ON FUZZY HYPERIDEALS OF Γ-HYPERRINGS

R. AMERI, H. HEDAYATI AND A. MOLAEE

Abstract. The aim of this paper is the study of fuzzy Γ-hyperrings. In this regard the notion of ν-fuzzy hyperideals of Γ-hyperrings are introduced and basic properties of them are investigated. In particular, the representation theorem for ν-fuzzy hyperideals are given and it is shown that the image of a ν-fuzzy hyperideal of a Γ-hyperring under a certain conditions is two-valued. Finally, the product of ν-fuzzy hyperideals are studied.

1. Introduction

Hyperstructure theory was born in 1934 when Marty defined hypergroups, began to analysis their properties and applied them to groups, rational algebraic functions [16]. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties and applied mathematics (for example see [5], [6], [22]).

Also, following the introduction of fuzzy sets by L. A. Zadeh in 1965 [23], the fuzzy set theory were developed by Zadeh himself and many researchers in mathematics and it was applied in many pure and applied areas. For example the concept of a fuzzy group was introduced by A. Rosenfeld and the notion of fuzzy ideal in a ring introduced and studied by W. J. Liu [15]. Recently fuzzy set theory have been had good develop in hyperstructures theory (for example see [7], [8], [9], [10], [11],[24]).

The notion of $\Gamma-$rings introduced by N. Nobosawa in [19] and immediately after him in 1966, Barnes extended this notion and obtained more results [4]. Kyuno investigated the new aspects of Γ-rings such as, prime Γ-rings and left and right unities of Γ-rings. Also in recent years Ozturk, Y. B. Jun and C. Y. Lee in [12] and [20] applied the concept of fuzzy sets to the theory of Γ-rings.

In this paper, first we introduce the notion of (ν-)fuzzy hyperideals of Γ-hyperrings and, then we obtain some related basic results. We characterize (ν-)fuzzy hyperideals based on their level subsets and associate a new (ν-fuzzy) hyperideal from a given fuzzy hyperideal of a Γ-hyperring. In particular, we show that under certain conditions ν-fuzzy hyperideals of Γ-hyperrings are two-valued. Finally we describe ν-fuzzy hyperideals of product of Γ-hyperrings.

Received: October 2007; Revised: April 2008; Accepted: December 2008

Key words and phrases: Γ- hyperring, (ν-fuzzy) hyperideal, Fuzzy polygroup, Canonical hypergroup, Fuzzy product.

The first author has been supported in part by Fuzzy System Research Center, University of Sistan and Bluchestan, Zahedan, Iran.

2. **Preliminaries**

In this section we gather all definitions and simple properties of $\Gamma-$hyperrings that we require in the next notions.

Let H be a nonempty set. A map $+ : H \times H \longrightarrow P_*(H)$ is called *hyperoperation* or *join operation*, where $P_*(H)$ denotes the set of all nonempty subsets of H.

Definition 2.1. [6] A nonempty set M together a hyperoperation $+$ is called a *polygroup* if the following conditions are satisfied:
(1) for all $x, y, z \in M$, $(x + y) + z = x + (y + z)$;
(2) for all $x \in M$ there exist an unique element $e \in M$ such that $e+x = x = x+e$ (we denote e by 0) ;
(3) for all $x \in M$ there exists an unique element $x' \in M$ such that $e \in x+x' \cap x'+x$ (we denote x' by $-x$);
(4) for all $x, y, z \in M$, $z \in x + y \Longrightarrow x \in z - y \Longrightarrow y \in z - x$.
By $U <_p M$, we mean U is a subpolygroup of M. We denote the set of all subpolygroup of M, by $SP(M)$. A *canonical hypergroup* is a commutative polygroup.

Definition 2.2. [15, 19] An algebraic structure $(R, +, .)$ is called a *hyperring* if the following statements are satisfied:
(i) $(R, +)$ is a canonical hypergroup ;
(ii) $(R, .)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x.0 = 0 = 0.x$;
(iii) The multiplication is distributive with respect to the hyperoperation $+$, i.e., $x.(y + z) = x.y + x.z$ and $(y + z).x = y.x + z.x$ $\forall x, y, z \in R$.

Remark 2.3. (i) It can be easily proved that zero is unique.
(ii) For simplicity of notation, sometimes we write xy instead of $x.y$ in Definition 2.2.
(iii) If $A, B \subseteq R$ and $x \in R$, then $A + B = \bigcup\{a + b | a \in A, b \in B\}$. Also, $A + x$ is used for $A + \{x\}$.
(iv) By axioms of Definition 2.2, it is easy to see that, $-(-x) = x$ and $-(x+y) = -x - y$, where $-A = \{-a \ \mid \ a \in A\}$. Also, $(a + b).(c + d) \subseteq a.c + b.c + a.d + b.d$.

Definition 2.4. Let R be a hyperring. Then
(i) R is *commutative* if $x.y = y.x$ $\forall x, y \in R$;
(ii) R is called *with identity*, if there exists an element, say $1 \in R$, such that $1.x = x = x.1, \forall x \in R$;
(iii) A nonempty subset A of R is said to be a subhyperring of R if $(A, +, .)$ is itself a hyperring. If $R \setminus \{0\}$ is a multiplicative group, then $(R, +, .)$ is a hyperfield.

Example 2.5. [18] (i) Let $(A, +, .)$ be a ring and N a normal semigroup of $(A, .)$. Then the multiplicative classes $\overline{x} = xN, x \in A$ form a partition of A. Let $\overline{A} = A/N$ be the set of these classes. If we define the product $\overline{x} \odot \overline{y}$ in $\overline{A}$ of $\overline{x}, \overline{y} \in \overline{A}$ as equal to their product as subsets of A, and their sum $\overline{x} \oplus \overline{y}$ in $\overline{A}$ as the set of all $\overline{z} \in \overline{A}$

contained in their sum as subsets of A, i.e.,

$$\overline{x} \oplus \overline{y} = \{\overline{z} | z \in \overline{x} + \overline{y}\} \text{ and } \overline{x} \odot \overline{y} = \overline{x.y}.$$

Then $(\overline{A}, \oplus, \odot)$ is a hyperring.

(ii) Let R be a commutative ring with identity. Letting $\overline{R} = \{\overline{x} = \{x, -x\} | x \in R\}$. Then $\overline{R}$ is a hyperring with respect to the hyperoperation $\overline{x} \oplus \overline{y} = \{\overline{x+y}, \overline{x-y}\}$ and multiplication $\overline{x} \odot \overline{y} = \overline{x.y}$.

Definition 2.6. (i) A nonempty subset I of a hyperring R is called a (resp. left) *right hyperideal* of R if (resp. $x.r \in I$) $r.x \in I$ $\forall r \in R$, $\forall x \in I$;

(ii) I is called a *hyperideal* if I is both left and right hyperideal;

(iii) A proper hyperideal I of R ($I \neq R$) is called a *prime hyperideal* if $a.b \in I$ implies that $a \in I$ or $b \in I$ (for a study of prime hyperideals and prime subhypermodules see [36]). The set of all prime hyperideal of R is called the *prime spectrum* of R and it is denoted by $Spec(R)$.

Definition 2.7. Let $(M, +)$ and $(\Gamma, +)$ be canonical hypergroups. Then M is said to be a $\Gamma-$*hyperring* if there exists a mapping $. : M \times \Gamma \times M \to P_*(M)$ such that the following conditions are satisfied:

(1) $(x + y)\alpha z \subseteq x\alpha z + y\alpha z$, $x\alpha(y + z) \subseteq x\alpha y + x\alpha z$, $\forall x, y, z \in M$, $\forall \alpha \in \Gamma$;

(2) $x(\alpha + \beta)y \subseteq x\alpha y + x\beta y$, $\forall x, y \in M$, $\forall \alpha, \beta \in \Gamma$;

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$, $\forall x, y, z \in M$, $\forall \alpha, \beta \in \Gamma$.

If in Definition 2.2, we replace all inclusions by equality, then M is called a *strong* Γ-*hyperring*.

Definition 2.8. A right (resp. left) *hyperideal* of Γ-hyperring M is a subpolygroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). Also if Δ is a subpolygroup of Γ, then the subpolygroup I of M is said to be a right (left) Δ-hyperideal if $I\Delta M \subseteq I$ (resp. $M\Delta I \subseteq I$). By $U <_h M$, we mean U is a hyperideal of Γ-hyperring M. Also we denote the set of all hyperideals of M by $HI(M)$.

Clearly every hyperideal of a Γ-hyperring is a Δ-hyperideal for some $\Delta \subseteq \Gamma$.

We use $I = [0, 1]$, the real unit interval as a chain with the usual ordering, in which $\bigwedge$ stands for minimum or infimum (inf)(or intersection) and $\bigvee$ stands for maximum or supremum(sup) (or union), for the degree of membership. A fuzzy subset of a given set X is a mapping $\mu : X \longrightarrow I$. We denote the set of all fuzzy subset of X by $FS(X)$, that is $FS(X) = \{\mu | \ \mu : X \longrightarrow [0, 1]$ is a function$\}$. For $\mu \in FS(X)$, the level subset of μ is defined by $\mu_t = \{x \in X | \ \mu(x) \geq t\}$. For a fuzzy set μ of X we denote by $Im(\mu)$ the image of μ.

Definition 2.9. [20] Let $(M, +)$ be a canonical hypergroup and $\mu \in FS(M)$. Then μ is a *fuzzy subpolygroup* of M if for all $a, b \in M$ the following conditions hold:

(1) $\bigwedge_{z \in a+b} \mu(z) \geq \mu(a) \wedge \mu(b)$;

(2) $\mu(-a) \geq \mu(a)$.

By $\mu <_{FP} M$, we mean μ is a fuzzy subpolygroup of M. Also we denote the set of all fuzzy subpolygroups of M, by $FP(M)$.

3. ν-Fuzzy Hyperideals of Γ-Hyperrings

In the sequel by M we mean a $\Gamma-$hyperring.

Definition 3.1. (i) A fuzzy subset μ of M is said to be a left (resp. right) *fuzzy hyperideal* of M if and only if for all $x, y \in M$ and $\gamma \in \Gamma$ we have

(1) $\mu \in FP(M)$;

(2) $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(y)$ (resp. $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(x)$).

By $\mu <_{FHI} M$, we mean μ is a fuzzy hyperideal of M. Also we denote the set of all fuzzy hyperideals of M by $FHI(M)$.

(ii) A fuzzy subset μ of M is said to be a left (resp. right) ν*-fuzzy hyperideal* of M if and only if for all $x, y \in M$ and $\gamma \in \Gamma$ we have

(1) $\mu \in FP(M)$ and $\nu \in FP(\Gamma)$;

(2) $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(y) \wedge \nu(\gamma)$ (resp. $\bigwedge\limits_{z \in x\gamma y} \mu(z) \geq \mu(x) \wedge \nu(\gamma)$).

By $\mu <_{FHI_\nu} M$, we mean μ is a ν-fuzzy hyperideal of M. Also we denote the set of all ν-fuzzy hyperideals of M by $FHI_\nu(M)$.

Clearly, every fuzzy hyperideal is a ν-fuzzy hyperideal, for some $\nu \in FP(\Gamma)$, by letting $\nu = \chi_\Gamma$, where χ_Γ denotes the characteristic function of Γ.

Example 3.2. Let $(M, +, \cdot)$ be an hyperring and Γ be an hyperideal of M. Define $\circ : M \times \Gamma \times M \longrightarrow \mathcal{P}^*(M)$ by $(a, \gamma, b) \mapsto a \circ \gamma \circ b = \{z \in M \mid z \in a.\gamma.b\}$. Then it is easy to verify that M is a strong $\Gamma-$hyperring. Also if I and Δ are hyperideals of hyperring $(M, +, .)$ and $\Delta \subseteq \Gamma$, then I is a $\Delta-$hyperideal of $\Gamma-$hyperring M, since $I\Delta M \subseteq I$ and $M\Delta I \subseteq I$. Now define μ and ν on I and Δ respectively as follow:

$$\mu(x) = \begin{cases} 0.8 & \text{if } x \in I, \\ 0 & \text{Otherwise} \end{cases} \qquad \nu(\delta) = \begin{cases} 0.5 & \text{if } \delta \in \Delta, \\ 0 & \text{Otherwise} \end{cases}$$

It is easy to verify that μ and ν are fuzzy subpolygroups of M and Γ respectively. Suppose that $x, y \in M$ and $\delta \in \Delta$ and $z \in x \circ \delta \circ y$. We can consider two cases:

(1) $x \in I$ or $y \in I$ then we can say that $x \circ \delta \circ y \subseteq I$ and so for all $z \in x \circ \delta \circ y$, we have $\mu(z) = 0.8 \geq 0.5 = (\mu(x) \vee \mu(y)) \wedge \nu(\delta)$.

(2) $x, y \notin I$ then $\mu(z) \geq 0 = (\mu(x) \vee \mu(y)) \wedge \nu(\delta)$.

Therefore μ is a $\nu-$fuzzy hyperideal of M as a $\Gamma-$hyperring.

Example 3.3. Let R be a hyperring and let $M_{m,n}\ (R)$ be the set of all matrices by the size $m \times n$ with entries of R . Define $\circ : M_{m,n}\ (R) \times M_{n,m}\ (R) \times M_{m,n}\ (R) \longrightarrow \mathcal{P}^*(M_{m,n}\ (R))$ by:

$$A \circ B \circ C = \{Z \in M_{m,n}(R) \,|Z \in ABC, \ \ A, C \in M_{m,n}(R), \ B \in M_{n,m}\ (R)\} .$$

Then it easy to verify that $M_{m,n}(R)$ is a $M_{n,m}(R)-$hyperring. Also if I and J are hyperideal of hyperring $(R, +, .)$, then it is easy to verify that $M_{m,n}(I)$ is a $M_{n,m}(J)-$hyperideal of $M_{m,n}(R)$ since $M_{m,n}(I) \circ M_{n,m}(J) \circ M_{m,n}(R) \subseteq M_{m,n}(I)$ (by Definition 2.3) and $M_{m,n}(R) \circ M_{n,m}(J) \circ M_{m,n}(I) \subseteq M_{m,n}(I)$ (by Definition

2.3). Now define μ and ν on $M_{m,n}(I)$ and $M_{n,m}(J)$ respectively as follow:

$$\mu(X)=\begin{cases}4/5 & \text{if } X\in M_{m,n}(I),\\ 7/10 & \text{if } X\notin M_{m,n}(I)\end{cases}\qquad \nu(Y)=\begin{cases}1/2 & \text{if } Y\in M_{n,m}(J),\\ 1/4 & \text{if } Y\notin M_{n,m}(J)\end{cases}$$

It is routine to check that μ is a $\nu-$fuzzy hyperideal of $M_{m,n}(R)$ as an $M_{n,m}(R)-$ hyperring.

Lemma 3.4. Let μ be a ν-fuzzy hyperideal of M. Then $\mu(x)\leq\mu(0_M)$, for all $x\in M$.

Proof. For any $x\in M$ we have $0_M\in x-x$. Thus $\mu(0_M)\geq \mu(x)\wedge\mu(-x)=\mu(x)$. □

Theorem 3.5.(Representation Theorem) Let μ be a fuzzy set in a Γ-hyperring M. Then μ is a left (resp. right) ν-fuzzy hyperideal of M if and only if each level subset μ_t of μ is a left (resp. right) ν_t-hyperideal of M, for each $t\in[0,\mu(0_M)\wedge\nu(0_\Gamma)]$.

Proof. Suppose that μ is a left(resp. right) ν-fuzzy hyperideal of M and let $\mu_t\neq\emptyset$. We have $\mu_t\subseteq M$, then for any $x,y,z\in\mu_t$, $(x+y)+z=x+(y+z)$. We show that

$$\forall a\in\mu_t,\ \exists 0_M\in\mu_t:\ a+0_M=a.$$

Since $a\in\mu_t$ and $\mu_t\subseteq M$, so $a\in M$ then there exists an unique $0_M\in M$ such that $a+0_M=a$. Also we have $0_M\in a-a$, thus $\mu(0_M)\geq\mu(a)\wedge\mu(-a)\geq t$, therefore $0_M\in\mu_t$. Similarly for all $x\in\mu_t$, there exists $-x\in\mu_t$, such that $0_M\in x-x$. We now show that

$$M\nu_t\mu_t\subseteq\mu_t \text{ (resp. } \mu_t\nu_tM\subseteq\mu_t).$$

Let $m\in M,\gamma\in\nu_t$, $u\in\mu_t$, and $z\in m\gamma u$, then we have

$$\mu(z)\geq\bigwedge_{z\in m\gamma u}\mu(z)\geq\mu(u)\wedge\nu(\gamma)\geq t;$$

thus $z\in\mu_t$. Therefore $M\nu_t\mu_t\subseteq\mu_t$. Similarly we can prove that $\mu_t\nu_tM\subseteq\mu_t$.

Conversely, suppose that μ_t is a left (resp. right) ν_t-hyperideal of M. We show that for all $a,b\in M$, $\bigwedge_{z\in a+b}\mu(z)\geq\mu(a)\wedge\mu(b)$.

If $a,b\in M$, then there exist $t_1,t_2\in[0,1]$, $\mu(a)=t_1$, $\mu(b)=t_2$. Put $t=t_1\wedge t_2$, thus $a,b\in\mu_t$, and $a+b\subseteq\mu_t$. Also if $z\in a+b$, we have $\mu(z)\geq t=\mu(a)\wedge\mu(b)$, therefore $\bigwedge_{z\in a+b}\mu(z)\geq\mu(a)\wedge\mu(b)$. Obviously for all $x\in M$, we have $\mu(x)\geq\mu(-x)$.

Let $x,y\in M$ and $\gamma\in\Gamma$ and $\mu(y)=t_1$ and $\nu(\gamma)=t_3$. Put $t=t_1\wedge t_3$, thus $y\in\mu_t$ and $\gamma\in\nu_t$. So $x\gamma y\subseteq\mu_t$, since μ_t is a ν_t-hyperideal. Then for all $z\in x\gamma y$ we have

$$\mu(z)\geq t=\mu(y)\wedge\nu(\gamma).$$

Similarly, we obtain that $\bigwedge_{z\in x\gamma y}\mu(z)\geq\mu(x)\wedge\nu(\gamma)$. This completes the proof. □

Example 3.6.

(1) Let $I_1 \subset I_2 \subset ... \subset I_n \subset ...$ be a strictly increasing sequence of left (resp. right)hyperideals of an arbitrary Γ-hyperring M and $\{t_j\}_{j=1}^{\infty}$ be a strictly increasing sequence in $[0, 1]$. Define μ on M as follows:

$$\mu(x) = t_j \text{ if } x \in I_j \backslash I_{j-1}, \text{ } where \text{ } t_{j-1} < t_j, j = 1, 2, ... \text{ } and \text{ } \mu(x) = 0, if x \in M \backslash \cup_{j=1}^{\infty} I_j,$$

It is easy to verify that $\mu_{t_{j+1}} \subseteq \mu_{t_j}$ and the only level subsets of M are M, and $\mu_{t_j} = I_j, j = 1, 2, ...$ Then by Theorem 3.5 μ is a left (resp. right) fuzzy hyperideal of M.

(2) Let $I_1 \supset I_2 \supset ... \supset I_n \supset ...$ be a strictly decreasing sequence of left (resp. right)hyperideals of an arbitrary Γ-hyperring M and $\{t_j\}_{j=1}^{n}$ be a strictly decreasing sequence in $[0, 1]$.

Define fuzzy subset μ on V by $\mu(x) = t_{j-1}$, if $x \in I_{j-1} \backslash I_j$ where

$$t_{j-1} > t_j, j = 1, 2, 3, ... \text{ and } \mu(x) = 1 \text{ if } x \in \cap_j^{\infty} I_j.$$

Again by Theorem 3.5 it is easy to verify that μ is a left (resp. right) fuzzy hyperideal of M, since the only level subsets of M are M and $\mu_{t_j} = I_j, j = 1, 2, ...$

(3) In Example 3.3, μ is ν-fuzzy hyperideal of $M_{m,n}(R)$, since $\mu_{4/5} = M_{m,n}(R)$ and $\mu_{7/10} = M_{m,n}(I)$ and $\nu_{4/5} = \nu_{7/10} = \nu_{1/4} = M_{n,m}(J)$, which are hyperideals.

Lemma 3.7. If $\mu \in FHI_\nu(M)$ and $\bigwedge_{t \in x-y} \mu(t) = \mu(0_M)$, then $\mu(x) = \mu(y)$.

Proof. We have

$$\mu(x) \geq \bigwedge_{t \in x-y+y} \mu(t) \geq (\bigwedge_{t' \in x-y} \mu(t')) \wedge \mu(y) = \mu(0_M) \wedge \mu(y) = \mu(y).$$

Then, $\mu(x) \geq \mu(y)$. Similarly, we have $\mu(y) \geq \mu(x)$. Therefore $\mu(x) = \mu(y)$. □

In next propositions we construct new (ν-fuzzy) hyperideals by given fuzzy hyperideals of Γ-hyperrings.

Proposition 3.8. Let μ be a left (resp. right) ν-fuzzy hyperideal of M and $\mu(0_M) = \nu(0_\Gamma)$. Then the set

$$M_\mu = \{x \in M | \text{ } \mu(x) = \mu(0_M)\}$$

is a left (resp. right) $\nu_{\mu(0_M)}$-hyperideal of M.

Proof. A direct verification shows that M_μ is a canonical hypergroup and $M_\mu \subseteq M$. We show that $M\nu_{\mu(0_M)}M_\mu \subseteq M_\mu$. Let $z \in x\gamma y$ such that $x \in M, \gamma \in \nu_{\mu(0_M)}$ and $y \in M_\mu$. We have $\mu(z) \geq \mu(y) \wedge \nu(\gamma) \geq \mu(0_M)$. Then by Lemma 3.4, $\mu(z) = \mu(0_M)$, thus $z \in M_\mu$. Similarly, we obtain $M_\mu \nu_{\mu(0_M)} M \subseteq M_\mu$. □

Proposition 3.9. Let μ be a left (resp. right) ν-fuzzy hyperideal of M, then

$$supp(\mu) = \{x \in M | \text{ } \mu(x) > 0\}$$

is a left (resp. right) $supp(\nu)$-hyperideal of M.

Proof. The proof is similar to the proof of Proposition 3.5 by some modification. □

Proposition 3.10. If μ is a ν-fuzzy hyperideal of Γ-hyperring M, then

$$R(\mu)(x) = \bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(z) \mid z \in nx, \exists n \in \mathbb{N}\}\}$$

is a ν-fuzzy hyperideal of M.

Proof. Let $z \in x + y$. We prove that

$$R(\mu)(z) \geq R(\mu)(x) \wedge R(\mu)(y).$$

For this we have

$$\begin{aligned}
R(\mu)(z) &= \bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(a)|\ a \in nz, \exists n \in \mathbb{N}\}\} \\
&\geq \bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(a)|a \in nx + ny, \exists n \in \mathbb{N}\}\} \qquad (\text{since } z \in x + y) \\
&\geq \bigvee_{n\in\mathbb{N}}[\{\bigwedge\{\mu(t_1)|t_1 \in nx, \exists n \in \mathbb{N}\}\} \wedge \{\bigwedge\{\mu(t_2)|t_2 \in ny, \exists n \in \mathbb{N}\}\}] \\
&= [\bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(t_1)|t_1 \in nx, \exists n \in \mathbb{N}\}\}] \wedge [\bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(t_2)|t_2 \in ny, \exists n \in \mathbb{N}\}\}] \\
&= R(\mu)(x) \wedge R(\mu)(y).
\end{aligned}$$

Also we have

$$\begin{aligned}
R(\mu)(x) &= \bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(z)|z \in nx, \exists n \in \mathbb{N}\}\} \\
&\geq \bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(-z)| -z \in n(-x), \exists n \in \mathbb{N}\}\} \\
&= R(\mu)(-x).
\end{aligned}$$

Now suppose that $z \in x\gamma y$. We prove that

$$R(\mu)(z) \geq (R(\mu)(x) \vee R(\mu)(y)) \wedge \nu(\gamma).$$

For this we have

$$\begin{aligned}
R(\mu)(z) &= \bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(a)|a \in nz, \exists n \in \mathbb{N}\}\} \\
&\geq \bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(a)|a \in (nx)\gamma y, \exists n \in \mathbb{N}\}\} \qquad (\text{since } z \in x\gamma y) \\
&\geq [\bigvee_{n\in\mathbb{N}}\{\bigwedge\{\mu(b)|b \in nx, \exists n \in \mathbb{N}\}\}] \wedge \nu(\gamma) \qquad (\text{since } \mu \in FHI(M)) \\
&= R(\mu)(x) \wedge \nu(\gamma).
\end{aligned}$$

Similarly, we can prove that $R(\mu)(z) \geq R(\mu)(y) \wedge \nu(\gamma)$. Therefore $R(\mu) \in FHI_\nu(M)$. □

Proposition 3.11. Let $\mu \in FHI_\nu(M)$ and $\mu^+(x) = \mu(x) + 1 - \mu(0_M)$.

(i) Then μ^+ is a ν-fuzzy hyperideal of M.

(ii) If $\mu(0_M) = \nu(0_\Gamma)$, then μ^+ is a ν^+-fuzzy hyperideal of M, where $\nu^+(x) = \nu(x) + 1 - \nu(0_\Gamma)$.

Proof. (i) Let $z \in x + y$, then we have

$$\begin{aligned} \mu^+(z) &= \mu(z) + 1 - \mu(0_M) \\ &\geq (\mu(x) \wedge \mu(y)) + 1 - \mu(0_M) \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &= (\mu(x) + 1 - \mu(0_M)) \wedge (\mu(y) + 1 - \mu(0_M)) \\ &= \mu^+(x) \wedge \mu^+(y). \end{aligned}$$

Also we have

$$\begin{aligned} \mu^+(z) &= \mu(z) + 1 - \mu(0_M) \\ &\geq \mu(-z) + 1 - \mu(0_M) \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &= \mu^+(-z). \end{aligned}$$

Now suppose $z \in x\gamma y$, then we have

$$\mu^+(z) = \mu(z) + 1 - \mu(0_M) \geq (\mu(x) \wedge \nu(\gamma)) + 1 - \mu(0_M). \tag{1}$$

We consider the following cases.

Case 1. If $\mu(x) \geq \nu(\gamma)$, then

$$(\mu(x) \wedge \nu(\gamma)) + 1 - \mu(0_M) = \nu(\gamma) + 1 - \mu(0_M) \tag{2}$$

we have $\mu(x) + 1 - \mu(0_M) \geq \mu(x) \geq \nu(\gamma)$, then

$$(\mu(x) + 1 - \mu(0_M)) \wedge \nu(\gamma) = \nu(\gamma). \tag{3}$$

Then from (1), (2) and (3) it is concluded that $\mu^+(z) \geq \nu(\gamma) + 1 - \mu(0_M) \geq \nu(\gamma)$. Thus $\mu^+(z) \geq \mu^+(x) \wedge \nu(\gamma)$.

Case 2. If $\mu(x) \leq \nu(\gamma)$, then

$$\begin{aligned} \mu^+(z) &\geq (\mu(x) \wedge \nu(\gamma)) + 1 - \mu(0_M) \\ &= \mu(x) + 1 - \mu(0_M) \\ &= \mu^+(x) \\ &\geq \mu^+(x) \wedge \nu(\gamma). \end{aligned}$$

Similarly to the both cases 1 and 2 we can obtain $\mu^+(z) \geq \mu^+(y) \wedge \nu(\gamma)$. Thus

$$\mu^+(z) \geq (\mu^+(x) \vee \mu^+(y)) \wedge \nu(\gamma).$$

Therefore $\mu^+ \in FHI_\nu(M)$.

(ii) Let μ is a ν-fuzzy hyperideal of $\Gamma-$hyperring M and $\mu(0_M) = \nu(0_\Gamma)$ and $z \in x - y$, for all $x, y \in M$. Obviously $\mu^+(z) \geq \mu^+(x) \wedge \mu^+(y)$. Suppose that $z \in x\gamma y$, for $x, y \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} \mu^+(z) &= \mu(z) + 1 - \mu(0_M) \\ &\geq \{[\mu(x) \vee \mu(y)] \wedge \nu(y)\} + 1 - \mu(0_M) \\ &= [(\mu(x) + 1 - \mu(0_M)) \vee (\mu(y) + 1 - \mu(0_M))] \wedge (\nu(\gamma) + 1 - \mu(0_M)) \\ &= [\mu^+(x) \vee \mu^+(y)] \wedge \nu^+(\gamma) \quad (\text{since } \mu(0_M) = \nu(0_\Gamma)). \end{aligned}$$

Therefore μ^+ is a ν^+-fuzzy hyperideal of M. □

Proposition 3.13. Let M be a Γ-hyperring and $\mu \in FHI_\nu(M)$.

(i) If $f : [0, \mu(0_M) \vee \nu(0_\Gamma)] \longrightarrow [0, 1]$ is an increasing map, then $\mu_f : M \longrightarrow [0, 1]$ defined by $\mu_f(x) = f(\mu(x))$ for all $x \in M$ is a ν_f-fuzzy hyperideal of M, where $\nu_f : \Gamma \longrightarrow [0, 1]$ is defined by $\nu_f(\gamma) = f(\nu(\gamma))$ for all $\gamma \in \Gamma$.

(ii) If $\mu(0_M) = \nu(0_\Gamma)$ and $\widetilde{\mu} : M \longrightarrow [0, 1]$ defined by $\widetilde{\mu}(x) = \mu(x)\mu(0_M)$ for all $x \in M$ is a $\widetilde{\nu}$-fuzzy hyperideal of M, where $\widetilde{\nu} : \Gamma \longrightarrow [0, 1]$ is defined by $\widetilde{\nu}(\gamma) = \nu(\gamma)\nu(0_\Gamma)$ for all $\gamma \in \Gamma$.

Proof. (i) Let $z \in x + y$ then $\mu(z) \geq \mu(x) \wedge \mu(y)$. Since f is increasing then, $f(\mu(z)) \geq f(\mu(x)) \wedge f(\mu(y))$, therefore $\mu_f(z) \geq \mu_f(x) \wedge \mu_f(y)$. Also we have

$$\mu_f(z) = f(\mu(z)) \geq f(\mu(-z)) = \mu_f(-z).$$

Suppose that $z \in x\gamma y$, then we have

$$\begin{aligned} \mu(z) &\geq (\mu(x) \vee \mu(y)) \wedge \nu(\gamma) \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &\Longrightarrow f(\mu(z)) \geq [f(\mu(x)) \vee f(\mu(y))] \wedge f(\nu(\gamma)) \quad (\text{since } f \text{ is increasing}) \\ &\Longrightarrow \mu_f(z) \geq (\mu_f(x) \vee \mu_f(y)) \wedge \nu_f(\gamma). \end{aligned}$$

Therefore $\mu_f \in FHI_{\nu_f}(M)$.

(ii) Let $z \in x + y$ then we have

$$\begin{aligned} \widetilde{\mu}(z) &= \mu(z)/\mu(0_M) \\ &= (1/\mu(0_M))\mu(z) \\ &\geq (1/\mu(0_M))(\mu(x) \wedge \mu(y)) \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &= (\mu(x)/\mu(0_M)) \wedge (\mu(y)/\mu(0_M)) \\ &= \widetilde{\mu}(x) \wedge \widetilde{\mu}(y). \end{aligned}$$

Also we have

$$\begin{aligned} \widetilde{\mu}(z) &= (1/\mu(0_M))\mu(z) \\ &\geq (1/\mu(0_M))\mu(-z) \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &= \widetilde{\mu}(-z). \end{aligned}$$

Suppose that $z \in x\gamma y$, then we have

$$\begin{aligned}\widetilde{\mu}(z) &= (1/\mu(0_M))\mu(z) \\ &\geq (1/\mu(0_M))[(\mu(x) \vee \mu(y)) \wedge \nu(\gamma)] \qquad (\text{since } \mu \in FHI_\nu(M)) \\ &= [\widetilde{\mu}(x) \vee \widetilde{\mu}(y)] \wedge \widetilde{\nu}(\gamma).\end{aligned}$$

Therefore, $\widetilde{\mu} \in FHI_{\widetilde{\nu}}(M)$. □

In the next theorem, we prove that under certain conditions, fuzzy hyperideal of Γ-hyperring is two-valued.

Theorem 3.14. Let $\mu \in FHI_{\eta'}(M)$, $\eta = 1/2\eta'$ and μ be maximal in the set $X = \{\nu \in FHI_\eta(M) \mid \nu(x) = 1,\ \exists x \in M\}$ under conclusion. Then μ is two-valued fuzzy hyperideal of M and it takes just 0 and 1.

Proof. Clearly $\mu \in FHI_\eta(M)$. We know that there exists $x \in M$ such that $\mu(x) = 1$, thus $\mu(0_M) \geq \mu(x) = 1$, hence $\mu(0_M) = 1$.

Let $x \in M$ be such that $\mu(x) \neq 1$. We show that $\mu(x) = 0$. Suppose that there exists $a \in M$ such that $0 < \mu(a) < 1$. Define $\nu : M \to [0, 1]$ by $\nu(x) = 1/2(\mu(x) + \mu(a))$, for all $x \in M$. We show that $\nu \in FHI_\eta(M)$. Suppose that $z \in x + y$, then we have

$$\begin{aligned}\nu(z) &= 1/2(\mu(z) + \mu(a)) \\ &\geq 1/2[(\mu(x) \wedge \mu(y)) + \mu(a)] \qquad (\text{since } \mu \in FHI_\eta(M)) \\ &= 1/2[(\mu(x) + \mu(a)) \wedge (\mu(y) + \mu(a))] \\ &= 1/2(\mu(x) + \mu(a)) \wedge 1/2(\mu(y) + \mu(a)) \\ &= \nu(x) \wedge \nu(y).\end{aligned}$$

Also it is easy to verify that if $z \in M$, then $\nu(z) \geq \nu(-z)$. Now suppose $z \in x\gamma y$, we prove $\nu(z) \geq \nu(x) \wedge \eta(\gamma)$. We have

$$\begin{aligned}\nu(z) &= 1/2(\mu(z) + \mu(a)) \\ &\geq 1/2[(\mu(x) \wedge \eta^{'}(\gamma)) + \mu(a)] \qquad (\text{since } \mu \in FHI_{\eta'}(M)) \\ &= 1/2[\mu(x) + \mu(a)] \ \wedge \ 1/2[\eta^{'}(\gamma)) + \mu(a)] \\ &= \nu(x) \wedge (\eta(\gamma) + 1/2\mu(a)) \\ &\geq \nu(x) \wedge \eta(\gamma).\end{aligned}$$

Similarly we can prove that $\nu(z) \geq \nu(y) \wedge \eta(\gamma)$. Therefore $\nu \in FHI_\eta(M)$.

Hence, by Proposition 3.10, $\nu^+ \in FHI_\eta(M)$. Also we have

$$\begin{aligned}\nu^+(x) &= \nu(x) + 1 - \nu(0_M) \\ &= 1/2(\mu(x) + \mu(a)) + 1 - 1/2(\mu(0_M) + \mu(a)) \\ &= 1/2(\mu(x) + 1). \qquad (\text{since } \mu(0_M) = 1)\end{aligned}$$

So we have

$$\nu^+(0_M) = 1/2(\mu(0_M) + 1) = 1/2(1 + 1) = 1.$$

Thus $\nu^+ \in X$. Also we have

$$\nu^+(0_M) = 1 > \nu^+(a) = 1/2(\mu(a) + 1) > \mu(a) \neq 1.$$

Hence ν^+ is non-constant and $\nu^+(a) > \mu(a)$. So μ is not maximal, this is a contradiction. Therefore there is not any $a \in M$ such that $0 < \mu(a) < 1$. □

4. Fuzzy Product of ν-Fuzzy Hyperideals

Suppose that $(M_i, +_i)_{i\in I}$ is a family of canonical hypergroups. Then $\prod_{i\in I} M_i = \{(x_i)_{i\in I} \mid x_i \in M_i\}$, the cartesian product of $(M_i, +_i)_{i\in I}$, with following hyperoperation is a canonical hypergroup:

$$(x_i)_{i\in I} + (y_i)_{i\in I} = \{(z_i)_{i\in I} \mid z_i \in x_i +_i y_i\}.$$

It is easy to verify that if M_i is a Γ_i-hyperring, then $\prod_{i\in I} M_i$ is $\prod_{i\in I} \Gamma_i$-hyperring by the following rule:

$$\circ : (\prod_{i\in I} M_i) \times (\prod_{i\in I} \Gamma_i) \times (\prod_{i\in I} M_i) \longrightarrow P^*(\prod_{i\in I} M_i),$$

which is defined by

$$(x_i)_{i\in I} \circ (\gamma_i)_{i\in I} \circ (y_i)_{i\in I} = \{(z_i)_{i\in I} \mid z_i \in x_i\gamma_i y_i, \forall i \in I\}.$$

Notation. In the next proposition, by $\prod_{i\in I} \mu_i$, we mean the fuzzy product of μ_is, which is defined as follows:

$$(\prod_{i\in I} \mu_i)((x_i)_{i\in I}) = \bigwedge_{i\in I} \mu_i(x_i).$$

In the next proposition we describe fuzzy hyperideals of product of Γ-hyperrings.

Proposition 4.1. Let μ_i be ν_i-fuzzy hyperideal of M_i as Γ_i-hyperring ($\forall i \in I$). Then $\prod_{i\in I} \mu_i$ is a $\prod_{i\in I} \nu_i$- fuzzy hyperideal of $\prod_{i\in I} M_i$ as $\prod_{i\in I} \Gamma_i$-hyperring.

Proof. Suppose $(z_i)_{i\in I} \in (x_i)_{i\in I} + (y_i)_{i\in I}$. Then $z_i \in x_i +_i y_i$, so $\mu_i(z_i) \geq \mu_i(x_i) \wedge \mu_i(y_i)$, for all $i \in I$. Also we have

$$\begin{aligned}
(\prod_{i\in I} \mu_i)((z_i)_{i\in I}) &= \bigwedge_{i\in I} \mu_i(z_i) \\
&\geq \bigwedge_{i\in I} (\mu_i(x_i) \wedge \mu_i(y_i)) \qquad (\text{since } \mu_i \in FHI_{\nu_i}(M_i)) \\
&= (\bigwedge_{i\in I} \mu_i(x_i)) \wedge (\bigwedge_{i\in I} \mu_i(y_i)) \\
&= (\prod_{i\in I} \mu_i)((x_i)_{i\in I}) \wedge (\prod_{i\in I} \mu_i)((y_i)_{i\in I}).
\end{aligned}$$

Also it is easy to verify that $(\prod_{i\in I}\mu_i)((x_i)_{i\in I}) \geq (\prod_{i\in I}\mu_i)(-(x_i)_{i\in I})$.

Suppose that $(z_i)_{i\in I} \in (x_i)_{i\in I}(\gamma_i)_{i\in I}(y_i)_{i\in I}$, then we have
$z_i \in x_i\gamma_i y_i, \ \forall i \in I$

$$\Longrightarrow \quad \mu_i(z_i) \geq (\mu_i(x_i) \vee \mu_i(y_i)) \wedge \nu_i(\gamma_i), \ \forall i \in I \quad (\text{since } \mu_i \in FHI_{\nu_i}(M_i))$$
$$\Longrightarrow \quad \bigwedge_{i\in I}\mu_i(z_i) \geq \bigwedge_{i\in I}[(\mu_i(x_i) \vee \mu_i(y_i)) \wedge \nu_i(\gamma_i)]$$
$$\Longrightarrow \quad (\prod_{i\in I}\mu_i)((z_i)_{i\in I}) \geq [(\prod_{i\in I}\mu_i)((x_i)_{i\in I}) \vee (\prod_{i\in I}\mu_i)((y_i)_{i\in I})] \wedge (\prod_{i\in I}\nu_i)((\gamma_i)_{i\in I}).$$

Therefore $\prod_{i\in I}\mu_i$ is a $\prod_{i\in I}\nu_i$-fuzzy hyperideal of $\prod_{i\in I}M_i$. □

Acknowledgements. The first author is partially supported by the "Research Center on Algebraic Hyperstructures and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran" and "Fuzzy Systems and Its Applications Center of Excellence, Shahid Bahonar University of Kerman, Kerman, Iran".

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Reza Ameri*, Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: ameri@umz.ac.ir

Hossein Hedayati, Department of Basic Sciences,, Babol University of Technology, Babol, Iran
E-mail address: h.hedayati@umz.ac.ir

A. Molaee, Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran

*Corresponding author