of stratified L-topological spaces can be embedded in it as a reflective subcategory.

The aim of this paper is to propose the concept of L-fuzzifying convergence spaces and study the relations between L-fuzzifying convergence spaces and L-fuzzifying topological spaces. As is suggested in [12], we replace stratified L-filters in [11, 12] by L-filters of ordinary sets and define the concept of L-fuzzifying convergence spaces. This paper is organized as follows. In Section 2, some basic concepts and notions which will be used throughout this paper are listed. In Section 3, some basic notions of L-fuzzifying topological spaces are recalled and the definition of L-fuzzifying convergence spaces is presented. In Section 4, the relations between L-fuzzifying topological spaces and L-fuzzifying convergence spaces are studied. It is shown that the category of L-fuzzifying topological spaces can be embedded in the category of L-fuzzifying convergence spaces as a reflective subcategory. In Section 5, the category of L-fuzzifying convergence spaces is shown to be a cartesian closed topological category. In Section 6, the specialization L-preorder of L-fuzzifying convergence spaces and L-fuzzifying topological spaces is studied.

2. Preliminaries

A complete lattice L is a complete Heyting algebra or a frame if the binary meets are distributive over arbitrary joins, i.e.,

$$a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$$

holds for all $a,\ b_i(i \in I) \in L$. For a complete Heyting algebra L, an implicative operator $\rightarrow: L \times L \longrightarrow L$ can be defined as

$$\forall a,\ b \in L,\ a \rightarrow b = \bigvee \{c \in L |\ a \wedge c \leq b\}.$$

Then $(\wedge,\ \rightarrow)$ forms a Galois connection [6] on L, i.e.,

$$\forall a,\ b,\ c \in L,\ a \wedge c \leq b \Longleftrightarrow c \leq a \rightarrow b.$$

Theorem 2.1. ([9]) *Let L be a complete Heyting algebra. Then*
(H1) $a \rightarrow (\bigwedge_i b_i) = \bigwedge_i (a \rightarrow b_i)$;
(H2) $(\bigvee_i b_i) \rightarrow a = \bigwedge_i (b_i \rightarrow a)$;
(H3) $b \rightarrow c \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$;
(H4) $a = 1 \rightarrow a$;
(H5) $a \leq b$ *iff* $a \rightarrow b = 1$;
(H6) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$;
(H7) $a \rightarrow b \leq (a \wedge c) \rightarrow (b \wedge c)$;
(H8) $a \rightarrow b \geq b$;
(H9) $a \wedge b = a \wedge (a \rightarrow b)$;
(H10) $a \leq (a \rightarrow b) \rightarrow b$;
(H11) $(a \rightarrow c) \wedge (b \rightarrow d) \leq (a \wedge b) \rightarrow (c \wedge d)$.

A complete lattice L is said to be completely distributive if it satisfies the completely distributive law, i.e.,

$$\bigvee_i (\bigwedge_j a_i^j) = \bigwedge_{f \in \prod_{i \in I} J_i} (\bigvee_i a_i^{f(i)})$$

or

$$\bigwedge_i (\bigvee_j a_i^j) = \bigvee_{f \in \prod_{i \in I} J_i} (\bigwedge_i a_i^{f(i)})$$

hold for all $X_i = \{a_i^j |\ j \in J_i\} \subseteq 2^L (\forall i \in I)$. Clearly, every completely distributive complete lattice is a frame.

Lemma 2.2. ([1]) *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be two functors. Then the followings are equivalent:*

(1) *$\forall A \in ob(\mathcal{A}),\ B \in ob(\mathcal{B})$ and any $\mathcal{A}$-morphism $f : A \longrightarrow G(B)$, there exists a $\mathcal{A}$-morphism $\eta_A : A \longrightarrow GF(A)$ and a unique $\mathcal{B}$-morphism $g : F(A) \longrightarrow B$ such that $f = G(g) \circ \eta_A$;*

(2) *$\forall A \in ob(\mathcal{A}),\ B \in ob(\mathcal{B})$ and any $\mathcal{B}$-morphism $h : F(A) \longrightarrow B$, there exists a $\mathcal{B}$-morphism $\xi_B : FG(B) \longrightarrow B$ and a unique $\mathcal{A}$-morphism $t : A \longrightarrow G(B)$ such that $h = \xi_B \circ F(t)$;*

(3) *$\forall A \in ob(\mathcal{A}),\ B \in ob(\mathcal{B})$, there is a bijection between $hom_{\mathcal{A}}(A,\ G(B))$ and $hom_{\mathcal{B}}(F(A),\ B)$.*

$(F,\ G)$ is called an adjunction between $\mathcal{A}$ and $\mathcal{B}$ if it satisfies any of the conditions in Lemma 2.2, in symbols $F \vdash G$. F is called the left adjoint of G and G the right adjoint functor of F. If $\mathcal{A}$ is a subcategory of $\mathcal{B}$ and the inclusion functor $i : \mathcal{A} \longrightarrow \mathcal{B}$ has a left (resp., right) adjoint F, then $\mathcal{A}$ is called a reflective (resp., coreflective) subcategory of $\mathcal{B}$ and F is called the reflector (resp., coreflector).

Throughout this paper, unless otherwise statement, L always denotes a complete Heyting algebra and $\to$ is the implicative operator induced by the binary meets. For other notions related to category theory we refer you to [1].

3. L-fuzzifying Topological Space and L-fuzzifying Convergence Space

Definition 3.1. ([21]) An L-fuzzifying topology on a nonempty set X is a function $\tau : 2^X \longrightarrow L$ which satisfies:

(FO1) $\tau(\emptyset) = \tau(X) = 1$;

(FO2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(FO3) $\tau(\bigcup_i A_i) \geq \bigwedge_i \tau(A_i)$.

For an L-fuzzifying topology τ on X, the pair $(X,\ \tau)$ is called an L-fuzzifying topological space.

Let $(X,\ \tau)$ be an L-fuzzifying topological space. We define $\mathcal{U}_\tau^x : 2^X \longrightarrow L$ by

$$\forall A \subseteq X,\ \mathcal{U}_\tau^x(A) = \bigvee_{B \in \dot{x}|A} \tau(B),$$

where $B \in \dot{x}|A$ means $x \in B \subseteq A$. $\mathcal{U}_\tau^x(A)$ can be interpreted as the degree of A to be a neighborhood of x. $\{\mathcal{U}_\tau^x |\ x \in X\}$ is called the L-fuzzifying neighborhood system [25] of $(X,\ \tau)$.

Theorem 3.2. ([25]) $\forall A \subseteq X,\ \bigwedge_{x\in A} \mathcal{U}_\tau^x(A) \geq \tau(A)$ *and if L is completely distributive then* $\bigwedge_{x\in A} \mathcal{U}_\tau^x(A) = \tau(A)$.

Definition 3.3. ([25]) An L-generalized neighborhood system on X is defined to be a set $P = \{p_x |\ x \in X\}$ of maps $p_x : 2^X \longrightarrow L$ such that $\forall\ U,\ V \in X$,

(GN1) $p_x(X) = 1,\ p_x(\emptyset) = 0$;

(GN2) $p_x(U) > 0$ implies $x \in U$;

(GN3) $p_x(U \cap V) = p_x(U) \wedge p_x(V)$;

(GN4) $p_x(U) = \bigvee_{V\in\dot{x}|U} \bigwedge_{y\in V} p_x(V)$.

It was proved in [25] that for any completely distributive complete lattice L, L-generalized neighborhood systems and L-fuzzifying topologies are conceptually equivalent with the transferring process $p_x = \mathcal{U}_\tau^x$ and $\tau(A) = \bigwedge_{x\in A} p_x(A)$.

A map $f : X \longrightarrow Y$ is called continuous with respect to the given L-fuzzifying topological spaces $(X,\ \tau_1)$ and $(Y,\ \tau_2)$ iff $\forall B \in 2^Y,\ \tau_1(f^{-1}(B)) \geq \tau_2(B)$. The category of L-fuzzifying topological spaces with continuous maps as morphisms will be denoted by L-**FYS**. It is well-known that L-**FYS** is a topological category, but not cartesian closed (for a completely distributive complete lattice L)[1].

[1]**Top** is not cartesian closed [1]. If L is completely distributive, then **Top** can be embedded in L-**FYS** as a simultaneously reflective and coreflective isomorphism-closed full subcategory [23, 24]. By Proposition 27.9 in [1], L-**FYS** is not cartesian closed.

Definition 3.4. ([10]) A map $\mathcal{F} : 2^X \longrightarrow L$ is called an L-filter of ordinary subsets on a nonempty set X if it satisfies

(F1) $\mathcal{F}(\emptyset) = 0,\ \mathcal{F}(X) = 1$;

(F2) $\forall A,\ B \subseteq X,\ A \subseteq B$ implies $\mathcal{F}(A) \leq \mathcal{F}(B)$;

(F3) $\forall A,\ B \subseteq X,\ \mathcal{F}(A \cap B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$.

An L-filter of ordinary subsets is called an L-filter in brief if no confusion arises.

Theorem 3.5. ([6]) *$\mathcal{F} : 2^X \longrightarrow L$ is an L-filter if and only if it fulfils $(F1)$ and*

(F4) $\forall A,\ B \subseteq X,\ \mathcal{F}(A \cap B) = \mathcal{F}(A) \wedge \mathcal{F}(B)$.

In the following discussion, we just call an L-filter of ordinary subsets an L-filter.

Example 3.6. (1) Define $[x] : 2^X \longrightarrow L$ by $[x](A) = 1$ if $x \in A$ and 0 otherwise. Then $[x]$ is an L-filter.

(2) Let $(X,\ \tau)$ be an L-fuzzifying topological space. Then $\forall x \in X,\ \mathcal{U}_\tau^x$ is an L-filter.

The family of all L-filters on X will be denoted by $\mathbb{F}_L(X)$. Then $\mathbb{F}_L(X)$ is a poset under pointwise order. The smallest element of $\mathbb{F}_L(X)$ is $\mathcal{F}_0(A) = 1$ if $A = X$ and 0 otherwise. Furthermore $\bigwedge_i \mathcal{F}_i$ is also an L-filter for $\{\mathcal{F}_i|\ i \in I\} \subseteq \mathbb{F}_L(X)$. Thus $\mathbb{F}_L(X)$ is a complete semilattice.

$\forall \mathcal{F} \in \mathbb{F}_L(X),\ \mathcal{G} \in \mathbb{F}_L(Y),\ \mathcal{F} \times \mathcal{G} \in \mathbb{F}_L(X \times Y)$ is defined by

$$\forall C \subseteq X \times Y,\ (\mathcal{F} \times \mathcal{G})(C) = \bigvee_{A \times B \subseteq C} \mathcal{F}(A) \wedge \mathcal{G}(B).$$

Let $\mathcal{F}$ be an L-filter on X and $f : X \longrightarrow Y$ be a map. Define $f^{\Rightarrow}(\mathcal{F}) : L^X \longrightarrow L$ by

$$\forall B \subseteq Y,\ f^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(f^{-1}(B)).$$

Theorem 3.7. *Let $\mathcal{F}$ be an L-filter on X and $f : X \longrightarrow Y$ be a map. Then $f^{\Rightarrow}(\mathcal{F})$ is an L-filter on Y.*

Proof. Straightforward. □

Theorem 3.8. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two maps. Then*

(1) $\forall x \in X,\ f^{\Rightarrow}([x]) = [f(x)]$;

(2) $\forall \mathcal{F} \in \mathbb{F}_L(X),\ g^{\Rightarrow}(f^{\Rightarrow}(\mathcal{F})) = (g \circ f)^{\Rightarrow}(\mathcal{F})$.

Proof. Straightforward. □

Definition 3.9. ([7], or [11] for L-fuzzy version) An L-fuzzifying convergence structure on X is a map $R : \mathbb{F}_L(X) \times X \longrightarrow L$ satisfying that

(FCS1) $\forall x \in X,\ R([x],\ x) = 1$;

(FCS2) $\forall x \in X,\ \mathcal{F},\ \mathcal{G} \in \mathbb{F}_L(X),\ \mathcal{F} \leq \mathcal{G}$ implies $R(\mathcal{F},\ x) \leq R(\mathcal{G},\ x)$.

The pair $(X,\ R)$ is called an L-fuzzifying convergence space.

For an L-fuzzifying convergence space $(X,\ R)$, define $\mathcal{U}_R^x : L^X \longrightarrow L$ by

$$\forall A \subseteq X,\ \mathcal{U}_R^x(A) = \bigwedge_{\mathcal{F} \in \mathbb{F}_L(X)} R(\mathcal{F},\ x) \rightarrow \mathcal{F}(A).$$

Then $\mathcal{U}_R^x$ is an L-filter (See Proposition 6.1 in [11] for stratified L-fuzzy convergence spaces).

A map $f : X \longrightarrow Y$ is called continuous with respect to the given L-fuzzifying convergence spaces $(X,\ R_1)$ and $(Y,\ R_2)$ iff $\forall(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X,\ R_1(\mathcal{F},\ x) \leq R_2(f^{\Rightarrow}(\mathcal{F}),\ f(x))$. The category of L-fuzzifying convergence spaces with continuous maps as morphisms will be denoted by L-**FYCS**.

Let $\{R_i|\ i \in I\}$ be a family of L-fuzzifying convergence structures on X. Then $(\bigwedge_i R_i)(\mathcal{F},\ x) := \bigwedge_i (R_i(\mathcal{F},\ x))$ also is an L-fuzzifying convergence structure on X. By L-**FYCS**(X) we denote the set of all L-fuzzifying convergence structures on X. Then L-**FYCS**(X) is a complete lattice under the partial order of $R_1 \preceq R_2$ iff $\forall(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X,\ R_1(\mathcal{F},\ x) \leq R_2(\mathcal{F},\ x)$. Thus L-**FYCS** is fibre-complete [1].

4. Relations Between Categories of L-FYS and L-FYCS

In this section, we will construct an L-fuzzifying topology by an L-fuzzifying convergence structure and conversely construct an L-fuzzifying convergence structure by an L-fuzzifying topology. Then we will study the relations between categories of L-**FYS** and L-**FYCS**.

Theorem 4.1. *Let R be an L-fuzzifying convergence structure on X. Define $\tau_R : 2^X \longrightarrow L$ by*

$$\forall A \subseteq X,\ \tau_R(A) = \bigwedge_{x \in A} \mathcal{U}_R^x(A) = \bigwedge_{(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times A} (R(\mathcal{F},\ x) \to \mathcal{F}(A)).$$

Then τ_R is an L-fuzzifying topology on X.

Proof. (FO1) Obviously, $\tau_R(\emptyset) = \tau(X) = 1$.

(FO2) $\forall A,\ B \subseteq X$,

$$\begin{aligned} \tau_R(A) \wedge \tau_R(B) &= (\bigwedge_{x \in A} \mathcal{U}_R^x(A)) \wedge (\bigwedge_{y \in B} \mathcal{U}_R^y(B)) \\ &\leq \bigwedge_{x \in A \cap B} (\mathcal{U}_R^x(A) \wedge \mathcal{U}_R^x(B)) \\ &= \bigwedge_{x \in A \cap B} \mathcal{U}_R^x(A \cap B) \\ &= \tau_R(A \cap B). \end{aligned}$$

(FO3) $\forall \{A_i|\ i \in I\} \subseteq 2^X (I \neq \emptyset)$,

$$\begin{aligned} \tau_R(\bigcup_i A_i) &= \bigwedge_{x \in \bigcup_i A_i} \mathcal{U}_R^x(\bigcup_i A_i) \\ &= \bigwedge_i \bigwedge_{x \in A_i} \mathcal{U}_R^x(\bigcup_j A_j) \\ &\geq \bigwedge_i \bigwedge_{x \in A_i} \mathcal{U}_R^x(A_i) \\ &= \bigwedge_i \tau_R(A_i). \end{aligned}$$

□

Conversely, for an L-fuzzifying topology τ on X, define $R_\tau : \mathbb{F}_L(X) \times X \longrightarrow L$ [9] by $\forall(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X$,

$$R_\tau(\mathcal{F},\ x) = \bigwedge_{A \subseteq X} (\mathcal{U}_\tau^x(A) \to \mathcal{F}(A)) = \bigwedge_{A \in \dot{x}} (\mathcal{U}_\tau^x(A) \to \mathcal{F}(A)).$$

Theorem 4.2. (1) $\forall x \in L,\ R_\tau(\mathcal{U}_\tau^x,\ x) = 1$;

(2) $\forall x \in L,\ \mathcal{U}_\tau^x = \mathcal{U}_{R_\tau}^x$;

(3) $\forall(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X,\ R_\tau(\mathcal{F},\ x) = \bigwedge_{A \in \dot{x}} (\tau(A) \to \mathcal{F}(A))$.

(4) R_τ *is an L-fuzzy convergence structure on X.*

Proof. (1) and (4) are trivial and straightforward.

(2) $\forall A \subseteq X$, on one hand,

$$\mathcal{U}_{R_\tau}^x(A) = \bigwedge_{\mathcal{F} \in \mathbb{F}_L(X)} (R_\tau(\mathcal{F},\ x) \to \mathcal{F}(A)) \leq R_\tau(\mathcal{U}_\tau^x,\ x) \to \mathcal{U}_\tau^x(A) = 1 \to \mathcal{U}_\tau^x(A) = \mathcal{U}_\tau^x(A).$$

On the other hand,

$$\begin{aligned} \mathcal{U}_{R_\tau}^x(A) &= \bigwedge_{\mathcal{F} \in \mathbb{F}_L(X)} (R_\tau(\mathcal{F},\ x) \to \mathcal{F}(A)) \\ &= \bigwedge_{\mathcal{F} \in \mathbb{F}_L(X)} ((\bigwedge_{B \subseteq X} \mathcal{U}_\tau^x(B) \to \mathcal{F}(B)) \to \mathcal{F}(A)) \\ &\geq \bigwedge_{\mathcal{F} \in \mathbb{F}_L(X)} ((\mathcal{U}_\tau^x(A) \to \mathcal{F}(A)) \to \mathcal{F}(A)) \\ &\geq \mathcal{U}_\tau^x(A). \end{aligned}$$

(3) $\forall(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X$,

$$\begin{aligned} R_\tau(\mathcal{F},\ x) &= \bigwedge_{A \in \dot{x}} (\mathcal{U}_\tau^x(A) \to \mathcal{F}(A)) \\ &= \bigwedge_{A \in \dot{x}} ((\bigvee_{x \in B \subseteq A} \tau(B)) \to \mathcal{F}(A)) \\ &= \bigwedge_{A \in \dot{x}} \bigwedge_{x \in B \subseteq A} (\tau(B) \to \mathcal{F}(A)) \\ &\geq \bigwedge_{A \in \dot{x}} \bigwedge_{B \leq A} (\tau(B) \to \mathcal{F}(B)) \\ &\geq \bigwedge_{A \in \dot{x}} (\tau(A) \to \mathcal{F}(A)). \end{aligned}$$

and

$$R_\tau(\mathcal{F},\ x) = \bigwedge_{A \in \dot{x}} ((\bigvee_{x \in B \subseteq A} \tau(B)) \to \mathcal{F}(A)) \leq \bigwedge_{A \in \dot{x}} (\tau(A) \to \mathcal{F}(A)).$$ □

In the following, we will study the relations between L-fuzzifying topologies and L-fuzzy convergence structures.

Theorem 4.3. *$\tau_{R_\tau} \geq \tau$ and if L is completely distributive then $\tau_{R_\tau} = \tau$.*

Proof. $\forall A \subseteq X$, since $\mathcal{U}_{R_\tau}^x = \mathcal{U}_\tau^x$, $\tau_{R_\tau}(A) = \bigwedge_{x \in A} \mathcal{U}_{R_\tau}^x(A) = \bigwedge_{x \in A} \mathcal{U}_\tau^x(A) \geq \tau(A)$. By Theorem 3.2, the equation holds obvious if L is completely distributive. □

Theorem 4.4. *$R_{\tau_R} \geq R$.*

Proof. $\forall(\mathcal{F},\ x)\in\mathbb{F}_L(X)\times X$, we have

$$\begin{array}{rcl} R_{\tau_R}(\mathcal{F},\ x) & = & \bigwedge\limits_{A\in\dot{x}}(\tau_R(A)\to\mathcal{F}(A)) \\ & = & \bigwedge\limits_{A\in\dot{x}}((\bigwedge\limits_{(\mathcal{G},\ y)\in\mathbb{F}_L(X)\times A} R(\mathcal{G},\ y)\to\mathcal{G}(A))\to\mathcal{F}(A)) \\ & \geq & \bigwedge\limits_{A\in\dot{x}}((R(\mathcal{F},\ x)\to\mathcal{F}(A))\to\mathcal{F}(A)) \\ & \geq & R(\mathcal{F},\ x). \end{array}$$

□

Remark 4.5. $R_{\tau_R}\leq R$ does not hold even if $L=\{0,\ 1\}$. For example, let $X=\{x,\ y\}$. Then $\mathbb{F}(X)=\{[x],\ [y],\ \{X\}\}$. Suppose that $R=\{([x],\ x),\ ([y],\ y),\ ([y],\ x)\}$. It's easy to verify that $\tau_R=\{\emptyset,\ \{y\},\ X\}$ and $R_{\tau_R}=R\cup\{(\{X\},\ x)\}$.

An L-fuzzifying convergence structure R on X is called topological if it can be induced by an L-fuzzifying topology, that is there exists an L-fuzzifying topology τ on X such that $R=R_\tau$.

Theorem 4.6. *The followings are equivalent:*

(1) $R_{\tau_R}=R$;

(2) $\forall\mathcal{F}\in\mathbb{F}_L(X),\ \forall x\in X,\ \mathcal{U}_R^x=\mathcal{U}_{\tau_R}^x$ *and* $R(\mathcal{F},\ x)=\bigwedge\limits_{A\subseteq X}\mathcal{U}_R^x(A)\to\mathcal{F}(A)$,

which imply

(3) *R is topological.*

If L is completely distributive, then the above three are equivalent.

Proof. Obviously (1)$\Longrightarrow$(3) holds.

(1)$\Longrightarrow$(2): By Theorem 4.2(3), we have

$$\mathcal{U}_{\tau_R}^x=\mathcal{U}_{R_{\tau_R}}^x=\mathcal{U}_R^x$$

and then

$$R(\mathcal{F},\ x)=R_{\tau_R}(\mathcal{F},\ x)=\bigwedge_{A\subseteq X}(\mathcal{U}_{\tau_R}^x(A)\to\mathcal{F}(A))=\bigwedge_{A\subseteq X}(\mathcal{U}_R^x(A)\to\mathcal{F}(A)).$$

(2)$\Longrightarrow$(1): $\forall(\mathcal{F},\ x)\in\mathbb{F}_L(X)\times X$,

$$R_{\tau_R}(\mathcal{F},\ x)=\bigwedge_{A\subseteq X}\mathcal{U}_{\tau_R}^x(A)\to\mathcal{F}(A)=\bigwedge_{A\subseteq X}\mathcal{U}_R^x(A)\to\mathcal{F}(A)=R(\mathcal{F},\ x).$$

If L is completely distributive, then (3)$\Longrightarrow$(1) can be easily implied by Theorem 4.3. □

Corollary 4.7. *If L is completely distributive, then R is topological if and only if*

$$\forall(\mathcal{F},\ x)\in\mathbb{F}_L(X)\times X,\ R(\mathcal{F},\ x)=\bigwedge_{A\subseteq X}\mathcal{U}_R^x(A)\to\mathcal{F}(A)$$

and

$$\forall A\subseteq X,\ \mathcal{U}_R^x(A)=\bigvee_{B\in\dot{x}|A}\bigwedge_{y\in B}\mathcal{U}_R^y(B).$$

Proof. The necessity is obvious by Theorem 4.6 and the properties of neighborhood system of L-fuzzifying topological spaces. To show the sufficiency, we only need to prove that $\forall x \in X$, $\mathcal{U}_R^x = \mathcal{U}_{\tau_R}^x$. In fact, $\forall A \subseteq X$, $\mathcal{U}_{\tau_R}^x(A) = \bigvee_{B \in \dot{x} | A} \tau_R(B) = \bigvee_{B \in \dot{x} | A} \bigwedge_{y \in B} \mathcal{U}_R^y(B) = \mathcal{U}_R^x(A)$. This completes the proof. □

Theorem 4.8. (1) $R_T : L$-**FYS** $\longrightarrow L$-**FYCS**$(\tau \mapsto R_\tau)$ *is a functor.*
(2) $T_R : L$-**FYCS** $\longrightarrow L$-**FYS**$(R \mapsto \tau_R)$ *is a functor.*

Proof. (1) Suppose that $f : (X,\ \tau_X) \longrightarrow (Y,\ \tau_Y)$ is continuous. $\forall (\mathcal{F}, x) \in \mathbb{F}_L(X) \times X$,

$$\begin{aligned} R_{\tau_Y}(f^{\Rightarrow}(\mathcal{F}),\ f(x)) &= \bigwedge_{B \in \dot{f(x)}} (\tau_Y(B) \rightarrow f^{\Rightarrow}(\mathcal{F})(B)) \\ &\geq \bigwedge_{f^{-1}(B) \in \dot{x}} (\tau_X(f^{-1}(B)) \rightarrow \mathcal{F}(f^{-1}(B))) \\ &\geq \bigwedge_{A \in \dot{x}} (\tau_X(A) \rightarrow \mathcal{F}(A)) \\ &= R_{\tau_X}(\mathcal{F},\ x). \end{aligned}$$

(2) Suppose that $f : (X,\ R_X) \longrightarrow (Y,\ R_Y)$ is a continuous map. $\forall B \subseteq Y$,

$$\begin{aligned} \tau_{R_X}(f^{-1}(B)) &= \bigwedge_{(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times f^{-1}(B)} (R_X(\mathcal{F},\ x) \rightarrow \mathcal{F}(f^{-1}(B))) \\ &\geq \bigwedge_{(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times f^{-1}(B)} (R_Y(f^{\Rightarrow}(\mathcal{F}),\ f(x)) \rightarrow f^{\Rightarrow}(\mathcal{F})(B)) \\ &\geq \bigwedge_{(\mathcal{G},\ y) \in \mathbb{F}_L(Y) \times B} (R_Y(\mathcal{G},\ y) \rightarrow \mathcal{G}(B)) \\ &= \tau_{R_Y}(B). \end{aligned}$$

□

Corollary 4.9. *R_T is the right adjoint of T_R. Thus L-**FYS** can be embedded in L-**FYCS** as a reflective subcategory.*

5. L-FYCS Is a Cartesian Closed Topological Category

The aim of this section is to show that L-**FYCS** is a cartesian closed topological category.

A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is called topological [1] provided every T-source $\{f_j : X \longrightarrow (X_j,\ D_j)\}_{j \in J}$ has a unique T-initial lift. A concrete category on **Set** is called a construct. A construct $\mathcal{C}$ is called topological if the forgetful functor $U : \mathcal{C} \longrightarrow$ **Set** is topological.

Theorem 5.1. *L-**FYCS** is topological over **Set** with respect to the expected forgetful functor.*

Let $U : L$-**FYCS** $\longrightarrow$ **Set** be the forgetful functor. Let $(X;\ (f_i,\ (Y_i,\ R_i))_{i \in I})$ be a U-structured source, i.e., X is a set and $(Y_i,\ R_i)$ is a family of L-fuzzifying convergence spaces and $\forall i \in I,\ f_i : X \longrightarrow Y_i$ is a map. We only need to prove that there exists an L-fuzzifying convergence structure R_X on X such that for any L-fuzzifying convergence space $(Z,\ R_Z)$, a map $g : (Z,\ R_Z) \longrightarrow (X,\ R_X)$

is an L-**FYCS**-morphism if and only if $\forall i \in I,\ f_i \circ g : (Z, R_Z) \longrightarrow (Y_i,\ R_i)$ is L-**FYCS**-morphism since L-**FYCS** is amnestic clearly.

In fact, defining $R_X : \mathbb{F}_L(X) \times X \longrightarrow L$ by $\forall (\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X$,

$$R_X(\mathcal{F},\ x) = \bigwedge_i R_i(f_i^{\Rightarrow}(\mathcal{F}),\ f_i(x)).$$

It's easy to verify that R_X is an L-fuzzifying convergence structure on X. On one hand, if $g : (Z,\ R_Z) \longrightarrow (X,\ R_X)$ is an L-**FYCS**-morphism, then $\forall (\mathcal{F},\ x) \in \mathbb{F}_L(Z) \times Z$,

$$\begin{array}{rcl} R_Z(\mathcal{F},\ x) & \le & R_X(g^{\Rightarrow}(\mathcal{F}),\ g(x)) \\ & = & \bigwedge_i R_i(f_i^{\Rightarrow}(g^{\Rightarrow}(\mathcal{F})),\ f_i(g(x))) \\ & = & \bigwedge_i R_i((f_i \circ g)^{\Rightarrow}(\mathcal{F}),\ (f_i \circ g)(x)), \end{array}$$

which implies that $\forall i \in I,\ R_Z(\mathcal{F},\ x) \le R_i((f_i \circ g)^{\Rightarrow}(\mathcal{F}),\ (f_i \circ g)(x))$ and $f_i \circ g : (Z,\ R_Z) \longrightarrow (Y_i,\ R_i)$ is also an L-**FYCS**-morphism. On the other hand, if $\forall i \in I,\ f_i \circ g : (Z,\ R_Z) \longrightarrow (Y_i\ R_i)$ is an L-**FYCS**-morphism, then $\forall (\mathcal{F},\ x) \in \mathbb{F}_L(Z) \times Z$,

$$\begin{array}{rcl} R_Z(\mathcal{F},\ x) & \le & \bigwedge_i R_i((f_i \circ g)^{\Rightarrow}(\mathcal{F}),\ (f_i \circ g)(x)) \\ & = & \bigwedge_i R_i(f_i^{\Rightarrow}(g^{\Rightarrow}(\mathcal{F})),\ f_i(g(x))) \\ & = & R_X(g^{\Rightarrow}(\mathcal{F}),\ g(x)). \end{array}$$

Hence $g : (Z,\ R_Z) \longrightarrow (X,\ R_X)$ is an L-**FYCS**-morphism.

Let $\{(X_i,\ R_i)|\ i \in I\}$ be a family of L-fuzzifying convergence spaces and $X = \prod_i X_i$ the cartesian product of $\{X_i|\ i \in I\}$. Define $R_X : \mathbb{F}_L(X) \times X \longrightarrow L$ by $\forall (\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X$,

$$R_X(\mathcal{F},\ x) = \bigwedge_i R_i(p_i^{\Rightarrow}(\mathcal{F}),\ x_i),$$

where $\forall i \in I,\ p_i : X \longrightarrow X_i$ is the projection and $x = (x_i)_{i \in I}$. It's easy to verify that R_X is an L-fuzzifying convergence structure on X and $\forall i \in I,\ p_i : X \longrightarrow X_i$ is continuous. Thus $(X,\ R_X)$ is the product object of $\{(X_i,\ R_i)|\ i \in I\}$ in the category L-**FYCS**.

A category with finite products is cartesian closed [1] if and only if for each pair $(A,\ B)$ of objects there exists an object $[A \to B]$ and an evaluation morphism $ev : [A \to B] \times A \longrightarrow B$ with the following universal property: for each morphism $f : C \times A \longrightarrow B$ there exists a unique morphism $\hat{f} : C \longrightarrow [A \to B]$ such that the following diagram commutes

$$\begin{tikzcd} C \times A \arrow[d, "\hat{f} \times id_A"'] \arrow[dr, "f"] & \\ {[A \to B] \times A} \arrow[r, "ev"] & B \end{tikzcd}$$

In the following, using the ideas in [11][2], we will show that L-**FYCS** is cartesian closed.

For two L-fuzzifying convergence spaces $(X,\ R_X)$ and $(Y,\ R_Y)$, let $[X \to Y]$ denote the set of all continuous maps from $(X,\ R_X)$ to $(Y,\ R_Y)$. Define $R_{[X\to Y]} : \mathbb{F}_L([X \to Y]) \times [X \to Y] \longrightarrow L$ (See the L-fuzzy version in [11]) by $\forall(\mathcal{F},\ f) \in \mathbb{F}_L([X \to Y]) \times [X \to Y]$,

$$R_{[X\to Y]}(\mathcal{F},\ f) = \bigwedge_{(\mathcal{G},\ x)\in\mathbb{F}_L(X)\times X} R_X(\mathcal{G},\ x) \to R_Y(ev^{\Rightarrow}(\mathcal{F}\times\mathcal{G}),\ f(x)).$$

Lemma 5.2. *Let $g : X \longrightarrow Y$ be a map and $\mathcal{G} \in \mathbb{F}_L(X)$. Then*

$$g^{\Rightarrow}(\mathcal{G}) \le ev^{\Rightarrow}([g]\times\mathcal{G}),$$

where $ev : [X \to Y] \times X \longrightarrow Y$ is the evaluation map.

Proof. $\forall C \subseteq Y$,

$$\begin{aligned}
ev^{\Rightarrow}([g]\times\mathcal{G})(C) &= ([g]\times\mathcal{G})(ev^{-1}(C))\\
&= \bigvee_{A\times B\subseteq ev^{-1}(C)} ([g](A)\wedge\mathcal{G}(B))\\
&= \bigvee_{A\times B\subseteq ev^{-1}(C),\ g\in A} \mathcal{G}(B)\\
&\le \mathcal{G}(g^{-1}(C))\\
&= g^{\Rightarrow}(\mathcal{G})(C),
\end{aligned}$$

where $A \subseteq [X \to Y],\ B \subseteq X$. Hence $g^{\Rightarrow}(\mathcal{G}) \le ev^{\Rightarrow}([g]\times\mathcal{G})$. □

Theorem 5.3. *$R_{[X\to Y]}$ is an L-fuzzifying convergence structure on $[X \to Y]$.*

Proof. Obviously, (FCS1) holds. To show (FCS2), $\forall g \in [X \to Y]$, by Lemma 5.2,

$$\begin{aligned}
R_{[X\to Y]}([g],\ g) &= \bigwedge_{(\mathcal{G},\ x)\in\mathbb{F}_L(X)\times X} R_X(\mathcal{G},\ x) \to R_Y(ev^{\Rightarrow}([g]\times\mathcal{G}),\ g(x))\\
&\ge \bigwedge_{(\mathcal{G},\ x)\in\mathbb{F}_L(X)\times X} R_X(\mathcal{G},\ x) \to R_Y(g^{\Rightarrow}(\mathcal{G}),\ g(x))\\
&= 1,
\end{aligned}$$

since $g : X \longrightarrow Y$ is continuous. □

Lemma 5.4. *Let $\mathcal{G} \in \mathbb{F}_L(X \times Y)$. Then ${p_X}^{\Rightarrow}(\mathcal{G}) \times {p_Y}^{\Rightarrow}(\mathcal{G}) \le \mathcal{G}$.*

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[2]Reference [11] is a fundamental reference in this paper, especially in this section. The approaches used in this section are mainly imitated from it.

Proof. $\forall C \subseteq X \times Y$,

$$\begin{aligned}(p_X{}^{\Rightarrow}(\mathcal{G}) \times p_Y{}^{\Rightarrow}(\mathcal{G}))(C) &= \bigvee_{A\times B\subseteq C} (p_X{}^{\Rightarrow}(\mathcal{G})(A) \wedge p_Y{}^{\Rightarrow}(\mathcal{G})(B)) \\ &= \bigvee_{A\times B\subseteq C} (\mathcal{G}(p_X^{-1}(A)) \wedge \mathcal{G}(p_Y^{-1}(B))) \\ &\le \bigvee_{A\times B\subseteq C} \mathcal{G}((A\times Y)\cap (X\times B)) \\ &= \bigvee_{A\times B\subseteq C} \mathcal{G}(A\times B) \\ &\le \mathcal{G}(C).\end{aligned}$$

□

Theorem 5.5. *The evaluation map $ev : [X \to Y] \times X \longrightarrow Y$ is continuous.*

Proof. Suppose that $p_1 : [X \to Y] \times X \longrightarrow [X \to Y]$ and $p_2 : [X \to Y] \times X \longrightarrow X$ are the corresponding projections. $\forall \mathcal{H} \in \mathbb{F}_L([X \to Y]\times X)$, $\forall (g,\ x) \in [X \to Y]\times X$,

$$\begin{aligned}&R_{[X\to Y]\times X}(\mathcal{H},\ (g,\ x)) \\ =\ & R_{[X\to Y]}(p_1{}^{\Rightarrow}(\mathcal{H}),\ g) \wedge R_X(p_2{}^{\Rightarrow}(\mathcal{H}),\ x) \\ \le\ & (R_X(p_2{}^{\Rightarrow}(\mathcal{H}),\ x) \to R_Y(ev^{\Rightarrow}(p_1{}^{\Rightarrow}(\mathcal{H}) \times p_2{}^{\Rightarrow}(\mathcal{H})),\ g(x))) \wedge R_X(p_2{}^{\Rightarrow}(\mathcal{H}),\ x) \\ \le\ & R_Y(ev^{\Rightarrow}(p_1{}^{\Rightarrow}(\mathcal{H}) \times p_2{}^{\Rightarrow}(\mathcal{H})),\ g(x)) \\ \le\ & R_Y(ev^{\Rightarrow}(\mathcal{H}),\ g(x)).\end{aligned}$$

□

Now let's consider the following situation. Let $f : X \times Y \longrightarrow Z$ be a map. Define for $x \in X$ the map $f_x : Y \longrightarrow Z,\ y \longmapsto f(x,\ y)$ and with this the map $f^* : X \longrightarrow Z^Y,\ x \longmapsto f_x$. The map $\varphi : Z^{X\times Y} \longrightarrow (Z^Y)^X,\ f \longmapsto f^*$ is called the exponential map.

Lemma 5.6. *Let $f : X \times Y \longrightarrow Z$ be a map and let $x \in X$. Then for an L-filter $\mathcal{F} \in \mathbb{F}_L(Y)$, it holds that $f_x{}^{\Rightarrow}(\mathcal{F}) \ge f^{\Rightarrow}([x] \times \mathcal{F})$.*

Proof. $\forall C \subseteq Y$,

$$\begin{aligned}f^{\Rightarrow}([x]\times\mathcal{F})(C) &= \bigvee_{A\times B\subseteq f^{-1}(C)} [x](A) \wedge \mathcal{F}(B) \\ &= \bigvee_{A\times B\subseteq f^{-1}(C),\ x\in A} \mathcal{F}(B) \\ &\le \mathcal{F}(f_x^{-1}(C)) \\ &= f_x{}^{\Rightarrow}(\mathcal{F})(C).\end{aligned}$$

□

Lemma 5.7. *$\forall \mathcal{F} \in \mathbb{F}_L(X),\ \mathcal{G} \in \mathbb{F}_L(Y)$, we have $p_X{}^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) = \mathcal{F}$ and $p_Y{}^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) = \mathcal{G}$.*

Proof. Straightforward. □

Lemma 5.8. *Let $f : (X,\ R_X) \times (Y,\ R_Y) \longrightarrow (Z,\ R_Z)$ be continuous. Then for each $x \in X$ also $f_x : (Y,\ R_Y) \longrightarrow (Z,\ R_Z)$ is continuous.*

Proof. $\forall(\mathcal{F},\ y)\in\mathbb{F}_L(Y)\times Y$,

$$\begin{array}{rcl} R_Z(f_x{}^{\Rightarrow}(\mathcal{F}),\ f_x(y)) & \geq & R_Z(f^{\Rightarrow}([x]\times\mathcal{F}),\ f(x,\ y)) \\ & \geq & R_{X\times Y}([x]\times\mathcal{F},\ (x,\ y)) \\ & = & R_X(p_X{}^{\Rightarrow}([x]\times\mathcal{F}),\ x)\wedge R_Y(p_Y{}^{\Rightarrow}([x]\times\mathcal{F}),\ y) \\ & = & R_X([x],\ x)\wedge R_Y(\mathcal{F},\ y) \\ & = & R_Y(\mathcal{F},\ y). \end{array}$$

□

Lemma 5.9. *$\forall\mathcal{F}\in\mathbb{F}_L(X),\ \mathcal{G}\in\mathbb{F}_L(Y)$, we have $ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F})\times\mathcal{G})=f^{\Rightarrow}(\mathcal{F}\times\mathcal{G})$.*

Proof. $\forall C\subseteq Z$,

$$\begin{array}{rcl} ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F})\times\mathcal{G})(C) & = & \bigvee\limits_{D\times B\subseteq ev^{-1}(C)}\varphi(f)^{\Rightarrow}(\mathcal{F})(D)\wedge\mathcal{G}(B) \\ & = & \bigvee\limits_{ev(D\times B)\subseteq C}\mathcal{F}(\varphi(f)^{-1}(D))\wedge\mathcal{G}(B) \\ & = & \bigvee\limits_{ev(D\times B)\subseteq C}\mathcal{F}(\{x\in X|\ f_x\in D\})\wedge\mathcal{G}(B), \end{array}$$

and

$$\begin{array}{rcl} f^{\Rightarrow}(\mathcal{F}\times\mathcal{G})(C) & = & \bigvee\limits_{A\times B\subseteq f^{-1}(C)}\mathcal{F}(A)\wedge\mathcal{G}(B) \\ & = & \bigvee\limits_{ev(D\times B)\subseteq C}\mathcal{F}(\varphi(f)^{-1}(D))\wedge\mathcal{G}(B) \\ & = & \bigvee\limits_{f(A\times B)\subseteq C}\mathcal{F}(A)\wedge\mathcal{G}(B). \end{array}$$

If $ev(D\times B)\subseteq C$, put $A=\{x\in X|\ f_x\in D\}$, then we have $f(A\times B)\subseteq C$. And if $f(A\times B)\subseteq C$, put $D=\{f_x|\ x\in A\}$, then we have $ev(D\times B)\subseteq C$. Hence $ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F})\times\mathcal{G})=f^{\Rightarrow}(\mathcal{F}\times\mathcal{G})$. □

Lemma 5.10. *If the map $f:X\times Y\longrightarrow Z$ is continuous, then also $\varphi(f):X\longrightarrow[Y\to Z]$ is continuous.*

Proof. By Lemma 5.9, $\varphi(f)$ is well-defined. $\forall(\mathcal{F},\ x)\in\mathbb{F}_L(X)\times X$, $\forall(\mathcal{G},\ y)\in\mathbb{F}_L(Y)\times Y$, we have

$$\begin{array}{rcl} R_X(\mathcal{F},\ x)\wedge R_Y(\mathcal{G},\ y) & = & R_{X\times Y}(\mathcal{F}\times\mathcal{G},\ (x,\ y)) \\ & \leq & R_Z(f^{\Rightarrow}(\mathcal{F}\times\mathcal{G}),\ f(x,\ y)) \\ & = & R_Z(ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F})\times\mathcal{G}),\ \varphi(f)(x)(y)). \end{array}$$

By the arbitrariness of $(\mathcal{G},\ y)\in\mathbb{F}_L(Y)\times Y$, we have

$$\begin{array}{rl} & R_X(\mathcal{F},\ x) \\ \leq & \bigwedge\limits_{(\mathcal{G},\ y)\in\mathbb{F}_L(Y)\times Y}(R_Y(\mathcal{G},\ y)\to R_Z(ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F})\times\mathcal{G}),\ \varphi(f)(x)(y))) \\ = & R_{[Y\to Z]}(\varphi(f)^{\Rightarrow}(\mathcal{F}),\ \varphi(f)(x)). \end{array}$$

Hence $\varphi(f):X\longrightarrow[Y\to Z]$ is continuous. □

By Theorem 5.3, Theorem 5.5 and Lemma 5.10, we have

Theorem 5.11. *L-**FYCS** is cartesian closed.*

Remark 5.12. (1) In [12], Jäger defined and studied several subcategories of the category of stratified L-fuzzy convergence spaces. One of which was called stratified L-limit spaces. It was shown that the category of stratified L-limit spaces can be embedded in the category of stratified L-fuzzy convergence spaces as a reflective subcategory and is also cartesian closed.

(2) An L-fuzzifying space $(X,\ R)$ is called an L-fuzzifying limit space if it satisfies

(FLim) $\forall \mathcal{F},\ \mathcal{G} \in \mathbb{F}_L(X),\ \forall x \in X,\ R(\mathcal{F},\ x) \wedge R(\mathcal{G},\ x) \leq R(\mathcal{F} \wedge \mathcal{G},\ x)$.

It's easy to verify that for any L-fuzzifying topological space $(X,\ \tau)$, $(X,\ R_\tau)$ is an L-fuzzifying limit space. Denote by L-**FYLim** the full subcategory of L-**FYCS** formed by all L-fuzzifying limit spaces. Restricting the two functors R_T and T_R to L-**FYLim**, we get that L-**FYS** can be embedded in L-**FYLim** as a reflective subcategory. Also, it can be shown that L-**FYLim** is cartesian closed by a similar fashion.

6. Specialization L-preorder

For a classical topological space X, define a binary relation $\leq$ on X by

$$x \leq y \text{ iff } x \in \{y\}^-,$$

where $\{y\}^-$ is the closure of $\{y\}$ in X. Then $\leq$ is a preorder on X, called the specialization preorder [6] of X. If X satisfies the T_0 axiom, then the specialization preorder is a partial order. Likewise, each L-topological space (resp., L-fuzzifying spaces, L-fuzzy spaces) can induced an L-preorder (See Definition 6.1), called the specialization L-preorder of the corresponding spaces. For $L = [0,\ 1]$ with a left continuous t-norm, from viewpoint of category theory, the specialization L-preorder of L-topological spaces (resp., L-fuzzifying topological spaces, L-fuzzy topological spaces) was studied in [14] (resp., [3], [4]).

In this section, we will study the specialization L-preorder of L-fuzzifying convergence spaces.

Definition 6.1. ([5] for L a residuated lattice, or [3, 4, 14] for $L = [0,\ 1]$) An L-relation on a set X is a map $P : X \times X \longrightarrow L$. An L-relation P is called an L-preorder if

(Pr1) (reflexivity) $P(x,\ x) = 1$ for all $x \in X$;

(Pr2) (transitivity) $P(x,\ y) \wedge P(y,\ z) \leq P(x,\ z)$ for all $x,\ y,\ z \in X$.

The pair $(X,\ P)$ is called an L-preordered set. For two L-preordered sets $(X,\ P_X)$ and $(Y,\ P_Y)$, a map $f : X \longrightarrow Y$ is called order-preserving if $P_X(x,\ y) \leq P_Y(f(x),\ f(y))$ for all $x,\ y \in X$. The category of all the L-preordered sets and order-preserving maps is denoted by L-**PrOrd**.

Theorem 6.2. *Let $(X,\ R)$ be an L-fuzzifying convergence space, define a map $P_R : X \times X \longrightarrow L$ by*

$$\forall x,\ y \in X,\ P_R(x,\ y) = \bigwedge_{\mathcal{F} \in \mathbb{F}_L(X)} R(\mathcal{F},\ y) \to R(\mathcal{F},\ x).$$

Then P_R is an L-preorder.

Proof. (Pr1) holds clearly. To show (Pr2), $\forall x,\ y,\ z \in X$,

$$\begin{array}{rl} & P_R(x,\ y) \wedge P_R(y,\ z) \\ = & (\bigwedge\limits_{\mathcal{F}\in\mathbb{F}_L(X)} R(\mathcal{F},\ y) \to R(\mathcal{F},\ x)) \wedge (\bigwedge\limits_{\mathcal{G}\in\mathbb{F}_L(X)} R(\mathcal{G},\ z) \to R(\mathcal{G},\ y)) \\ \leq & \bigwedge\limits_{\mathcal{F}\in\mathbb{F}_L(X)} (R(\mathcal{F},\ y) \to R(\mathcal{F},\ x)) \wedge (R(\mathcal{F},\ z) \to R(\mathcal{F},\ y)) \\ \leq & \bigwedge\limits_{\mathcal{F}\in\mathbb{F}_L(X)} R(\mathcal{F},\ z) \to R(\mathcal{F},\ x) \\ = & P_R(x,\ z). \end{array}$$

□

Remark 6.3. $P_R : L$-**FYCS** $\longrightarrow L$-**PrOrd** is not a functor even if $L = \{0,\ 1\}$ as the following example shows. Let $X = \{x,\ y\}$, $Y = \{a,\ b,\ c\}$ and $f : X \longrightarrow Y$ be defined by $f(x) = a,\ f(y) = b$. Then $\mathbb{F}(X) = \{[x],\ [y],\ \{X\}\}$ and $\mathbb{F}(Y) = \{[a],\ [b],\ [c],\ \uparrow\{a,\ b\},\ \uparrow\{b,\ c\},\ \uparrow\{c,\ a\},\ \{X\}\}$. Suppose that $R_X = \{([x],\ x),\ ([y],\ y),\ ([y],\ x)\}$ and $R_Y = \{([a],\ a),\ ([a],\ c),\ ([b],\ b),\ ([b],\ a),\ ([c],\ c),\ ([c],\ b),\ (\uparrow\{a,\ b\},\ a),\ (\uparrow\{b,\ c\},\ b),\ (\uparrow\{c,\ a\},\ c)\}$. Then R_X and R_Y are $\{0,\ 1\}$-fuzzifying convergence structures on X and Y respectively and $f : (X,\ R_X) \longrightarrow (Y,\ R_Y)$ is continuous. It's easy to verify that $x \leq y$ in P_{R_X} while $a \not\leq b$ in P_{R_Y} since $(\uparrow\{b,\ c\},\ b) \in R_Y$ and $(\uparrow\{b,\ c\},\ a) \notin R_Y$.

In [14], based on a left continuous t-norm $*$ on the unit interval $[0,\ 1]$, Lai and Zhang studied the relations between the categories of **FPrOrd** (i.e., I-**FPrOrd**) and **FTS** (i.e., the category of stratified fuzzy topological spaces). Recently Fang and Chen analogized Lai and Zhang's results to fuzzifying setting in [3].

For a fuzzy preordered set $(X,\ P)$, its associated I-fuzzifying topology $\nabla(P) : 2^X \longrightarrow [0,\ 1]$ is defined by

$$\forall A \subseteq X,\ \nabla(P)(A) = \bigwedge_{x\in A,\ y\notin A} 1 - P(x,\ y).$$

Conversely, for an I-fuzzy topology τ on X, its associated fuzzy preorder $\Theta : X \times X \longrightarrow I$ is defined by

$$\forall x,\ y \in X,\ \Theta(\tau)(x,\ y) = \bigwedge_{A\in\dot{x},\ A\notin\dot{y}} 1 - \tau(A).$$

Fang and Chen showed that $(\nabla,\ \Theta)$ is a Galois connection [1] between the categories **FPrOrd** and **FYS**.

The results in [3] can be easily extended to any complete Heyting algebra.

Theorem 6.4. (1) *Let τ be an L-fuzzifying topology on X. Then the map $\Theta_\tau : X \times X \longrightarrow L$ defined by*

$$\forall x,\ y \in X,\ \Theta_\tau(x,\ y) = \bigwedge_{A\in\dot{x},\ A\notin\dot{y}} \tau(A) \to 0$$

is an L-preorder on X.

(2) *Let P be an L-preorder on X. Then the map $\nabla_P : 2^X \longrightarrow L$ defined by*

$$\forall A \subseteq X,\ \nabla_P(A) = \bigwedge_{x \in A,\ y \notin A} P(x,\ y) \to 0$$

is an L-fuzzifying topology on X.

Proof. Trivial. □

Remark 6.5. Since there need not exist a reverse involution on L, we replace $1 - \tau(A)$ (resp., $1 - P(x,\ y)$) by $\tau(A) \to 0$ (resp., $P(x,\ y) \to 0$). Still we can prove that $\tau \leq \nabla(\Theta(\tau))$ and $P \leq \Theta(\nabla(P))$ for any L-fuzzifying topology τ and any L-preorder P on X. Thus $(\nabla,\ \Theta)$ is a Galois connection between the categories L-**PrOrd** and L-**FYS**.

Theorem 6.6. *The following diagram commutes, where $|\cdot|$ stands for the class of objects of the category $\cdot$.*

$$\begin{tikzcd}
{|L\text{-}\mathbf{FYS}|} \arrow[rr, "R_T"] \arrow[dr, "\Theta"'] & & {|L\text{-}\mathbf{FYCS}|} \arrow[dl, "P_R"] \\
& {|L\text{-}\mathbf{FPrOrd}|} &
\end{tikzcd}$$

Proof. Let $(X,\ \tau)$ be an L-fuzzifying topological space. Then $\forall(\mathcal{F},\ x) \in \mathbb{F}_L(X) \times X$,

$$R_\tau(\mathcal{F},\ x) = \bigwedge_{x \in A} \tau(A)) \to \mathcal{F}(A)$$

and

$$R_\tau([y],\ x) = \bigwedge_{A \in \dot{x},\ A \notin \dot{y}} \tau(A) \to 0 = \Theta(\tau)(x,\ y).$$

We only need to show that $P_{R_\tau}(x,\ y) = R_\tau([y],\ x)$ in the following. In fact, it is easy to see that

$$P_{R_\tau}(x,\ y) = \bigwedge_{\mathcal{F} \in \mathbb{F}_L(X)} R_\tau(\mathcal{F},\ y) \to R_\tau(\mathcal{F},\ x) \leq R_\tau([y],\ x).$$

Secondly, $\forall \mathcal{F} \in \mathbb{F}_L(X)$,

$$\begin{aligned}
R_\tau(\mathcal{F},\ y) \to R_\tau(\mathcal{F},\ x) &= R_\tau(\mathcal{F},\ y) \to (\bigwedge_{A \in \dot{x}} (\tau(A)) \to \mathcal{F}(A)) \\
&= \bigwedge_{A \in \dot{x}} R_\tau(\mathcal{F},\ y) \to (\tau(A) \to \mathcal{F}(A)) \\
&\geq \bigwedge_{A \in \dot{x}} \tau(A) \to (R_\tau(\mathcal{F},\ y) \to \mathcal{F}(A)).
\end{aligned}$$

If $A \in \dot{y}$, then $R_\tau(\mathcal{F},\ y) \le \tau(A) \to \mathcal{F}(A)$ and $R_\tau(\mathcal{F},\ y) \to \mathcal{F}(A) \ge (\tau(A) \to \mathcal{F}(A)) \to \mathcal{F}(A) \ge \tau(A)$ and $\tau(A) \to (R_\tau(\mathcal{F},\ y) \to \mathcal{F}(A)) = 1$. Thus

$$\begin{aligned}
& \bigwedge_{A \in \dot{x}} \tau(A) \to (R_\tau(\mathcal{F},\ y) \to \mathcal{F}(A)) \\
= & \bigwedge_{A \in \dot{x},\ A \notin \dot{y}} \tau(A) \to (R_\tau(\mathcal{F},\ y) \to \mathcal{F}(A)) \\
\ge & \bigwedge_{A \in \dot{x},\ A \notin \dot{y}} \tau(A) \to 0 \\
= & R_\tau([y],\ x).
\end{aligned}$$

Hence $P_{R_\tau}(x,\ y) = R_\tau([y],\ x)$ and by the arbitrariness of $(x,\ y)$, $P_R \circ R_T = \Theta$. $\square$

Corollary 6.7. *If $(X,\ R)$ is a topological L-fuzzifying convergence space, then it's specialization L-preorder is $P_R(x,\ y) = R([y],\ x)(\forall x,\ y \in L)$.*

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