Iranian Journal of Fuzzy Systems **Vol. 6, No. 2,** (2009) pp. 61-74

HYPERGROUPS AND GENERAL FUZZY AUTOMATA

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ABSTRACT. In this paper, we first define the notion of a complete general fuzzy automaton with threshold c and construct an H_ν- group, as well as commutative hypergroups, on the set of states of a complete general fuzzy automaton with threshold c. We then define invertible general fuzzy automata, discuss the notions of "homogeneity, "separation, "thresholdness connected, "thresholdness inner irreducible and "principal and strongly connected, as applied to them and use these concepts to construct a quasi-order hypergroup on an invertible general fuzzy automaton. Finally, we derive relationships between the properties of an invertible general fuzzy automaton and the induced hypergroup.

1. Introduction and Preliminaries

Zadeh [20] introduced the theory of fuzzy sets and, soon after, Wee [18] introduced the concept of fuzzy automata. Automata have a long history both in theory and application [2],[3] and are the prime examples of general computational systems over discrete spaces [9]. In the conventional spectrum of automata (i.e. deterministic finite-state automata, non-deterministic finite-state automata, probabilistic automata and fuzzy finite-state automata), deterministic finite-state automata have found the most application in different areas [4], [11], [12], [16]. Fuzzy automata not only provide a systematic approach for handling uncertainty in such systems, but are can also be used in continuous spaces [17]. Moreover, they are able to create capabilities which are not easily achievable by other mathematical tools [19].

A fuzzy finite-state automaton (FFA) is a six-tuple $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the intitial state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the fuzzy transition function which is maps the current state into the next state and $\omega : Q \to Z$ is the output function. Associated with each fuzzy transition there is a membership value in $[0, 1]$ called the weight of the transition. The transition from the state q_i, to the state q_j, upon the input a_k, is denoted by $\delta(q_i, a_k, q_j)$. We use this notation to refer both to a transition and its weight. In other words, whenever $\delta(q_i, a_k, q_j)$ is a value, it refers to the weight of the transition. The set of all transitions of $\tilde{F}$ is denoted by Δ. The above definition is generally accepted as a formal definition of FFA [13], [14], [15].

Received: December 2007; Revised: July 2008; Accepted: October 2008

Key words and phrases: (General) Fuzzy automata, (Quasi-order) Hypergroup, Invertibility, Connectedness.

The question of assignment of membership values to the "next states is an important problem which should be clarified in the definition of FFA. When assigning membership values to states, there are two issues which must be dealt with: the assignment of a membership value to a state upon the completion of a transition and the cases where a state is forced to take several membership values simultaneously via overlapping transitions. In 2004, M. Doostfatemeh and S. C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [8]. We followed their line of study in [21]. In this paper, we introduce several new concepts and derive related results.

Definition 1.1. [13] Let Σ be a set. A word of Σ is the product of a finite sequence of elements in Σ, Λ is the empty word and Σ^* is the set of all words on Σ. In fact, Σ^* is the free monoid on Σ. The length $\ell(x)$ of the word $x \in \sum^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X, $\tilde{P}(X)$ will denote the set of all fuzzy subsets on X.

Definition 1.2. [8] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where
(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy starting states, $\tilde{R} \subset \tilde{P}(Q)$,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \to Z$ is the output function,
(vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$ is the augmented transition function,
(vii) $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is the membership assignment function,
(viii) $F_2 : [0, 1]^* \to [0, 1]$ is the multi-membership resolution function.

We note that the function $F_1(\mu, \delta)$, has two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the transition from state q_i to q_j on input a_k may be represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

In other words, membership value (mv) of the state q_j at time $t + 1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

The function $F_1(\mu, \delta)$ may have several forms, the most common being $\max\{\mu, \delta\}$ and $\min\{\mu, \delta\}$, $(\mu + \delta)/2$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

For all $i \geq 0$, let $Q_{act}(t_i)$ be the set of all active states at time t_i. We have $Q_{act}(t_0) = \tilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

$Q_{act}(t_i)$ is a fuzzy set. Hence, if a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we have:

$$q \in Domain(Q_{act}(t_i)) \text{ and } T \subset Domain(Q_{act}(t_i)).$$

We briefly write: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm 1.3. [8] (Multi-membership resolution) The following algorithm assigns a unified membership value to the active state q_j if there are several simultaneous transitions to it at time $t+1$.
(1) Process each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, by the membership assignment function F_1, and hence produce a membership value

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

These membership values are not necessarily equal.
(2) Process these values using the function F_2. The result will be assigned as the instantaneous membership value of the active state q_j. We have

$$\mu^{t+1}(q_j) = \underset{i=1}{\overset{n}{F_2}}[v_i] = \underset{i=1}{\overset{n}{F_2}}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

Where
- n: is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$: is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$: is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t+1$.

Definition 1.4. [21] Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton, as defined above. Then a max-min general fuzzy automaton is defined as

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

such that :

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0, 1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and let for every i, $i \geq 0$

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & otherwise \end{cases}$$

and for every i, $i \geq 1$
$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$,
$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$,
and recursively
$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots$
$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1})\}$,
in which $\forall 1 \leq i \leq n$, $u_i \in \Sigma$, and $\forall 1 \leq i \leq n-1$, the input at time t_i is u_i.

Definition 1.5. [21] Let $\tilde{F}^*$ be a max-min general fuzzy automaton. The response function $r^{\tilde{F}^*}: \Sigma^* \times Q \to [0,1]$ of $\tilde{F}^*$ is defined by

$$r^{\tilde{F}^*}(x,q) = \bigvee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, q)$$

for any $x \in \Sigma^*$, $q \in Q$.

Definition 1.6. [21] Let $q \in Q$, $0 \leq c < 1$. Then q is called an accessible state of $\tilde{F}^*$ with threshold c if there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x,q) > c$.

Definition 1.7. [21] Let $A \subseteq Q$ and $0 \leq c < 1$. Then $\tilde{F}^*$ is said to be connected on A, with threshold c, if $A = \overline{Q_c}$, where $\overline{Q}_c$ is the set of all accessible states with threshold c.

Definition 1.8. [5] A nonempty set H endowed with a hyperoperation $\circ : H^2 \to P^*(H)$ is called a hypergroupoid, where $P^*(H)$ is the set of all nonempty subsets of H. The image of the pair $(a,b) \in H^2$ is denoted by $a \circ b$ and called the hyperproduct of a and b. If A and B are nonempty subsets of H, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$.

Definition 1.9. [5] The hypergroupoid $\langle H, \circ \rangle$ is called a semihypergroup if the hyperoperation "$\circ$ is associative. A semihypergroup $\langle H, \circ \rangle$ is called a hypergroup if

$$H \circ a = a \circ H = H, \quad \forall a \in H.$$

Definition 1.10. [5] Let $\langle H, \circ \rangle$ be a hypergroup, K a nonempty subset of H. Then $\langle K, \circ \rangle$ is called a subhypergroup if

$$K \circ a = a \circ K = K, \quad \forall a \in K.$$

Definition 1.11. [5] Let $\langle H, \circ \rangle$ and $\langle K, * \rangle$ be hypergroupoids and $f : H \to K$ be a function. We say that
(i) f is a homomorphism if for all $(a,b) \in H^2$, $f(a \circ b) \subset f(a) * f(b)$;
(ii) f is a good homomorphism if for all $(a,b) \in H^2$, $f(a \circ b) = f(a) * f(b)$.

Definition 1.12. [5] Let $\langle H, \circ \rangle$ be a hypergroupoid and let R be an equivalence relation on H. We say that R is regular to the right if the following implication holds:

$$aRb \Rightarrow \forall u \in H, a \circ u \overline{\overline{R}} b \circ u$$

$$\text{(i.e. } \forall x \in a \circ u, \exists y \in b \circ u : xRy \;\text{ and }\; \forall y \in b \circ u, \exists x \in a \circ u : xRy.$$

Regularity to the left is defined similarly. We say that R is regular if it is regular both to the right and to the left.

Theorem 1.13. [5] Let H be a semihypergroup and R be an equivalence on H.
(i) If R is regular, a semihypergroup structure may be defined on H/R with respect to the hyperproduct $\overline{x} \otimes \overline{y} = \{\overline{z} : z \in x \circ y\}$.
(ii) If H is a hypergroup and the canonical projection $\prod : H \to H/R$ is a good

epimorphism, then $\langle H/R, \otimes\rangle$ is a hypergroup.

Definition 1.14. [6] Let $\langle H, \circ\rangle$ be a hypergroupoid. We say that H is a quasi-order hypergroup if

$$\forall (a,b) \in H^2, a \in a^3 \subseteq a^2 \quad \text{and} \quad a \circ b = a^2 \cup b^2.$$

Definition 1.15. [6] The hypergroupoid $\langle H, \circ\rangle$ is called an H_ν-group if
(i) weak associativity is satisfied:

$$x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset, \forall (x,y,z) \in H^3$$

and
(ii) the reproductive axiom holds:

$$H \circ x = x \circ H = H, \quad \forall x \in H.$$

2. Hypergroups and General Fuzzy Automata

Definition 2.1. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and $0 \leq c < 1$. We say that $\tilde{F}^*$ is complete with threshold c if, for a fixed element q of Q and for all x in $\Sigma^* \backslash \{\Lambda\}$, there exists at most one fixed element $p \in Q$ such that

$$\tilde{\delta}^*((q, \mu^{t_q}(q)), x, p) > c$$

where $\mu^{t_q}(q)$ is the membership value of the state q at time t_q.
Example 2.2. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, where
$Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b\}$, $Q_{act}(t_0) = \tilde{R} = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}$,
$F_1(\mu, \delta) = Min(\mu, \delta)$, $Z = \emptyset$, ω and F_2 are not applicable, $\delta(q_0, a, q_1) = 0.4$, $\delta(q_0, b, q_2) = 0.5$, $\delta(q_1, a, q_2) = 0.3$, $\delta(q_2, a, q_2) = 0.2$.

If we choose the input string $x = aa \ldots a$, then

$$Q_{act}(t_1) = \{(q_1, \mu^{t_1}(q_1))\}, \qquad Q_{act}(t_i) = \{(q_2, \mu^{t_i}(q_2))\}, \forall i \geq 2,$$
$$\mu^{t_1}(q_1) = \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = F_1(1, 0.4) = 0.4,$$
$$\mu^{t_2}(q_2) = \tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = F_1(0.4, 0.3) = 0.3,$$
$$\mu^{t_3}(q_2) = \tilde{\delta}((q_2, \mu^{t_2}(q_2)), a, q_2) = F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_2)) = F_1(0.3, 0.2) = 0.2,$$
$$\mu^{t_4}(q_2) = \tilde{\delta}((q_2, \mu^{t_3}(q_2)), a, q_2) = F_1(\mu^{t_3}(q_2), \delta(q_2, a, q_2)) = F_1(0.2, 0.2) = 0.2,$$
$$\mu^{t_i}(q_2) = 0.2, \forall i \geq 5,$$
$$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a, q_1) = 0.4,$$
$$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q_2) = 0.4 \wedge 0.3 = 0.3,$$
$$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aaa, q_2) = 0.4 \wedge 0.3 \wedge 0.2 = 0.2,$$
$$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a^n, q_2) = 0.2, \forall n \geq 4,$$
$$\tilde{\delta}^*((q_1, \mu^{t_1}(q_1)), a, q_2) = 0.3,$$
$$\tilde{\delta}^*((q_2, \mu^{t_2}(q_2)), a, q_2) = 0.2,$$
$$\tilde{\delta}^*((q_2, \mu^{t_3}(q_2)), a, q_2) = 0.2, \ldots .$$

If we choose the input string $x = baa\dots a$, then

$$\begin{aligned}
&Q_{act}(t_i) = \{(q_2, \mu^{t_i}(q_2))\}, \forall i \geq 1,\\
&\mu^{t_1}(q_2) = \tilde{\delta}((q_0, \mu^{t_0}(q_0)), b, q_2) = F_1(\mu^{t_0}(q_0), \delta(q_0, b, q_2)) = F_1(1, 0.5) = 0.5,\\
&\mu^{t_2}(q_2) = \tilde{\delta}((q_2, \mu^{t_1}(q_2)), a, q_2) = F_1(\mu^{t_1}(q_2), \delta(q_2, a, q_2)) = F_1(0.5, 0.2) = 0.2,\\
&\mu^{t_3}(q_2) = \tilde{\delta}((q_2, \mu^{t_2}(q_2)), a, q_2) = F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_2)) = F_1(0.2, 0.2) = 0.2,\\
&\mu^{t_i}(q_2) = 0.2, \forall i \geq 4,\\
&\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), b, q_2) = 0.5,\\
&\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), ba, q_2) = 0.5 \wedge 0.2 = 0.2,\\
&\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), baa, q_2) = 0.5 \wedge 0.2 \wedge 0.2 = 0.2,\\
&\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), ba^n, q_2) = 0.2, \forall n \geq 3,\\
&\tilde{\delta}^*((q_2, \mu^{t_1}(q_2)), a, q_2) = 0.2,\\
&\tilde{\delta}^*((q_2, \mu^{t_2}(q_2)), a, q_2) = 0.2,\\
&\tilde{\delta}^*((q_2, \mu^{t_3}(q_2)), a, q_2) = 0.2, \dots\ .
\end{aligned}$$

So $\tilde{F}^*$ is complete with threshold c, for all c and $0.4 \leq c < 1$.

Theorem 2.3. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a complete general fuzzy automaton with threshold c such that $0 \leq c < 1$. For all $x \in \Sigma^*$, define a hyperoperation on Q as follows:

$$p \underset{x}{o} q = \begin{cases} \{p_1, q_1\}, & \text{if } \tilde{\delta}^*((p, \mu^{t_p}(p)), x, p_1) > c, \tilde{\delta}^*((q, \mu^{t_q}(q)), x, q_1) > c,\\ \{p_1\}, & \text{if } \tilde{\delta}^*((p, \mu^{t_p}(p)), x, p_1) > c, \text{ and } \tilde{\delta}^*((q, \mu^{t_q}(q)), x, q_1) \leq c \text{ or} \\ & \qquad \text{does not exist}\\ \{q_1\}, & \text{if } \tilde{\delta}^*((q, \mu^{t_q}(q)), x, q_1) > c, \text{ and } \tilde{\delta}^*((p, \mu^{t_p}(p)), x, p_1) \leq c \text{ or} \\ & \qquad \text{does not exist}\\ \emptyset, & \quad \text{otherwise.} \end{cases}$$

Now let

$$p \circ q = (p \underset{x}{\circ} q) \cup (p \underset{\Lambda}{\circ} q), \forall x \in \Sigma^* \backslash \{\Lambda\}.$$

Then $\langle Q, \circ \rangle$ is a commutative H_ν-group.

Proof. We first show that the hyperoperation " $\circ$ is weakly associative.
By Definition 1.4, we have

$$\tilde{\delta}^*((p, \mu^{t_p}(p)), \Lambda, p) = 1 > c \ \text{ and } \ \tilde{\delta}^*((q, \mu^{t_q}(q)), \Lambda, q) = 1 > c\ ,$$

then $p \underset{\Lambda}{\circ} q = \{p, q\}, \forall p, q \in Q$. So for all x in $\Sigma^* \backslash \{\Lambda\}$, we have

$$\begin{aligned}(p \circ q) \circ r &= [(p \underset{x}{\circ} q) \cup (p \underset{\Lambda}{\circ} q)] or = [(p \underset{x}{\circ} q) \circ r] \cup [(p \underset{\Lambda}{\circ} q) or] \\ &= [\bigcup_{t \in p \underset{x}{\circ} q} (t \circ r)] \cup [\bigcup_{s \in p \underset{\Lambda}{\circ} q} (s \circ r)] \\ &\supseteq (p \circ r) \cup (q \circ r) \\ &\supseteq \{p, q, r\}.\end{aligned}$$

Similarly,

$$p \circ (q \circ r) \supseteq \{p, q, r\}.$$

Thus $p \circ (q \circ r) \cap (p \circ q) \circ r \neq \emptyset$, $\forall (p, q, r) \in Q^3$.

Therefore the hyperoperation " $\circ$ is weakly associative. We claim that

$$Q \circ q = q \circ Q = Q, \forall q \in Q.$$

It is clear that $Q \circ q \subseteq Q$. For the reverse inclusion, let $p \in Q$. Since $(p \underset{\Lambda}{\circ} q) = \{p, q\}$, hence $p \in p \underset{\Lambda}{\circ} q \subseteq p \circ q \subseteq Q \circ q$. Therefore $Q \subseteq Q \circ q$. □

Example 2.4. In Example 2.2, suppose that "$\circ$" is the hyperoperation introduced in Theorem 2.3 and $c = 0.4$. Then for all x in $\Sigma^* \backslash \{\Lambda\}$,
$q_0 \underset{x}{\circ} q_0 = \{q_2\}$, $q_0 \underset{x}{\circ} q_1 = q_1 \underset{x}{\circ} q_0 = \{q_2\}$, $q_0 \underset{x}{\circ} q_2 = q_2 \underset{x}{\circ} q_0 = \{q_2\}$, $q_1 \underset{x}{\circ} q_2 = q_2 \underset{x}{\circ} q_1 = \emptyset$, $q_1 \underset{x}{\circ} q_1 = \emptyset$ and $q_2 \underset{x}{\circ} q_2 = \emptyset$. So we have

$\circ$	q_0	q_1	q_2
q_0	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_2\}$
q_1	$\{q_0, q_1, q_2\}$	$\{q_1\}$	$\{q_1, q_2\}$
q_2	$\{q_0, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$

Theorem 2.5. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q, $\overline{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$ and $p \in Q$. We define a hyperoperation on Q as follows:

$$p \circ p = \{r \in Q : \overline{D}(\lambda)(p) \geq \overline{D}(\lambda)(r)\},$$

and

$$p \circ q = (p \circ p) \cup (q \circ q), \text{ where } p \neq q.$$

Then $\langle Q, \circ \rangle$ is a commutative hypergroup.

Proof. We first show that the hyperoperation "$\circ$ is associative. We have

$$\begin{aligned}(p\circ q)\circ s &= [\{r_1\in Q:\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(r_1)\}\cup\{r_2\in Q:\overline{D}(\lambda)(q)\geq\overline{D}(\lambda)(r_2)\}]\circ s\\ &= \{r_1'\in Q:\overline{D}(\lambda)(r_1)\geq\overline{D}(\lambda)(r_1'),\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(r_1)\}\\ &\cup \{r_2'\in Q:\overline{D}(\lambda)(r_2)\geq\overline{D}(\lambda)(r_2'),\overline{D}(\lambda)(q)\geq\overline{D}(\lambda)(r_2)\}\\ &\cup \{r_3\in Q:\overline{D}(\lambda)(s)\geq\overline{D}(\lambda)(r_3)\}\subseteq\{r_1'\in Q:\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(r_1')\}\\ &\cup \{r_2'\in Q:\overline{D}(\lambda)(q)\geq\overline{D}(\lambda)(r_2')\}\\ &\cup \{r_3\in Q:\overline{D}(\lambda)(s)\geq\overline{D}(\lambda)(r_3)\}.\end{aligned}$$

Let

$$\begin{aligned}T=\{r_1\in Q:\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(r_1)\} &\cup \{r_2\in Q:\overline{D}(\lambda)(q)\geq\overline{D}(\lambda)(r_2)\}\\ &\cup \{r_3\in Q:\overline{D}(\lambda)(s)\geq\overline{D}(\lambda)(r_3)\}.\end{aligned}$$

Thus $(p\circ q)\circ s\subseteq T$. Now, let $r\in\{r_1\in Q:\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(r_1)\}$. Then $\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(r)$. Hence $r\in p\circ p\subseteq(p\circ q)\circ s$. So $(p\circ q)\circ s\supseteq T$. Therefore

$$(p\circ q)\circ s=T.$$

Similarly,

$$p\circ(q\circ s)=T.$$

Therefore the hyperoperation "$\circ$ is associative. We claim that

$$Q\circ q=q\circ Q=Q,\quad\forall q\in Q.$$

It is clear that $Q\circ q\subseteq Q$. For the reverse inclusion, let $p\in Q$. Since $\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(p)$, hence

$$p\in\{r_1\in Q:\overline{D}(\lambda)(p)\geq\overline{D}(\lambda)(r_1)\}\cup\{r_2\in Q:\overline{D}(\lambda)(q)\geq\overline{D}(\lambda)(r_2)\}=p\circ q.$$

Thus $p\in Q\circ q$. So $Q\subseteq Q\circ q$. □

Theorem 2.6. Let $\tilde{F}^*=(Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2)$ be a max-min general fuzzy automaton and

$$B=\bigcup_{i\geq0}Q_{act}(t_i).$$

Also let $(p,\mu^{t_i}(p))\in Q_{act}(t_i)$ and $(q,\mu^{t_j}(q))\in Q_{act}(t_j)$. Define a hyperoperation on B as follows:

$$(p,\mu^{t_i}(p))\circ(q,\mu^{t_j}(q))=Q_{act}(t_i)\cup Q_{act}(t_j)$$

Then $\langle B,\circ\rangle$ is a commutative hypergroup.

Proof. We first show that the hyperoperation "$\circ$ is associative.
Let $(p,\mu^{t_i}(p))\in Q_{act}(t_i),(q,\mu^{t_j}(q))\in Q_{act}(t_j)$ and $(s,\mu^{t_k}(s))\in Q_{act}(t_k)$. We have

$$\begin{aligned}[(p,\mu^{t_i}(p))\circ(q,\mu^{t_j}(q))]\circ(s,\mu^{t_k}(s)) &= [Q_{act}(t_i)\cup Q_{act}(t_j)]\circ(s,\mu^{t_k}(s))\\ &= Q_{act}(t_i)\cup Q_{act}(t_j)\cup Q_{act}(t_k).\end{aligned}$$

On the other hand,

$$\begin{aligned}(p,\mu^{t_i}(p))\circ[(q,\mu^{t_j}(q))\circ(s,\mu^{t_k}(s))] &= (p,\mu^{t_i}(p))\circ[Q_{act}(t_j)\cup Q_{act}(t_k)]\\ &= Q_{act}(t_i)\cup Q_{act}(t_j)\cup Q_{act}(t_k).\end{aligned}$$

Therefore the hyperoperation "$\circ$ is associative. We claim that

$$B \circ (q, \mu^{t_j}(q)) = (q, \mu^{t_j}(q)) \circ B = B, \quad \forall (q, \mu^{t_j}(q)) \in B.$$

It is clear that $B \circ (q, \mu^{t_j}(q)) \subseteq B$. For the reverse inclusion, let $(p, \mu^{t_i}(p)) \in B$. Then we have

$$(p, \mu^{t_i}(p)) \in (p, \mu^{t_i}(p)) \circ (q, \mu^{t_j}(q)) = Q_{act}(t_i) \cup Q_{act}(t_j) \subseteq B \circ (q, \mu^{t_j}(q)).$$

Therefore $B \subseteq B \circ (q, \mu^{t_j}(q))$. □

Theorem 2.7. In Theorem 2.6, let $(p, \mu^{t_i}(p))R(q, \mu^{t_j}(q)) \Longleftrightarrow t_i = t_j$. Then the equivalence relation R on B is regular.

Proof. It is easy to see that R is an equivalence relation. Now, let $(s, \mu^{t_k}(s)) \in B$ and $(p, \mu^{t_i}(p))R(q, \mu^{t_j}(q))$. Then it is clear that

$$((p, \mu^{t_i}(p)) \circ (s, \mu^{t_k}(s)))\overline{R}((q, \mu^{t_j}(q)) \circ (s, \mu^{t_k}(s))).$$

Thus R is regular on B. □

Theorem 2.8. $\langle B/R, \otimes \rangle$ is a hypergroup where

$$\overline{(p, \mu^{t_i}(p))} \otimes \overline{(q, \mu^{t_j}(q))} = \{\overline{(r, \mu^{t_k}(r))} : (r, \mu^{t_k}(r)) \in (p, \mu^{t_i}(p)) \circ (q, \mu^{t_j}(q))\}.$$

Proof. By Theorem 1.13, it is sufficient to show that $\prod : B \to B/R$ is a good epimorphism. We have

$$\begin{aligned} \prod((p, \mu^{t_i}(p))) \otimes \prod((q, \mu^{t_j}(q))) &= \{\overline{(r, \mu^{t_k}(r))} : (r, \mu^{t_k}(r)) \in (p, \mu^{t_i}(p)) \circ (q, \mu^{t_j}(q))\} \\ &= \{\overline{(r, \mu^{t_k}(r))} : (r, \mu^{t_k}(r)) \in Q_{act}(t_i) \cup Q_{act}(t_j)\} \\ &= \prod((p, \mu^{t_i}(p)) \circ (q, \mu^{t_j}(q))). \end{aligned}$$

Therefore $\prod$ is a good epimorphism. □

3. Quasi-Order Hypergroups and Invertible General Fuzzy Automata

Definition 3.1. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton. We say that $\tilde{F}^*$ is invertible, if there exist t_i and t_j such that

$$\tilde{\delta}((q, \mu^{t_i}(q)), a, q_1) = \tilde{\delta}((q, \mu^{t_j}(q)), a, q_2),$$

then $q_1 = q_2$, $\forall q \in Q$, $a \in \Sigma$.

In an invertible general fuzzy automaton, let

$$A = (\bigcup_{i \geq 0} \{w_{1i}, w_{2i}, \ldots, w_{ni} \text{ and products of } w_{ji}s : w_{ji} \text{ is the membership value of an active state at time } t_i, \forall 1 \leq j \leq n,\ n_i = |Q_{act}(t_i)|\}) \cup \{1\}.$$

We now define the partial map $\tilde{\delta}_1 : (Q \times [0,1]) \times \Sigma \times A \to Q$ by

$$\tilde{\delta}_1((q, \mu^t(q)), a, w) = q' \ \text{ if } \ \tilde{\delta}((q, \mu^t(q)), a, q') = w = \mu^{t+1}(q') \tag{1}$$

and consider the invertible general fuzzy automaton of the form :

$$\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$$

such that $\tilde{\delta}_* : (Q \times [0,1]) \times \Sigma^* \times A \to Q$ is a partial map define by :
$\forall q \in Q_{act}(t)$, $a \in \Sigma$, $w \in A$ and $w = \tilde{\delta}((q, \mu^t(q)), a, q') = \mu^{t+1}(q')$,
$\tilde{\delta}_*((q, \mu^t(q)), a, w) = \tilde{\delta}_1((q, \mu^t(q)), a, w) = q'$,
$\forall q \in Q_{act}(t), \tilde{\delta}_*((q, \mu^t(q)), \Lambda, 1) = q$,
$\forall q_{i-1} \in Q_{act}(t_{i-1})$, $x_i \in \Sigma$, $w_i \in A$, $\tilde{\delta}((q_{i-1}, \mu^{t_{i-1}}(q_{i-1})), x_i, q_i) = w_i$ and w_i satisfies in (1), $\forall 1 \leq i \leq n$ and

$$\begin{aligned}
&\tilde{\delta}_*((q_0, \mu^{t_0}(q_0)), x_1 x_2 \dots x_n, w_1 \cdot w_2 \cdot \dots \cdot w_n) \\
&= \tilde{\delta}_*((\tilde{\delta}_1((q_0, \mu^{t_0}(q_0)), x_1, w_1), \mu^{t_1}(\tilde{\delta}_1((q, \mu^{t_0}(q_0)), x_1, w_1))), x_2 \dots x_n, w_2 \cdot \dots \cdot w_n) \\
&= \tilde{\delta}_*((q_1, \mu^{t_1}(q_1)), x_2 \dots x_n, w_2 \cdot \dots \cdot w_n) = \dots \\
&= \tilde{\delta}_1((q_{n-1}, \mu^{t_{n-1}}(q_{n-1})), x_n, w_n) = q_n.
\end{aligned}$$

If $Q' \subseteq Q$, write :

$\tilde{\delta}_*((Q', \mu(Q'), \Sigma^*, A) = \{\tilde{\delta}_*((q, \mu^t(q)), x, w) : q \in Q', x \in \Sigma^*, w \in A$, the factors of w satisfy in (1) and $\ell(x)$ equal to the number of factors of w $\}$.

Also, for simplicity, we write $\tilde{\delta}_*((q, \mu(q)), \Sigma^*, A)$ instead of $\tilde{\delta}_*((\{q\}, \mu(\{q\})), \Sigma^*, A)$.

Definition 3.2. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton, $0 \leq c < 1$ and $\overline{Q}_c$ be the set of all accessible states with threshold c of $\tilde{F}^*$. We say that $\overline{Q}_c$ is homogeneous if

$$\tilde{\delta}_*((\overline{Q}_c, \mu(\overline{Q}_c)), \Sigma^*, A) \subseteq \overline{Q}_c.$$

Example 3.3. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be the max-min general fuzzy automaton in Example 2.2. Since $\tilde{\delta}((q, \mu^{t_i}(q)), a, q') = \tilde{\delta}((q, \mu^{t_j}(q)), a, q'')$ implies that $q' = q''$, hence $\tilde{F}^*$ is invertible. We have

$$\begin{aligned}
\tilde{\delta}^*((q_2, \mu^{t_3}(q_2)), a^2, q_2) &= \bigvee_{q \in Q_{act}(t_4)} [\tilde{\delta}((q_2, \mu^{t_3}(q_2)), a, q) \wedge \tilde{\delta}((q, \mu^{t_4}(q)), a, q_2)] \\
&= \tilde{\delta}((q_2, \mu^{t_3}(q_2)), a, q_2) \wedge \tilde{\delta}((q_2, \mu^{t_4}(q_2)), a, q_2) \\
&= 0.2 \wedge 0.2 = 0.2,
\end{aligned}$$

$\tilde{\delta}^*((q_2, \mu^{t_i}(q_2)), a^n, q_2) = 0.2, \ \forall i \geq 3, \forall n \geq 1,$

$r^{\tilde{F}^*}(a, q_1) = \bigvee_{q \in Q_{act}(t_0)} \tilde{\delta}^*((q, \mu^{t_0}(q)), a, q_1) = \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1) = 0.4,$

$r^{\tilde{F}^*}(b, q_2) = \bigvee_{q \in Q_{act}(t_0)} \tilde{\delta}^*((q, \mu^{t_0}(q)), b, q_2) = \tilde{\delta}((q_0, \mu^{t_0}(q_0)), b, q_2) = 0.5.$

Thus if $c = 0.4$, then $\overline{Q}_c = \{q_2\}$. Also,
$\tilde{\delta}_*((q_0, \mu^{t_0}(q_0)), a, 0.4) = q_1$, $\tilde{\delta}_*((q_0, \mu^{t_0}(q_0)), b, 0.5) = q_2$,
$\tilde{\delta}_*((q_1, \mu^{t_1}(q_1)), a, 0.3) = q_2$, $\tilde{\delta}_*((q_2, \mu^{t_3}(q_2)), a, 0.2) = q_2$, and
$\tilde{\delta}_*((q_2, \mu^{t_i}(q_2)), a^n, (0.2)^n) = q_2, \ \forall i \geq 3, \forall n \geq 1.$
So $\tilde{\delta}_*((\overline{Q}_c, \mu(\overline{Q}_c)), \Sigma^*, A) = \{q_2\} \subseteq \overline{Q}_c$. Therefore $\overline{Q}_c$ is homogeneous.

Definition 3.4. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton and $\overline{Q}_c$ be homogeneous. We say that $\overline{Q}_c$ is separated if

$$\tilde{\delta}_*((Q\backslash\overline{Q}_c, \mu(Q\backslash\overline{Q}_c)), \Sigma^*, A) \cap \overline{Q}_c = \emptyset.$$

Example 3.5. In Example 3.3, for $c = 0.4$, we have $\tilde{\delta}_*((q_0, \mu^{t_0}(q_0)), b, 0.5) = q_2$. On the other hand, since $\overline{Q}_c = \{q_2\}$, hence

$$\tilde{\delta}_*((q_0, \mu^{t_0}(q_0)), b, 0.5) \in \tilde{\delta}_*((Q\backslash\overline{Q}_c, \mu(Q\backslash\overline{Q}_c)), \Sigma^*, A).$$

Thus $\tilde{\delta}_*((Q\backslash\overline{Q}_c, \mu(Q\backslash\overline{Q}_c)), \Sigma^*, A) \cap \overline{Q}_c \neq \emptyset$. So $\overline{Q}_c$ is not separated.

Definition 3.6. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton. $\tilde{F}_*$ is called thresholdness connected if there does not exist c, $0 \leq c < 1$, such that $\overline{Q}_c$ is separated.

Theorem 3.7. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton. We define the hyperoperation "$\circ$ on Q by

$$q \circ r = \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A) \cup \tilde{\delta}_*((r, \mu(r)), \Sigma^*, A).$$

Then $\langle Q, \circ \rangle$ is a quasi-order hypergroup.

Proof. We have $q \circ q = q^2 = \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A)$ and $q = \tilde{\delta}_*((q, \mu^t(q)), \Lambda, 1) \in \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A) = q^2$. Thus $q \in q^2$ and $q \circ r = q^2 \cup r^2$. Also,
$q^3 = \bigcup_{s \in q^2} q \circ s = \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A) \cup (\bigcup_{s \in \tilde{\delta}_*((q,\mu(q)),\Sigma^*,A)} \tilde{\delta}_*((s, \mu(s)), \Sigma^*, A)) = q^2 \cup$
$\{\tilde{\delta}_*((\tilde{\delta}_*((q, \mu^t(q)), a, w_1), \mu^{t+1}(\tilde{\delta}_*((q, \mu^t(q)), a, w_1))), b, w_2) : a, b \in \Sigma^*, w_1, w_2 \in A\}$
$= q^2 \cup \{\tilde{\delta}_*((q, \mu^t(q)), ab, w_1 w_2) : ab \in \Sigma^*, w_1 w_2 \in A\}$
$= q^2 \cup \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A) = q^2 \cup q^2 = q^2.$ □

Definition 3.8. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton. The quasi-order hypergroup $\langle Q, \circ \rangle$ is called thresholdness inner irreducible if, for every c_1, c_2, such that $0 \leq c_1 < 1$, $0 \leq c_2 < 1$, $\overline{Q}_{c_1}$ and $\overline{Q}_{c_2}$ are homogeneous, $Q = \overline{Q}_{c_1} \circ \overline{Q}_{c_2}$ and $\overline{Q}_{c_1} \cap \overline{Q}_{c_2} \neq \emptyset$.

Definition 3.9. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton. $\tilde{F}_*$ is called principal if for every homogeneous $\overline{Q}_c$, where $0 \leq c < 1$, there exists c_1 such that $0 \leq c_1 < 1$ and $Q\backslash\overline{Q}_c = \overline{Q}_{c_1}$.

Theorem 3.10. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton and "$\circ$ be the hyperoperation defined in Theorem 3.7. Then
(i) If $\tilde{F}_*$ is thresholdness connected, then $\langle Q, \circ \rangle$ is thresholdness inner irreducible.
(ii) If $\tilde{F}_*$ is principal and $\langle Q, \circ \rangle$ is thresholdness inner irreducible, then $\tilde{F}_*$ is thresholdness connected.

Proof. (i) Let $\tilde{F}_*$ be thresholdness connected, $\overline{Q}_{c_1}$ and $\overline{Q}_{c_2}$ be homogeneous and $Q = \overline{Q}_{c_1} \circ \overline{Q}_{c_2}$, where $0 \le c_1 < 1$, $0 \le c_2 < 1$. By Definitions 3.4, 3.6, we have

$$\tilde{\delta}_*((Q\backslash\overline{Q}_{c_1}, \mu(Q\backslash\overline{Q}_{c_1})), \Sigma^*, A) \cap \overline{Q}_{c_1} \neq \emptyset.$$

Then there exist $q_1 \in Q\backslash\overline{Q}_{c_1}$, $a \in \Sigma^*$, $w \in A$ and $q_2 \in \overline{Q}_{c_1}$ such that

$$q_2 = \tilde{\delta}_*((q_1, \mu^t(q_1)), a, w) \in \tilde{\delta}_*((q_1, \mu(q_1)), \Sigma^*, A) = q_1 \circ q_1.$$

Since $q_1 \in Q = \overline{Q}_{c_1} \circ \overline{Q}_{c_2}$, hence there exist $u \in \overline{Q}_{c_1}$ and $v \in \overline{Q}_{c_2}$ such that

$$q_1 \in u \circ v = \tilde{\delta}_*((u, \mu(u)), \Sigma^*, A) \cup \tilde{\delta}_*((v, \mu(v)), \Sigma^*, A).$$

Since $\overline{Q}_{c_1}$ and $\overline{Q}_{c_2}$ are homogeneous, hence

$$\tilde{\delta}_*((u, \mu(u)), \Sigma^*, A) = u \circ u \subseteq \overline{Q}_{c_1} \quad \text{and} \quad \tilde{\delta}_*((v, \mu(v)), \Sigma^*, A) = v \circ v \subseteq \overline{Q}_{c_2}.$$

Now, since $q_1 \in Q\backslash\overline{Q}_{c_1}$, hence $q_1 \in \tilde{\delta}_*((v, \mu(v)), \Sigma^*, A) = v \circ v$. Thus, we have:

$$\begin{aligned} q_2 \in \tilde{\delta}_*((q_1, \mu(q_1)), \Sigma^*, A) &= q_1 \circ q_1 \subseteq (v \circ v) \circ (v \circ v) \\ &= v^3 \circ v = v^2 \circ v = v^3 = v^2 = v \circ v \subseteq \overline{Q}_{c_2}. \end{aligned}$$

Therefore $q_2 \in \overline{Q}_{c_1} \cap \overline{Q}_{c_2}$ and $\overline{Q}_{c_1} \cap \overline{Q}_{c_2} \neq \emptyset$. Consequently $\langle Q, \circ \rangle$ is thresholdness inner irreducible.

(ii) Let $\langle Q, \circ \rangle$ be thresholdness inner irreducible and $\tilde{F}_*$ be thresholdness disconnected. Then there exists c such that $0 \le c < 1$, $\overline{Q}_c$ is homogeneous and

$$\tilde{\delta}_*((Q\backslash\overline{Q}_c, \mu(Q\backslash\overline{Q}_c)), \Sigma^*, A) \cap \overline{Q}_c = \emptyset.$$

So $\tilde{\delta}_*((Q\backslash\overline{Q}_c, \mu(Q\backslash\overline{Q}_c)), \Sigma^*, A) \subseteq Q\backslash\overline{Q}_c$. Since $\tilde{F}_*$ is principal, hence there exists c_1 such that $0 \le c_1 < 1$ and $Q\backslash\overline{Q}_c = \overline{Q}_{c_1}$. Then $\tilde{\delta}_*((\overline{Q}_{c_1}, \mu(\overline{Q}_{c_1})), \Sigma^*, A) \subseteq \overline{Q}_{c_1}$, so $\overline{Q}_{c_1}$ is homogeneous. We have $\overline{Q}_c \circ \overline{Q}_{c_1} \subseteq Q$. Now, let $q \in Q$. If $q \in \overline{Q}_c$, then we consider an arbitrary $r \in \overline{Q}_c$ and if $q \in \overline{Q}_{c_1}$, we consider an arbitrary $r \in \overline{Q}_c$. We have :

$$q = \tilde{\delta}_*((q, \mu^t(q)), \Lambda, 1) \in \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A) = q^2.$$

Thus

$$q \in \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A) \cup \tilde{\delta}_*((r, \mu(r)), \Sigma^*, A) = q \circ r = r \circ q \subseteq \overline{Q}_c \circ \overline{Q}_{c_1}.$$

Therefore $Q = \overline{Q}_c \circ \overline{Q}_{c_1}$. Also, we have $\overline{Q}_c$ and $\overline{Q}_{c_1}$ are homogeneous and $\overline{Q}_c \cap \overline{Q}_{c_1} = \emptyset$, which is in contradiction with the fact $\langle Q, \circ \rangle$ is thresholdness inner irreducible. □

Definition 3.11. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton. $\tilde{F}_*$ is called strongly connected if

$$\forall (q, p) \in Q^2, \exists x \in \Sigma^*, \exists w \in A : \tilde{\delta}_*((q, \mu^t(q)), x, w) = p.$$

Theorem 3.12. Let $\tilde{F}_* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}_*, F_1, F_2, A)$ be an invertible general fuzzy automaton and "$\circ$ be the hyperoperation defined in Theorem 3.7. Then $\tilde{F}_*$ is strongly connected if and only if the quasi-order hypergroup $\langle Q, \circ \rangle$ satisfies the condition

$$Q = q \circ q, \forall q \in Q.$$

Proof. Let $\tilde{F}_*$ be strongly connected and $q \in Q$. It is clear that $q \circ q \subseteq Q$. If $p \in Q$, then there exist $x \in \Sigma^*$ and $w \in A$ such that $\tilde{\delta}_*((q, \mu^t(q)), x, w) = p$. Thus we have

$$p = \tilde{\delta}_*((q, \mu^t(q)), x, w) \in \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A) = q \circ q.$$

So $Q \subseteq q \circ q$. Therefore $Q = q \circ q$, $\forall q \in Q$.

Conversely, let $Q = q \circ q$, $\forall q \in Q$. Then we have $Q = q \circ q = \tilde{\delta}_*((q, \mu(q)), \Sigma^*, A)$. Thus if $p \in Q$, then there exist $x \in \Sigma^*$ and $w \in A$ such that $\tilde{\delta}_*((q, \mu^t(q)), x, w) = p$. Therefore $\tilde{F}_*$ is strongly connected. □

Acknowledgements. The authors would like to thank the reviewers and editorial board for their valuable comments and suggestions.

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