ABSORBENT ORDERED FILTERS AND THEIR FUZZIFICATIONS IN IMPLICATIVE SEMIGROUPS

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ABSTRACT. The notion of absorbent ordered filters in implicative semigroups is introduced, and its fuzzification is considered. Relations among (fuzzy) ordered filters, (fuzzy) absorbent ordered filters, and (fuzzy) positive implicative ordered filters are stated. The extensionproperty for (fuzzy) absorbent ordered filters is established. Conditions for (fuzzy) ordered filters to be (fuzzy) absorbent ordered filters are provided. The notions of normal/maximal fuzzy absorbent ordered filters and complete absorbent ordered filters are introduced and their properties are investigated.

1. Introduction

The notions of implicative semigroups and ordered filters were introduced by Chan and Shum [3]. The first is a generalization of implicative semilattices (see Nemitz [13] and Blyth [2]) and is closely related to implication in mathematical logic and set theoretic differences (see Birkhoff [1] and Curry [4]). As shown by Nemitz [13], ordered filters play an important role in the general development of implicative semilattice theory. Motivated by this fact, Chan and Shum [3] established some elementary propositions and constructed the quotient structure of implicative semigroups via ordered filters. Jun et al. [10] discussed ordered filters of implicative semigroups. To study implicative semigroups in depth, it is necessary to establish a more complete theory of ordered filters. Jun [7] investigated further properties of implicative semigroups and of their ordered filters. In particular, he introduced the notion of n-fold implicative ordered filters in implicative semigroups, stated some equivalent conditions for an ordered filter to be an implicative ordered filter and obtained the so called extension property for implicative ordered filters. He also stated a condition for the ordered filter $\{1\}$ to be n-fold implicative.

In this paper, we introduce the notion of (fuzzy) absorbent ordered filters in implicative semigroups. We give relations among (fuzzy) ordered filters, (fuzzy) absorbent ordered filters, and (fuzzy) positive implicative ordered filters and provide conditions for (fuzzy) ordered filters to be (fuzzy) absorbent ordered filters. Then, using the notion of level sets, we establish a characterization of fuzzy absorbent ordered filters. Furthermore, we demonstrate an extension property for (fuzzy) absorbent

ordered filters and derive a fuzzy absorbent ordered filter from a collection of absorbent ordered filters with additional conditions. We alao prove a characterization theorem for the set of values of any fuzzy absorbent ordered filter which is a well ordered subset of $[0, 1]$. Finally, we introduce the concept of normal/maximal fuzzy absorbent ordered filters and complete absorbent ordered filters, and investigate their properties.

2. Preliminaries

We first recall some definitions and results. By a *negatively partially ordered semigroup* (briefly, *n.p.o. semigroup*) we mean a set S with a partial ordering "$\leq$" and a binary operation "$\cdot$" such that for all $x, y, z \in S$, we have:

(a1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(a2) $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$,
(a3) $x \cdot y \leq x$ and $x \cdot y \leq y$.

A n.p.o. semigroup $(S; \leq, \cdot)$ is said to be *implicative* if there is an additional binary operation $* : S \times S \to S$ such that

$$(2.1) \qquad (\forall x, y, z \in S)\,(z \leq x * y \iff z \cdot x \leq y).$$

The operation $*$ is called *implication*. Henceforth, an implicative n.p.o. semigroup will be simply called an *implicative semigroup*.

An implicative semigroup $(S; \leq, \cdot, *)$ is said to be *commutative* if it satisfies

$$(2.2) \qquad (\forall x, y \in S)\,(x \cdot y = y \cdot x).$$

In other words, $(S, \cdot)$ is a commutative semigroup.

In any implicative semigroup $(S; \leq, \cdot, *)$, $x * x = y * y$ and, in fact, this is the greatest element in S. We shall denote it by 1. Some elementary properties of implicative semigroups are summarized in the following proposition.

Proposition 2.1. [3, Theorem 1.4] *Let S be an implicative semigroup. Then for every $x, y, z \in S$, the following hold:*

(b1) $x \leq 1$, $x * x = 1$, $x = 1 * x$,
(b2) $x \leq y * (x \cdot y)$,
(b3) $x \leq x * (x \cdot x)$,
(b4) $x \leq y * x$,
(b5) *if $x \leq y$ then $x * z \geq y * z$ and $z * x \leq z * y$,*
(b6) $x \leq y$ *if and only if $x * y = 1$,*
(b7) $x * (y * z) = (x \cdot y) * z$,
(b8) *if S is commutative then $x * y \leq (s \cdot x) * (s \cdot y)$ for all s in S.*

Definition 2.2. [3, Definition 2.1] Let S be an implicative semigroup and let F be a nonempty subset of S. Then F is called an *ordered filter* of S if

(F1) $x \cdot y \in F$ for every $x, y \in F$; i.e. F is a subsemigroup of S.
(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

The following proposition gives a characterization of ordered filters.

Proposition 2.3. [10, Proposition 2] *Suppose S is an implicative semigroup. Then a non-empty subset F of S is an ordered filter if and only if it satisfies the following conditions:*

(F3) $1 \in F$,
(F4) $x * y \in F$ *and* $x \in F$ *imply* $y \in F$.

Proposition 2.4. [12] *If S is a commutative implicative semigroup, then for any $x, y, z \in S$,*

(b9) $x * (y * z) = y * (x * z)$.
(b10) $y * z \leq (x * y) * (x * z)$.
(b11) $x \leq (x * y) * y$.

Proposition 2.5. [5] *Let S be a commutative implicative semigroup and let F be a non-empty subset of S. Then F is an ordered filter of S if and only if*

$$(2.3) \qquad (\forall x, y \in F)\,(\forall z \in S)\,(x \leq y * z \Rightarrow z \in F).$$

Definition 2.6. [5] Let S be an implicative semigroup. A non-empty subset F of S is called an *implicative ordered filter* of S if it satisfies (F3) and

(F5) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$

for all $x, y, z \in S$.

Definition 2.7. [8] Let S be an implicative semigroup. A non-empty subset F of S is called a *positive implicative ordered filter* of S if it satisfies (F3) and

(F6) $(\forall x, y, z \in S)\,(x * ((y * z) * y) \in F, x \in F \Rightarrow y \in F)$.

Proposition 2.8. [8] *Let S be an implicative semigroup and let F be an ordered filter of S. Then F is a positive implicative ordered filter of S if and only if*

$$(2.4) \qquad (\forall x, y \in S)\,((x * y) * x \in F \Rightarrow x \in F).$$

Definition 2.9. [11] A fuzzy set A in S is called a *fuzzy ordered filter* of S if the following conditions hold.

(c1) $(\forall x \in S)\,(\mathrm{A}(x) \leq \mathrm{A}(1))$.
(c2) $(\forall x, y \in S)\,(\mathrm{A}(y) \geq \min\{\mathrm{A}(x * y), \mathrm{A}(x)\})$.

We note that every fuzzy ordered filter is order preserving.

Proposition 2.10. [6] *Let A be a fuzzy set in a commutative implicative semigroup S. Then A is a fuzzy ordered filter of S if and only if*

$$(2.5) \qquad (\forall x, y, z \in S)\,(x \cdot y \leq z \Rightarrow \mathrm{A}(z) \geq \min\{\mathrm{A}(x), \mathrm{A}(y)\}).$$

Definition 2.11. [6] A fuzzy set A in S is called a *fuzzy implicative ordered filter* of S if it satisfies the conditions (c1) and

(c3) $(\forall x, y, z \in S)$ $(A(x * z) \geq \min\{A(x * (y * z)), A(x * y)\})$.

We may show that every fuzzy implicative ordered filter is a fuzzy ordered filter, but the converse is not true in general [6].

Definition 2.12. [9] A fuzzy set A in S is called a *fuzzy positive implicative ordered filter* of S if it satisfies the conditions (c1) and

(c4) $(\forall x, y, z \in S)$ $(A(y) \geq \min\{A(x * ((y * z) * y)), A(x)\})$.

3. Absorbent Ordered Filters

In this section, we introduce the notion of an absorbent ordered filter which lies somewhere between the notions of an ordered filter and a positive implicative ordered filter. In what follows S will denote an implicative semigroup unless otherwise specified.

Definition 3.1. A non-empty subset F of S is called an *absorbent ordered filter* of S if it satisfies (F3) and

(F7) $(\forall x, y, z \in S)$ $(z * (y * x) \in F, z \in F \Rightarrow ((x * y) * y) * x \in F)$.

Example 3.2. Consider an implicative semigroup $S = \{1, a, b, c, d, 0\}$ with Cayley tables and Hasse diagram as follows:

·	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

*	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $I = \{1, a, b\}$ is an absorbent ordered filter of S.

Theorem 3.3. *Every absorbent ordered filter is an ordered filter.*

Proof. Let F be an absorbent ordered filter of S and let $x, y, z \in S$ be such that $z * x \in F$ and $z \in F$. Then $z * (1 * x) \in F$ and $z \in F$. It follows from (F7) that $x = ((x * 1) * 1) * x \in F$ so that F is an ordered filter of S. $\square$

As the following example shows, the converse of Theorem 3.3 is not true in general.

Example 3.4. Consider an implicative semigroup $S = \{1, a, b, c, d\}$ with Cayley tables and Hasse diagram as follows:

·	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $F = \{1, b\}$ is an ordered filter of S, but it is not an absorbent ordered filter of S since $1 * (c * a) = 1 * 1 = 1 \in F$ and

$$((a * c) * c) * a = (c * c) * a = 1 * a = a \notin F.$$

The following theorem gives a necessary and sufficient condition for an ordered filter to be an absorbent ordered filter.

Theorem 3.5. *An ordered filter F of S is absorbent if and only if it satisfies:*

(3.1) $$(\forall x, y \in S)\,(y * x \in F \;\Rightarrow\; ((x * y) * y) * x \in F).$$

Proof. Assume that F is an absorbent ordered filter of S and let $x, y \in S$ be such that $y * x \in F$. Then $1 * (y * x) = y * x \in F$ and $1 \in F$. It follows from (F7) that $((x * y) * y) * x \in F$. Conversely, let F be an ordered filter of S satisfying (3.1) and let $x, y, z \in S$ be such that $z * (y * x) \in F$ and $z \in F$. Then $y * x \in F$ by (F4) and hence $((x * y) * y) * x \in F$ by (3.1). Therefore F is an absorbent ordered filter of S. $\qquad \square$

Theorem 3.6. *Every positive implicative ordered filter is an absorbent ordered filter.*

Proof. Let F be a positive implicative ordered filter of S. Then F is an ordered filter of S [8, Theorem 2.2]. Let $x, y \in S$ be such that $y * x \in F$. It is sufficient to show that $((x * y) * y) * x \in F$. Since $x \leq ((x * y) * y) * x$, hence $(((x * y) * y) * x) * y \leq x * y$. Putting $a = ((x * y) * y) * x$, we obtain

(3.2) $$\begin{aligned} (a * y) * a &= ((((x * y) * y) * x) * y) * (((x * y) * y) * x) \\ &\geq (x * y) * (((x * y) * y) * x) \\ &= ((x * y) * y) * ((x * y) * x) \;\geq\; y * x. \end{aligned}$$

It follows from (F2) that $(a * y) * a \in F$ so from Proposition 2.8, $a \in F$, i.e., $((x * y) * y) * x \in F$. Hence F is an absorbent ordered filter of S. $\qquad \square$

As the following example shows, the converse of Theorem 3.6 is not true in general.

Example 3.7. Let S be the implicative semigroup in Example 3.2. Then $I = \{1, a\}$ is an absorbent ordered filter of S, but it is not a positive implicative ordered filter of S since $a * ((b * a) * b) = a * (1 * b) = a * b = a \in I$, but $b \notin I$.

Theorem 3.8. (Extension property) *Let F and G be ordered filters of a commutative implicative semigroup S such that $F \subseteq G$. If F is absorbent, then so is G.*

Proof. Let $x, y \in S$ be such that $y * x \in G$. Then $y * ((y * x) * x) = (y * x) * (y * x) = 1 \in F$. Since F is absorbent, it follows from Theorem 3.5 that

(3.3) $$((((y * x) * x) * y) * y) * ((y * x) * x) \in F$$

so that $(y * x) * (((((y * x) * x) * y) * y) * x) \in F \subset G$. Since $y * x \in G$, therefore $((((y * x) * x) * y) * y) * x \in G$. But

$$
\begin{aligned}
(((((y * x) * x) * y) * y) * x) * (((x * y) * y) * x) \\
\geq ((x * y) * y) * ((((y * x) * x) * y) * y) \\
\geq (((y * x) * x) * y) * (x * y) \\
\geq x * ((y * x) * x) = (y * x) * (x * x) \\
= (y * x) * 1 = 1.
\end{aligned}
$$

(3.4)

Now, by Proposition 2.5, $((x * y) * y) * x \in G$. Hence, by Theorem 3.5, G is an absorbent ordered filter of S. $\qquad \square$

4. **Fuzzy Absorbent Ordered Filters**

Definition 4.1. A fuzzy set A in S is called a *fuzzy absorbent ordered filter* of S if it satisfies the conditions (c1) and

(c5) $(\forall x, y, z \in S)\ (\mathrm{A}(((x * y) * y) * x) \geq \min\{\mathrm{A}(z * (y * x)), \mathrm{A}(z)\})$.

Example 4.2. Let S be the implicative semigroup in Example 3.2. Define a fuzzy set A in S by $\mathrm{A}(1) = 0.6$ and $\mathrm{A}(x) = 0.3$ for all $x \in S$ with $x \neq 1$. Then A is a fuzzy absorbent ordered filter of S. Also a fuzzy set B in S defined by $\mathrm{B}(1) = \mathrm{B}(a) = \mathrm{B}(b) = 0.7$ and $\mathrm{B}(c) = \mathrm{B}(d) = \mathrm{B}(0) = 0.2$ is a fuzzy absorbent ordered filter of S.

Theorem 4.3. *Every fuzzy absorbent ordered filter is a fuzzy ordered filter.*

Proof. Let A be a fuzzy absorbent ordered filter of S. Taking $y = 1$ in (c5) and using (b1), we have

$$
\begin{aligned}
\mathrm{A}(x) &= \mathrm{A}(((x * 1) * 1) * x) \\
&\geq \min\{\mathrm{A}(z * (1 * x)), \mathrm{A}(z)\} \\
&= \min\{\mathrm{A}(z * x)), \mathrm{A}(z)\}
\end{aligned}
$$

for all $x, z \in S$. Hence A is a fuzzy ordered filter of S. $\qquad \square$

As the following example shows, the converse of Theorem 3.3 is not true in general.

Example 4.4. Let S be the implicative semigroup in Example 3.4. A fuzzy set A in S given by

$$\mathrm{A}(1) = 0.5 > 0.3 = \mathrm{A}(x)$$

for all $x \in S$ with $x \neq 1$, is a fuzzy ordered filter of S. But A is not a fuzzy absorbent ordered filter of S since

$$\mathrm{A}(((a * d) * d) * a) = 0.3 < 0.5 \min\{\mathrm{A}(1 * (d * a)), \mathrm{A}(1)\}.$$

The following theorem provides a condition for a fuzzy ordered filter to be a fuzzy absorbent ordered filter.

Theorem 4.5. *Let* A *be a fuzzy ordered filter of S. Then the following are equivalent.*

(i) A *is a fuzzy absorbent ordered filter of S.*
(ii) $(\forall x, y \in S)\,(\mathrm{A}(y * x) \leq \mathrm{A}(((x * y) * y) * x)).$

Proof. Assume that A is a fuzzy absorbent ordered filter of S. Putting $z = 1$ in (c5), by (b1) and (c1), we have

$$\mathrm{A}(((x * y) * y) * x) \geq \min\{\mathrm{A}(1 * (y * x)), \mathrm{A}(1)\} = \mathrm{A}(y * x)$$

for all $x, y \in S$. Conversely let A be a fuzzy ordered filter of S satisfying the condition (ii). Let $x, y, z \in S$. Then by (ii) and (c2),

$$\mathrm{A}(((x * y) * y) * x) \geq \mathrm{A}(y * x) \geq \min\{\mathrm{A}(z * (y * x)), \mathrm{A}(z)\}$$

Hence A is a fuzzy absorbent ordered filter of S. $\qquad\square$

Lemma 4.6. [9] *Let A be a fuzzy ordered filter of S. Then A is a fuzzy positive implicative ordered filter of S if and only if*

$$(4.1) \qquad\qquad (\forall x, y \in S)\,(\mathrm{A}(x) \geq \mathrm{A}((x * y) * x)).$$

Lemma 4.7. [6] *Let A be a fuzzy ordered filter of S. Then the following are equivalent:*

(i) A *is a fuzzy implicative ordered filter of S.*
(ii) $(\forall x, y \in S)\,(\mathrm{A}(x * y) \geq \mathrm{A}(x * (x * y))).$
(iii) $(\forall x, y, z \in S)\,(\mathrm{A}((x * y) * (x * z)) \geq \mathrm{A}(x * (y * z))).$

Theorem 4.8. *Let A be a fuzzy set in a commutative implicative semigroup S. Then A is a fuzzy positive implicative ordered filter of S if and only if it is both a fuzzy implicative ordered filter and a fuzzy absorbent ordered filter of S.*

Proof. Suppose that A is a fuzzy positive implicative ordered filter of S. Then A is a fuzzy implicative ordered filter of S (see [9, Theorem 3.6]). Since $x \leq ((x*y)*y)*x$ for all $x, y \in S$, we have

$$(4.2) \qquad\qquad (\forall x, y \in S)\,((((x * y) * y) * x) * y \leq x * y).$$

Setting $b := ((x * y) * y) * x$,, by (4.2), (b5), (b9), and (b10) we obtain

$$
\begin{aligned}
(b * y) * b &= (((((x * y) * y) * x) * y) * ((((x * y) * y) * x)) \\
&\geq (x * y) * (((x * y) * y) * x) \\
&= ((x * y) * y) * ((x * y) * x) \\
&\geq y * x.
\end{aligned}
$$

Since A is order preserving, it follows from Lemma 4.6 that

$$\mathrm{A}(((x * y) * y) * x) = \mathrm{A}(b) \geq \mathrm{A}((b * y) * b) \geq \mathrm{A}(y * x).$$

Hence by Theorem 4.5, A is a fuzzy absorbent ordered filter of S. Conversely, assume that A is both a fuzzy implicative ordered filter and a fuzzy absorbent ordered filter of S. By (b5) and (b11), we have $(x * y) * x \leq (x * y) * ((x * y) * y)$ for all $x, y \in S$. It follows from Lemma 4.7 that

$$(4.3) \qquad (\forall x, y \in S)\,(\mathrm{A}((x * y) * y) \geq \mathrm{A}((x * y) * ((x * y) * y)) \geq \mathrm{A}((x * y) * x)).$$

On the other hand, since $(x * y) * x \leq y * x$ for all $x, y \in S$, it follows from Theorem 4.5 that

$$A(((x * y) * y) * x) \geq A(y * x) \geq A((x * y) * x).$$

Therefore, by (c2) and (4.3) we have

$$A(x) \geq \min\{A((x * y) * y), A(((x * y) * y) * x)\} \geq A((x * y) * x)$$

for all $x, y \in S$. Hence, by Lemma 4.6, A is a fuzzy positive implicative ordered filter of S. □

Theorem 4.9. (Extension property) *Let* A *and* B *be fuzzy ordered filters of a commutative implicative semigroup* S *such that* $A(1) = B(1)$ *and* $A \subset B$, ; *i.e.* $A(x) \leq B(x)$ *for all* $x \in S$. *If* A *is a fuzzy absorbent ordered filter of* S, *then so is* B.

Proof. Let $x, y \in S$. By Theorem 3.5, (b9), (b1) and the assumption of the theorem, we have

$$(4.4)\quad\begin{aligned}
&B(((((y * x) * x) * y) * y) * ((y * x) * x)) \\
&\geq A(((((y * x) * x) * y) * y) * ((y * x) * x)) \\
&\geq A(y * ((y * x) * x)) \\
&= A((y * x) * (y * x)) \\
&= A(1) = B(1).
\end{aligned}$$

By (3.4), (2.1), Proposition 2.10, (c1), (c2), (b9) and (4.4), we have

$$(4.5)\quad\begin{aligned}
&B(((x * y) * y) * x) \geq \min\{B(1), B(((((y * x) * x) * y) * y) * x)\} \\
&= B((((((y * x) * x) * y) * y) * x) \\
&\geq \min\{B((y * x) * (((((y * x) * x) * y) * y) * x)), B(y * x)\} \\
&= \min\{B(((((y * x) * x) * y) * y) * ((y * x) * x)), B(y * x)\} \\
&\geq \min\{B(1), B(y * x)\} \\
&= B(y * x).
\end{aligned}$$

Hence it follows from Theorem 4.5 that B is a fuzzy absorbent ordered filter of S. □

Theorem 4.10. *Let* A *be a fuzzy set in* S. *Then* A *is a fuzzy absorbent ordered filter of* S *if and only if every nonemptythe level set*

$$U(A; t) := \{x \in S \mid A(x) \geq t\}, \, t \in [0, 1]$$

is an absorbent ordered filter of S .

Proof. Note that A is a fuzzy ordered filter of S if and only if $U(A; t) \neq \emptyset$ is an ordered filter of S for all $t \in [0, 1]$ [11, Theorem 1]. Assume that A is a fuzzy absorbent ordered filter of S. Let $x, y \in S$ be such that $y * x \in U(A; t)$. Then $t \leq A(y * x) \leq A(((x * y) * y) * x)$, and so $((x * y) * y) * x \in U(A; t)$. Hence $U(A; t)$ is an absorbent ordered filter of S. Conversely, suppose that A is not a fuzzy absorbent ordered filter of S. Then we can take $a, b \in S$ such that $A(b * a) > A(((a * b) * b) * a)$. Putting

$$t_0 := \frac{1}{2}\Big(A(b * a) + A(((a * b) * b) * a)\Big),$$

we have $\mathrm{A}(((a * b) * b) * a) < t_0 < \mathrm{A}(b * a)$. It follows that $b * a \in U(\mathrm{A}; t_0)$, but $((a * b) * b) * a \notin U(\mathrm{A}; t_0)$. Thus $U(\mathrm{A}; t_0)$ is not an absorbent ordered filter. Hence A satisfies the inequality $\mathrm{A}(y * x) \leq \mathrm{A}(((x * y) * y) * x)$ for all $x, y \in S$, and A is a fuzzy absorbent ordered filter of S. $\qquad\square$

Theorem 4.11. *For $\emptyset \neq \Lambda \subset [0, 1]$, let $\{F_k \mid k \in \Lambda\}$, be a collection of absorbent ordered filters of S such that*

    (i) $S = \bigcup_{k \in \Lambda} F_k$,

    (ii) $(\forall k, r \in \Lambda)\ (k > r \Leftrightarrow F_k \subset F_r)$.

Then a fuzzy set A in S defined by

$$(4.6) \qquad\qquad \mathrm{A}(x) = \sup\{r \in \Lambda \mid x \in F_r\}$$

is a fuzzy absorbent ordered filter of S.

Proof. Let $k \in [0, 1]$ be such that $U(\mathrm{A}; k) \neq \emptyset$. Note that either

$$k = \sup\{r \in \Lambda \mid r < k\} = \sup\{r \in \Lambda \mid F_k \subset F_r\}$$

or

$$k \neq \sup\{r \in \Lambda \mid r < k\} = \sup\{r \in \Lambda \mid F_k \subset F_r\}.$$

In the first case, $U(\mathrm{A}; k) = \bigcap_{k > r} F_r$ which is an absorbent ordered filter, because

$$x \in U(\mathrm{A}; k) \Leftrightarrow x \in F_r \text{ for all } r < k \Leftrightarrow x \in \bigcap_{k > r} F_r.$$

In the second case, there exists $\varepsilon > 0$ such that $(k - \varepsilon, k) \cap \Lambda = \emptyset$. We prove that $U(\mathrm{A}; k) = \bigcup_{k \leq r} F_r$, which is an absorbent ordered filter. Indeed, if $x \in \bigcup_{k \leq r} F_r$ then $x \in F_r$ for some $r \geq k$. Hence $\mathrm{A}(x) \geq r \geq k$, and so $x \in U(\mathrm{A}; k)$. Now if $x \notin \bigcup_{k \leq r} F_r$, then $x \notin F_r$ for all $r \geq k$. Therefore $x \notin F_r$ for all $r > k - \varepsilon$, which shows that if $x \in F_r$ then $r \leq k - \varepsilon$. Thus $\mathrm{A}(x) \leq k - \varepsilon$, and so $x \notin U(\mathrm{A}; k)$. It follows from Theorem 4.10 that A is a fuzzy absorbent ordered filter of S. $\qquad\square$

Theorem 4.12. *Let A be a fuzzy set in S with $\mathrm{Im}(\mathrm{A}) = \{t_1, t_2, \cdots t_n\}$, where $t_i < t_j$ whenever $i > j$. Let $\{F_k \mid k = 1, 2, \cdots, n\}$ be a family of absorbent ordered filters of S such that*

    (i) $F_1 \subset F_2 \subset \cdots \subset F_n = S$

    (ii) $\mathrm{A}(\tilde{F}_k) = t_k$, *where* $\tilde{F}_k = F_k \setminus F_{k-1}$, $F_0 = \emptyset$ *for* $k = 1, 2, \cdots, n$.

Then A is a fuzzy absorbent ordered filter of S.

Proof. Since $U(\mathrm{A}; t_k) = F_k$ for all $k = 1, 2, \cdots, n$, the proof follows easily from Theorem 4.10. $\qquad\square$

Corollary 4.13. *Let A be a fuzzy set in S and let $\mathrm{Im}(\mathrm{A}) = \{t_1, t_2, \cdots, t_n\}$, where $t_1 > t_2 > \cdots > t_n$. If $F_1 \subset F_2 \subset \cdots \subset F_n = S$ are absorbent ordered filters of S such that $\mathrm{A}(F_k) \geq t_k$ for $k = 1, 2, \cdots, n$, then A is a fuzzy absorbent ordered filter of S.*

Theorem 4.14. *Let $\{F_k \mid k \in \mathbb{N}\}$ be a nested family of absorbent ordered filters of S; i.e. $F_1 \supset F_2 \supset \cdots$. Let A be a fuzzy set in S defined by*

$$\mathrm{A}(x) = \begin{cases} \dfrac{k}{k+1} & for \ x \in F_k \setminus F_{k+1}, \quad k = 0, 1, 2, \cdots, \\ 1 & for \ x \in \bigcap\limits_{k=0}^{\infty} F_k \end{cases}$$

for all $x \in S$, where F_0 stands for S. Then A is a fuzzy absorbent ordered filter of S.

Proof. Clearly $\mathrm{A}(1) \geq \mathrm{A}(x)$ for all $x \in S$. Let $x, y, z \in S$. Suppose that $z * (y * x) \in F_k \setminus F_{k+1}$ and $z \in F_r \setminus F_{r+1}$ for $k = 0, 1, 2, \cdots$; $r = 0, 1, 2, \cdots$. Without loss of generality, we may assume that $k \leq r$. Then, obviously, $z \in F_k$. Since F_k is a absorbent ordered filter, it follows from (F7) that $((x * y) * y) * x \in F_k$ so that

$$\mathrm{A}(((x * y) * y) * x) \geq \frac{k}{k+1} = \min\left\{\mathrm{A}(z * (y * x)), \mathrm{A}(z)\right\}.$$

If $z * (y * x) \in \bigcap\limits_{k=0}^{\infty} F_k$ and $z \in \bigcap\limits_{k=0}^{\infty} F_k$, then $((x * y) * y) * x \in \bigcap\limits_{k=0}^{\infty} F_k$. Hence

$$\mathrm{A}(((x * y) * y) * x) = 1 = \min\left\{\mathrm{A}(z * (y * x)), \mathrm{A}(z)\right\}.$$

If $z * (y * x) \notin \bigcap\limits_{k=0}^{\infty} F_k$ and $z \in \bigcap\limits_{k=0}^{\infty} F_k$, then there exists $i \in \mathbb{N}$ such that $z * (y * x) \in F_i \setminus F_{i+1}$. It follows that $((x * y) * y) * x \in F_i$ so that

$$\mathrm{A}(((x * y) * y) * x) \geq \frac{i}{i+1} = \min\left\{\mathrm{A}(z * (y * x)), \mathrm{A}(z)\right\}.$$

Finally, assume that $z * (y * x) \in \bigcap\limits_{k=0}^{\infty} F_k$ and $z \notin \bigcap\limits_{k=0}^{\infty} F_k$. Then $z \in F_j \setminus F_{j+1}$ for some $j \in \mathbb{N}$. Hence $((x * y) * y) * x \in F_j$, and thus

$$\mathrm{A}(((x * y) * y) * x) \geq \frac{j}{j+1} = \min\left\{\mathrm{A}(z * (y * x)), \mathrm{A}(z)\right\}.$$

Consequently, A is a fuzzy absorbent ordered filter of S. $\square$

Corollary 4.15. *If every fuzzy absorbent ordered filter A of S has a finite image, then every descending chain of absorbent ordered filters of S terminates after a finite number of steps.*

Theorem 4.16. *Assume that every descending chain $F_1 \supset F_2 \supset \cdots$ of absorbent ordered filters of S terminates after a finite number of steps; i.e. there exists $r \in \mathbb{N}$ such that $F_r = F_k$ for all $k \geq r$. Let A be a fuzzy absorbent ordered filter of S in which a sequence of elements of $\mathrm{Im}(\mathrm{A})$ is strictly increasing. Then A has a finite number of values.*

Proof. Assume that $\mathrm{Im}(\mathrm{A})$ is not finite. Let $\{t_k\}$ be a strictly increasing sequence of elements of $\mathrm{Im}(\mathrm{A})$, that is, $0 \leq t_1 < t_2 < \cdots \leq 1$. Then by Theorem 4.10, $U(\mathrm{A}; t_r)$ is an absorbent ordered filter of S for all $r \in \mathbb{N}$, and we get a strictly descending chain

$$U(\mathrm{A}; t_1) \supset U(\mathrm{A}; t_2) \supset U(\mathrm{A}; t_3) \supset \cdots$$

of absorbent ordered filters of S which is not terminating. This is a contradiction, and so A has finite number of values. $\qquad\square$

Theorem 4.17. *Every ascending chain of absorbent ordered filters of S terminates after a finite number of steps if and only if the set of values of any fuzzy absorbent ordered filter in S is a well-ordered subset of $[0, 1]$.*

Proof. Assume that the set of values of a fuzzy absorbent ordered filter A of S is not well ordered. Then there exists a strictly decreasing sequence $\{t_n\}$ such that $t_n = A(x_n)$ for some $x_n \in S$. But in this case the family $\{U(A; t_n)\}$ of level absorbent ordered filters form a strictly ascending chain, which is a contradiction. To prove the converse, suppose that there exists a strictly ascending chain $F_1 \subset F_2 \subset F_3 \subset \cdots$ of absorbent ordered filters of S. Then $F := \bigcup_{n \in \mathbb{N}} F_n$ is an absorbent ordered filter of S. Define a fuzzy set A in S by

$$A(x) = \begin{cases} 0 & \text{for } x \notin F, \\ \frac{1}{k} & \text{where } k = \min\{n \in \mathbb{N} \mid x \in F_n\}. \end{cases}$$

Since $1 \in F_n$ for all $n \in \mathbb{N}$, we have $A(1) = 1 \geq A(x)$ for all $x \in S$. For any $x, y, z \in S$, if $z * (y * x) \notin F$ or $z \notin F$ then

$$A\big(((x * y) * y) * x\big) \geq 0 = \min\{A(z * (y * x)), A(z)\}.$$

Now let $x, y, z \in S$ be such that $z * (y * x) \in F$ and $z \in F$. If $z * (y * x)$, $z \in F_n \setminus F_{n-1}$ for some $n \in \mathbb{N}$, then $((x * y) * y) * x \in F_n$. Hence

$$A\big(((x * y) * y) * x\big) \geq \frac{1}{n} = \min\{A(z * (y * x)), A(z)\}.$$

If $z * (y * x) \in F_n \setminus F_{n-1}$ and $z \notin F_n \setminus F_{n-1}$, then either $z \in F_{n-1} \subset F_n$ or $z \in F_m \setminus F_{m-1}$ for some $m > n$. In the first case, $((x * y) * y) * x \in F_n$ and so

$$A\big(((x * y) * y) * x\big) \geq \frac{1}{n} = A(z * (y * x)) \geq \min\{A(z * (y * x)), A(z)\}.$$

In the second case, we have $((x * y) * y) * x \in F_m$ and thus $A(z * (y * x)) = \frac{1}{n} > \frac{1}{m} = A(z)$. Therefore

$$A\big(((x * y) * y) * x\big) \geq \frac{1}{m} = \min\{A(z * (y * x)), A(z)\}.$$

Similarly, we get

$$A\big(((x * y) * y) * x\big) \geq \min\{A(z * (y * x)), A(z)\}$$

for $z * (y * x) \notin F_n \setminus F_{n-1}$ and $z \in F_n \setminus F_{n-1}$. This proves that A is a fuzzy absorbent ordered filter of S. Since the chain of absorbent ordered filters $F_1 \subset F_2 \subset F_3 \subset \cdots$ is not terminating, A has a strictly descending sequence of values. This contradicts that the value set of any fuzzy absorbent ordered filter is well ordered. This completes the proof. $\qquad\square$

Theorem 4.18. *Let A be a fuzzy absorbent ordered filter of S with $\text{Im}(A) = \{t_i \mid i \in \Lambda\}$ and let $\Omega := \{U(A; t) \mid t \in \text{Im}(A)\}$. Then*

(i) *there exists a unique $t_0 \in \text{Im}(A)$ such that $t_0 \geq t$ for all $t \in \text{Im}(A)$.*

  (ii) *S is the set-theoretic union of all $U(\mathrm{A}; t) \in \Omega$.*
  (iii) *the members of Ω form a chain.*
  (iv) *Ω contains all level absorbent ordered filters of A if and only if A attains its infimum on all absorbent ordered filters of S.*

Proof. (i) follows from the fact that $t_0 = \mathrm{A}(1) \geq \mathrm{A}(x)$ for all $x \in S$.
  (ii) If $x \in S$, then $\mathrm{A}(x) = t_x \in \mathrm{Im}(\mathrm{A})$. This implies that

$$x \in U(\mathrm{A}; t_x) \subset \bigcup_{t \in \mathrm{Im}(\mathrm{A})} U(\mathrm{A}; t) \subset S,$$

which proves (ii).
  (iii) Note that $U(\mathrm{A}; t_i) \subset U(\mathrm{A}; t_j) \Leftrightarrow t_i \geq t_j$ for $i, j \in \Lambda$. Hence Ω is totally ordered by inclusion.
  (iv) Suppose that Ω contains all level absorbent ordered filters of A. Let F be a absorbent ordered filter of S. If A is constant on F, then we are done. Now assume that A is not constant on F. We consider two cases: $F = S$ and $F \subset S$. If $F = S$, let $\beta = \inf \mathrm{Im}(\mathrm{A})$. Then $\beta \leq t \in \mathrm{Im}(\mathrm{A})$; i.e. $U(\mathrm{A}; \beta) \supseteq U(\mathrm{A}; t)$ for all $t \in \mathrm{Im}(\mathrm{A})$. But $U(\mathrm{A}; 0) = S \in \Omega$ because Ω contains all level absorbent ordered filters of A. Hence there exists $\alpha \in \mathrm{Im}(\mathrm{A})$ such that $U(\mathrm{A}; \alpha) = S$. It follows that $S = U(\mathrm{A}; \alpha) \subset U(\mathrm{A}; \beta)$ so that $U(\mathrm{A}; \beta) = U(\mathrm{A}; \alpha) = S$ because every level absorbent ordered filter of A is a absorbent ordered filter of S. Now it is sufficient to show that $\beta = \alpha$. If $\beta < \alpha$, then there exists $\gamma \in \mathrm{Im}(\mathrm{A})$ such that $\beta \leq \gamma < \alpha$. Thus $U(\mathrm{A}; \gamma) \supset U(\mathrm{A}; \alpha) = S$, a contradiction. Therefore $\beta = \alpha \in \mathrm{Im}(\mathrm{A})$. In the case $F \subset S$, we consider the fuzzy set A_F in S defined by

$$\mathrm{A}_F(x) = \begin{cases} \delta \neq 0 & \text{for } x \in F, \\ 0 & \text{for } x \in S \setminus F. \end{cases}$$

It is easily verified that A_F is a fuzzy absorbent ordered filter of S. Let

$$J := \{ i \in \Lambda \mid \mathrm{A}(y) = t_i \text{ for some } y \in F \}$$

and $\Omega_F := \{ U(\mathrm{A}_F; t_i) \mid i \in J \}$. Now Ω_F contains all level absorbent ordered filters. Hence there exists $x_0 \in F$ such that $\mathrm{A}(x_0) = \inf\{ \mathrm{A}_F(x) \mid x \in F \}$, which implies that $\mathrm{A}(x_0) = \{ \mathrm{A}(x) \mid x \in F \}$. This proves that A attains its infimum on all absorbent ordered filters of S. To prove the converse, let $U(\mathrm{A}; \alpha)$ be a level absorbent ordered filter of A. If $\alpha = t$ for some $t \in \mathrm{Im}(\mathrm{A})$, then clearly $U(\mathrm{A}; \alpha) \in \Omega$. If $\alpha \neq t$ for all $t \in \mathrm{Im}(\mathrm{A})$, then there does not exist $x \in S$ such that $\mathrm{A}(x) = \alpha$. Let $F = \{ x \in S \mid \mathrm{A}(x) > \alpha \}$. Obviously, $1 \in F$. Now let $x, y, z \in S$ be such that $z * (y * x) \in F$ and $z \in F$. Then $\mathrm{A}\big(z * (y * x)\big) > \alpha$ and $\mathrm{A}(z) > \alpha$. It follows from (c5) that

$$\mathrm{A}\Big(\big((x * y) * y\big) * x \Big) \geq \min\Big\{ \mathrm{A}\big(z * (y * x)\big), \mathrm{A}(z) \Big\} > \alpha$$

so that $((x*y)*y)*x \in F$. Hence F is a absorbent ordered filter of S. By hypothesis there exists $y \in F$ such that $\mathrm{A}(y) = \inf\{ \mathrm{A}(x) \mid x \in F \}$. But $\mathrm{A}(y) \in \mathrm{Im}(\mathrm{A})$ implies that $\mathrm{A}(y) = s$ for some $s \in \mathrm{Im}(\mathrm{A})$. Hence $\inf\{ \mathrm{A}(x) \mid x \in F \} = s > \alpha$. Note that there does not exist $z \in S$ such that $\alpha \leq \mathrm{A}(z) < s$. It follows that

$U(\mathrm{A}; \alpha) = U(\mathrm{A}; s)$, and so $U(\mathrm{A}; \alpha) \in \Omega$. Thus Ω contains all level absorbent ordered filters of A. $\qquad \square$

5. Normalization of Fuzzy Absorbent Ordered Filters

Definition 5.1. A fuzzy absorbent ordered filter A of S is said to be *normal* if there exists $a \in S$ such that $\mathrm{A}(a) = 1$.

We note that if A is a normal fuzzy absorbent ordered filter of S, then $\mathrm{A}(1) = 1$, hence a fuzzy absorbent ordered filter A of S is normal if and only if $\mathrm{A}(1) = 1$. Let $NFAO(S)$ denote the set of all normal fuzzy absorbent ordered filters of S.

Theorem 5.2. *Let* A *be a fuzzy absorbent ordered filter of* S *and let* A^+ *be a fuzzy set in* S *given by* $\mathrm{A}^+(x) = \mathrm{A}(x) + 1 - \mathrm{A}(1)$ *for all* $x \in S$. *Then* $\mathrm{A}^+ \in NFAO(S)$ *and* $\mathrm{A} \subset \mathrm{A}^+$.

Proof. For any $x, y, z \in S$ we have $\mathrm{A}^+(1) = \mathrm{A}(1) + 1 - \mathrm{A}(1) = 1 \geq \mathrm{A}^+(x)$, and

$$
\begin{aligned}
(5.1) \quad & \min\{\mathrm{A}^+(z * (y * x)), \mathrm{A}^+(z)\} \\
& = \min\{\mathrm{A}(z * (y * x)) + 1 - \mathrm{A}(1), \mathrm{A}(z) + 1 - \mathrm{A}(1)\} \\
& = \min\{\mathrm{A}(z * (y * x)), \mathrm{A}(z)\} + 1 - \mathrm{A}(1) \\
& \leq \mathrm{A}(((x * y) * y) * x) + 1 - \mathrm{A}(1) \\
& = \mathrm{A}^+(((x * y) * y) * x).
\end{aligned}
$$

Hence $\mathrm{A}^+ \in NFAO(S)$. Obviously, $\mathrm{A} \subset \mathrm{A}^+$. $\qquad \square$

Corollary 5.3. *If* A *is a fuzzy absorbent ordered filter of* S *satisfying* $\mathrm{A}^+(a) = 0$ *for some* $a \in S$, *then* $\mathrm{A}(a) = 0$.

It is clear that a fuzzy absorbent ordered filter A of S is normal if and only if $\mathrm{A}^+ = \mathrm{A}$, and for any fuzzy absorbent ordered filter A of S we have $(\mathrm{A}^+)^+ = \mathrm{A}^+$. Hence if A is a normal fuzzy absorbent ordered filter of S, then $(\mathrm{A}^+)^+ = \mathrm{A}$.

Theorem 5.4. *Let* A *be a fuzzy absorbent ordered filter of* S. *If there exists a fuzzy absorbent ordered filter* B *of* S *satisfying* $\mathrm{B}^+ \subset \mathrm{A}$, *then* $\mathrm{A} \in NFAO(S)$.

Proof. Suppose that there exists a fuzzy absorbent ordered filter B of S such that $\mathrm{B}^+ \subset \mathrm{A}$. Then $1 = \mathrm{B}^+(1) \leq \mathrm{A}(1)$, whence $\mathrm{A}(1) = 1$. Hence $\mathrm{A} \in NFAO(S)$. $\qquad \square$

Theorem 5.5. *Let* A *be a fuzzy absorbent ordered filter of* S *and let* $\phi : [0, \mathrm{A}(1)] \to [0, 1]$ *be an increasing function. Let* A_ϕ *be a fuzzy set in* S *defined by* $\mathrm{A}_\phi(x) = \phi(\mathrm{A}(x))$ *for all* $x \in S$. *Then* A_ϕ *is a fuzzy absorbent ordered filter of* S. *Moreover, if* $\phi(\mathrm{A}(1)) = 1$ *then* $\mathrm{A}_\phi \in NFAO(S)$; *and if* $\phi(t) \geq t$ *for all* $t \in [0, \mathrm{A}(1)]$, *then* $\mathrm{A} \subset \mathrm{A}_\phi$.

Proof. Since ϕ is increasing and $\mathrm{A}(1) \geq \mathrm{A}(x)$ for all $x \in S$, we have

$$\mathrm{A}_\phi(1) = \phi(\mathrm{A}(1)) \geq \phi(\mathrm{A}(x)) = \mathrm{A}_\phi(x), \ \forall x \in S.$$

Let $x, y, z \in S$. Then

$$
\begin{aligned}
\min\{\mathrm{A}_\phi(z*(y*x)), \mathrm{A}_\phi(z)\} &= \min\{\phi(\mathrm{A}(z*(y*x))), \phi(\mathrm{A}(z))\} \\
&= \phi(\min\{\mathrm{A}(z*(y*x)), \mathrm{A}(z)\}) \\
&\leq \phi(\mathrm{A}(((x*y)*y)*x)) \\
&= \mathrm{A}_\phi(((x*y)*y)*x).
\end{aligned}
$$

(5.2)

Hence A_ϕ is a fuzzy absorbent ordered filter of S. If $\phi(\mathrm{A}(1)) = 1$, then obviously A_ϕ is normal and so $\mathrm{A}_\phi \in NFAO(S)$. Assume that $\phi(t) \geq t$ for all $t \in [0, \mathrm{A}(1)]$. Then $\mathrm{A}_\phi(x) = \phi(\mathrm{A}(x)) \geq \mathrm{A}(x)$ for all $x \in S$, which proves that $\mathrm{A} \subset \mathrm{A}_\phi$. □

Theorem 5.6. *Let* $\mathrm{A} \in NFAO(S)$ *be a non-constant maximal element of the poset* $(NFAO(S), \subset)$. *Then* A *takes only the values* 0 *and* 1.

Proof. Note that $\mathrm{A}(1) = 1$. Let $x \in S$ be such that $\mathrm{A}(x) \neq 1$. It is sufficient to show that $\mathrm{A}(x) = 0$. If not, then there exists $a \in S$ such that $0 < \mathrm{A}(a) < 1$. Let B be a fuzzy set in S defined by $\mathrm{B}(x) = \frac{1}{2}(\mathrm{A}(x) + \mathrm{A}(a))$ for all $x \in S$. Then, clearly, B is well defined and we have

$$
\mathrm{B}(1) = \tfrac{1}{2}(\mathrm{A}(1) + \mathrm{A}(a)) \geq \tfrac{1}{2}(\mathrm{A}(x) + \mathrm{A}(a)) = \mathrm{B}(x), \forall x \in S.
$$

Let $x, y, z \in S$. Then

$$
\begin{aligned}
\mathrm{B}(((x*y)*y)*x) &= \tfrac{1}{2}(\mathrm{A}(((x*y)*y)*x) + \mathrm{A}(a)) \\
&\geq \tfrac{1}{2}(\min\{\mathrm{A}(z*(y*x)), \mathrm{A}(z)\} + \mathrm{A}(a)) \\
&= \min\{\tfrac{1}{2}(\mathrm{A}(z*(y*x)) + \mathrm{A}(a)), \tfrac{1}{2}(\mathrm{A}(z) + \mathrm{A}(a))\} \\
&= \min\{\mathrm{B}(z*(y*x)), \mathrm{B}(z)\}.
\end{aligned}
$$

(5.3)

Hence B is a fuzzy absorbent ordered filter of S, and so, by Theorem 5.2, $\mathrm{B}^+ \in NFAO(S)$. Now,

$$
\mathrm{B}^+(1) = 1 > \mathrm{B}^+(a) = \tfrac{1}{2}(\mathrm{A}(a) + 1) > \mathrm{A}(a).
$$

Hence B^+ is non-constant. From $\mathrm{B}^+(a) > \mathrm{A}(a)$ it follows that A is not maximal in $(NFAO(S), \subset)$, which is a contradiction. □

For an absorbent ordered filter F of S, if we define a fuzzy set A_F in S by

$$
\mathrm{A}_F(x) := \begin{cases} 1 & \text{for } x \in F, \\ 0 & \text{otherwise}, \end{cases}
$$

then $\mathrm{A}_F \in NFAO(S)$ and

$$
[1]_{\mathrm{A}_F} := \{x \in S \mid \mathrm{A}_F(x) = \mathrm{A}_F(1)\} = F.
$$

Let $AOF(S)$ denote the set of all absorbent ordered filters of S. For all $F \in AOF(S)$ and $\mathrm{A} \in NFAO(S)$, we define the mappings $\Phi : AOF(S) \to NFAO(S)$ and $\Psi : NFAO(S) \to AOF(S)$ by $\Phi(F) = \mathrm{A}_F$ and $\Psi(\mathrm{A}) = [1]_{\mathrm{A}}$. Then $\Psi\Phi = 1_{AOF(S)}$ and $\Phi\Psi(\mathrm{A}) = \Phi([1]_{\mathrm{A}}) = \mathrm{A}_{[1]_{\mathrm{A}}} \subset \mathrm{A}$.

Theorem 5.7. *If* $F, G \in AOF(S)$, *then* $\mathrm{A}_{F \cap G} = \mathrm{A}_F \cap \mathrm{A}_G$, *that is,* $\Phi(F \cap G) = \Phi(F) \cap \Phi(G)$. *If* $\mathrm{A}, \mathrm{B} \in NFAO(S)$, *then* $[1]_{\mathrm{A} \cap \mathrm{B}} = [1]_{\mathrm{A}} \cap [1]_{\mathrm{B}}$, *that is,* $\Psi(\mathrm{A} \cap \mathrm{B}) = \Psi(\mathrm{A}) \cap \Psi(\mathrm{B})$.

Proof. Let $x \in S$. If $x \in F \cap G$, then $\mathrm{A}_{F \cap G}(x) = 1$ and $\mathrm{A}_F(x) = 1 = \mathrm{A}_G(x)$. It follows that

$$\mathrm{A}_{F \cap G}(x) = 1 = \min\{\mathrm{A}_F(x), \mathrm{A}_G(x)\} = (\mathrm{A}_F \cap \mathrm{A}_G)(x).$$

If $x \notin F \cap G$, then $x \notin F$ or $x \notin G$, and thus

$$\mathrm{A}_{F \cap G}(x) = 0 = \min\{\mathrm{A}_F(x), \mathrm{A}_G(x)\} = (\mathrm{A}_F \cap \mathrm{A}_G)(x).$$

Therefore $\mathrm{A}_{F \cap G} = \mathrm{A}_F \cap \mathrm{A}_G$. Now let $\mathrm{A}, \mathrm{B} \in NFAO(S)$. Obviously $\mathrm{A} \cap \mathrm{B} \in NFAO(S)$ and so

$$
\begin{aligned}
\Psi(\mathrm{A} \cap \mathrm{B}) &= [1)_{\mathrm{A} \cap \mathrm{B}} \\
&= \big\{ x \in S \mid (\mathrm{A} \cap \mathrm{B})(x) = (\mathrm{A} \cap \mathrm{B})(1) \big\} \\
&= \big\{ x \in S \mid \min\{\mathrm{A}(x), \mathrm{B}(x)\} = 1 \big\} \\
&= \big\{ x \in S \mid \mathrm{A}(x) = 1 = \mathrm{B}(x) \big\} \\
&= \big\{ x \in S \mid \mathrm{A}(x) = 1 \big\} \cap \big\{ x \in S \mid \mathrm{B}(x) = 1 \big\} \\
&= \big\{ x \in S \mid \mathrm{A}(x) = \mathrm{A}(1) \big\} \cap \big\{ x \in S \mid \mathrm{B}(x) = \mathrm{B}(1) \big\} \\
&= [1)_{\mathrm{A}} \cap [1)_{\mathrm{B}} = \Psi(\mathrm{A}) \cap \Psi(\mathrm{B}).
\end{aligned}
$$

$\square$

Definition 5.8. A fuzzy absorbent ordered filter A of S is said to be *maximal* if

(c6) A is non-constant,
(c7) A^+ is a maximal element of $(NFAO(S), \subset)$.

Theorem 5.9. *If* A *is a maximal fuzzy absorbent ordered filter of S, then*

(i) A *is normal,*
(ii) A^+ *takes only the values 0 and 1,*
(iii) $\mathrm{A}_{[1)_{\mathrm{A}}} = \mathrm{A}$,
(iv) $[1)_{\mathrm{A}}$ *is a maximal absorbent ordered filter of S.*

Proof. Let A be a maximal fuzzy absorbent ordered filter of S. Then A^+ is a non-constant maximal element of the poset $(NFAO(S), \subset)$. It follows from Theorem 5.6 that A^+ takes only the values 0 and 1. Note that $\mathrm{A}^+(x) = 1$ if and only if $\mathrm{A}(x) = \mathrm{A}(1)$; and $\mathrm{A}^+(x) = 0$ if and only if $\mathrm{A}(x) = \mathrm{A}(1) - 1$. By Corollary 5.3, we get $\mathrm{A}(x) = 0$, that is, $\mathrm{A}(1) = 1$. Hence A is normal. This proves (i) and (ii).

(iii) Obviously $\mathrm{A}_{[1)_{\mathrm{A}}} \subset \mathrm{A}$ and $\mathrm{A}_{[1)_{\mathrm{A}}}$ takes only the values 0 and 1. Let $x \in S$. If $\mathrm{A}(x) = 0$, then clearly $\mathrm{A} \subset \mathrm{A}_{[1)_{\mathrm{A}}}$. If $\mathrm{A}(x) = 1$, then $x \in [1)_{\mathrm{A}}$ and so $\mathrm{A}_{[1)_{\mathrm{A}}}(x) = 1$. This shows that $\mathrm{A} \subset \mathrm{A}_{[1)_{\mathrm{A}}}$.

(iv) Since A is non-constant, $[1)_{\mathrm{A}}$ is a proper absorbent ordered filter of S. Let G be a absorbent ordered filter of S such that $[1)_{\mathrm{A}} \subset G$. Now, $\mathrm{A}_F \subset \mathrm{A}_G$ if and only if $F \subset G$ for every absorbent ordered filters F and G of S. Hence. by (iii), $\mathrm{A} = \mathrm{A}_{[1)_{\mathrm{A}}} \subset \mathrm{A}_G$. Since $\mathrm{A}, \mathrm{A}_G \in NFAO(S)$ and $\mathrm{A} = \mathrm{A}^+$ is a maximal element of $NFAO(S)$, it follows that either $\mathrm{A}_G = \mathrm{A}$ or $\mathrm{A}_G = \mathbf{1}$ where $\mathbf{1} : S \to [0, 1]$ is a fuzzy set defined by $\mathbf{1}(x) = 1$ for all $x \in S$. If $\mathrm{A}_G = \mathrm{A}$, then $[1)_{\mathrm{A}} = [1)_{\mathrm{A}_G} = G$. If $\mathrm{A}_G = \mathbf{1}$, then $G = S$. Hence $[1)_{\mathrm{A}}$ is a maximal absorbent ordered filter of S. $\square$

Definition 5.10. A fuzzy absorbent ordered filter A of S is said to be *complete* if it is normal and there exists $z \in S$ such that $A(z) = 0$.

Note that A_F is a complete fuzzy absorbent ordered filter of S for every absorbent ordered filter F of S.

Theorem 5.11. *Let* A *be a fuzzy absorbent ordered filter of S and let w be a fixed element of S such that $A(1) \neq A(w)$. Define a fuzzy set A^* in S by $A^*(x) = \frac{A(x) - A(w)}{A(1) - A(w)}$ for all $x \in S$. Then A^* is a complete fuzzy absorbent ordered filter of S.*

Proof. For any $x \in S$, we have

$$A^*(1) = \frac{A(1) - A(w)}{A(1) - A(w)} = 1 \geq A^*(x).$$

Let $x, y, z \in S$. Then

$$\min\{A^*\big(z * (y * x)\big), A^*(z)\}$$
$$= \min\left\{\frac{A(z*(y*x)) - A(w)}{A(1) - A(w)}, \frac{A(z) - A(w)}{A(1) - A(w)}\right\}$$
$$= \frac{1}{A(1) - A(w)}\Big(\min\big\{A(z * (y * x)) - A(w), A(z) - A(w)\big\}\Big)$$
$$= \frac{1}{A(1) - A(w)}\Big(\min\{A(z * (y * x)), A(z)\} - A(w)\Big)$$
$$\leq \frac{1}{A(1) - A(w)}\Big(A\big(((x * y) * y) * x\big) - A(w)\Big)$$
$$= A^*\big(((x * y) * y) * x\big).$$

Hence $A^* \in NFAO(S)$. Since $A^*(w) = 0$, we conclude that A^* is a complete fuzzy absorbent ordered filter of S. $\square$

Let $C(S)$ denote the set of all complete fuzzy absorbent ordered filters of S. Note that $C(S) \subset NFAO(S)$ and the restriction of the partial ordering "$\subset$" of $NFAO(S)$ gives a partial ordering of $C(S)$. Note that if $A \in C(S)$, then $A^* = A$.

Theorem 5.12. *Every non-constant maximal element of $(NFAO(S), \subset)$ is also a maximal element of $(C(S), \subset)$.*

Proof. Let A be a non-constant maximal element of $(NFAO(S), \subset)$. Then A takes only the values 0 and 1(Theorem 5.6) and, in fact, $A(1) = 1$ and $A(w) = 0$ for some $w(\neq 1) \in S$. Hence A is complete. Assume that there exists $B \in C(S)$ such that $A \subset B$. It follows that $A \subset B$ in $NFAO(S)$. Since A is maximal in $(NFAO(S), \subset)$ and B is non-constant, we have $A = B$. Thus A is a maximal element of $(C(S), \subset)$. $\square$

Theorem 5.13. *Every maximal fuzzy absorbent ordered filter of S is complete.*

Proof. Let A be a maximal fuzzy absorbent ordered filter of S. Then A is normal and $A = A^+$ takes only the values 0 and 1. Since A is non-constant and $A(1) = 1$, it is clear that there exists $w(\neq 1) \in S$ such that $A(w) = 0$. Hence A is complete. $\square$

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