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FUZZY CONVEX SUBALGEBRAS OF COMMUTATIVE RESIDUATED LATTICES

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Abstract. In this paper, we define the notions of fuzzy congruence relations and fuzzy convex subalgebras on a commutative residuated lattice and we obtain some related results. In particular, we will show that there exists a one to one correspondence between the set of all fuzzy congruence relations and the set of all fuzzy convex subalgebras on a commutative residuated lattice. Then we study fuzzy convex subalgebras of an integral commutative residuated lattice and will prove that fuzzy filters and fuzzy convex subalgebras of an integral commutative residuated lattice coincide.

1. Introduction

The concept of fuzzy sets was firstly introduced by Zadeh in 1965 ([15]). The theory of fuzzy sets has been developed in a wide variety of fields such as fuzzy mathematics. One of the important branch of fuzzy mathematics is fuzzy algebras. Many researchers have studied some algebraic structure such as fuzzy group, fuzzy ring, fuzzy modules, ... Also, the concept of fuzzy sets was applied to BCI, BCK, MV, BL-algebras. The concept of a commutative residuated lattice was firstly introduced by M. Ward and R. P. Dilworth as generalization of ideal lattices of rings (See [4], [13] and [14]). A commutative residuated lattice is an ordered algebraic structure $L = (L, \wedge, \vee, *, :, e)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, *, e)$ is a commutative monoid and for all $x, y, z \in L$

$$x * y \leq z \Longleftrightarrow x \leq z : y \Longleftrightarrow y \leq z : x.$$

The class of commutative residuated lattices is denoted by *CRL*. It was shown that *CRL* is an ideal variety in the sense that its congruence correspond to convex subalgebra. An integral commutative residuated lattice is an algebraic structure $L = (L, \wedge, \vee, :, *, e)$ such that $L = (L, \wedge, \vee, :, *, e)$ is a commutative residuated lattice and e is the greatest element of this algebra. In this case, we define $1 := e$ ([1]). We denote the class of integral commutative residuated lattice by *ICRL* which is a subclass of *CRL*. As an example, every BL-algebra is a member of the class *ICRL*. In fact, we consider a BL-algebra as a member of the subvariety of *ICRL* such that 0 is the least element of this algebra and it satisfies the additional

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identities $x * (y : x) = x \wedge y$ and $(y : x) \vee (x : y) = 1$. In the next section, some preliminary definitions and theorems are stated. In section 3, first of all we define the notion of fuzzy congruence relations on a commutative residuated lattice, state and prove some related results. Then, using the concept of fuzzy sets to a commutative residuated lattice, we introduce the notion of fuzzy convex subalgebras of a commutative residuated lattice to investigate their properties. In particular, we will prove that there exists a bijection between the set of all fuzzy convex subalgebras and the set of all fuzzy congruence relations on a commutative residuated lattice. Then we will introduce the notion of fuzzy quotient algebras in a commutative residuated lattice and will prove that the fuzzy quotient algebras induced by fuzzy convex subalgebras are commutative residuated lattices.
Fuzzy filter on a BL-algebra was introduced and studied by L. Z. Liu and K. T.Li in ([10]). Finally, in section 4, using this definition for fuzzy filters of an integral commutative residuated lattice, we will show that each fuzzy filter of an integral commutative residuated lattice is a convex fuzzy subalgebra and vice versa, i.e., every convex fuzzy subalgebra of an integral commutative residuated lattice is a fuzzy filter.

2. Preliminaries

Here we review some definitions and results which are needed in the other sections.

Definition 2.1. A commutative binary operation $* : P \times P \to P$ on a poset $(P, \leq)$ is said to be residuated iff there exists a binary operation $:: P \times P \to P$ such that

$$x * y \leq z \Longleftrightarrow x \leq z : y$$

for all $x, y, z \in P$. Then : is called a residual of the $*$ and $(P, \leq)$ is called a *residuated poset* under the operation $*$.

Theorem 2.2. [5] *Let $(P, \leq)$ be a poset. Then $* : P \times P \to P$ is residuated if and only if it is order preserving in each component and $max\{p \in P : x * p \leq y\}$ exists for all $x, y \in P$. In this case, $y : x = max\{p \in P : x * p \leq y\}$ for all $x, y \in P$.*

Definition 2.3. [5] A *commutative residuated lattice* is an ordered algebraic structure $L = (L, \wedge, \vee, :, *, e)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, *, e)$ is a commutative monoid and the operation : serves as the residual for the monoid multiplication under the lattice ordering.

Theorem 2.4. [2] *Let $(L, \wedge, \vee, *, :, e)$ be a commutative residuated lattice. Then we have the following properties:*
(1) $x : e = x$, $e \leq x : x$,
(2) $x * (y : x) \leq y$,
(3) $y : x \leq (y * z) : (x * z)$,
(4) *if* $x \leq y$, *then* $x * z \leq y * z$,
(5) $(z : y) : x = z : (x * y)$,
(6) $(e : x) * (e : y) \leq e : (x * y)$,
(7) $x * (y \vee z) = (x * y) \vee (x * z)$,

(8) $x * (y \wedge z) \leq (x * y) \wedge (x * z)$,
(9) $y : x \leq (y : z) : (x : z)$,
(10) $y : x \leq (z : x) : (z : y)$,
(11) $y \leq (x * y) : x$,
for all $x, y, z \in L$.

Definition 2.5. [5] Let $(L, \wedge, \vee, *, :, e)$ be a commutative residuated lattice and $X \subseteq L$. Then we denote the upper set generated by X in L with $\uparrow X$, that is $\uparrow X = \{y \in L : \exists x \in X \ni x \leq y\}$. If $X = \{x\}$ is a singleton, we will use the symbol $\uparrow x$ in place of $\uparrow \{x\}$. The lower set $\downarrow X$ generated by X in L is defined dually.

Definition 2.6. A non-empty subset S of a commutative residuated lattice which is a commutative residuated lattice with respect to the operations of residuated lattice is called a *subalgebra* of the commutative residuated lattice.

Definition 2.7. [5] Let S be a subalgebra of a commutative residuated lattice L. S is called a *convex subalgebra* of L if $a, b \in S$, then $\uparrow a \cap \downarrow b \subseteq S$.

Theorem 2.8. [9, 12] *Let* $(L, \wedge, \vee, *, :, 1)$ *be an integral commutative residuated lattice. Then we have the following properties:*
(1) $x : 1 = x$, $x : x = 1$, $1 : x = 1$,
(2) $x * y \leq x \wedge y$,
(3) $x \leq y$ *if and only if* $y : x = 1$,
(4) $y \leq y : x$,
(5) $x * (y : x) \leq x, y$ *and hence* $x * (y : x) \leq x \wedge y$,
(6) $(z : y) : x = (z : x) : y$,
for all $x, y, z \in L$.

Definition 2.9. [9, 12] A non-empty subset F of an integral commutative residuated lattice $(L, \wedge, \vee, *, :, 1)$ is called *filter* if
$(F1)$ $1 \in F$,
$(F2)$ if $x \in F$ and $y : x \in F$, then $y \in F$,
for all $x, y \in L$.

Theorem 2.10. [9, 12] *A non-empty subset* F *of an integral commutative residuated lattice* $(L, \wedge, \vee, *, :, 1)$ *is a filter of* L *iff*
$(F1)'$ *if* $x, y \in F$*, then* $x * y \in F$,
$(F2)'$ *if* $x \in F$*,* $y \in L$ *and* $x \leq y$*, then* $y \in F$.

Filters of a commutative residuated lattice are also called congruence filters or deductive systems in literature.

Definition 2.11. [15] A *fuzzy set* of a non-empty set X is a mapping $\mu : X \to [0, 1]$.

Definition 2.12. [3] Let μ be a *fuzzy set* of a non-empty set X. For each $t \in [0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called *t-level subset* of μ.

Definition 2.13. let $\{\mu_i\}_{i\in I}$ be a family of fuzzy sets of a non-empty set X. Then the fuzzy sets $\bigcup_{i\in I} \mu_i$ and $\bigcap_{i\in I} \mu_i$ are defined by

$$\bigcup_{i\in I} \mu_i(x) = \sup_{i\in I} \mu_i(x) \qquad \text{and} \qquad \bigcap_{i\in I} \mu_i(x) = \inf_{i\in I} \mu_i(x),$$

for all $x \in X$.

Definition 2.14. [11] A *fuzzy equivalence relation* R on a non-empty set X is a fuzzy subset of $X \times X$ satisfying the following conditions:
$(R1)$ $R(x,x) = Sup\{R(y,z) : y,z \in X\}$ (reflexive),
$(R2)$ $R(x,y) = R(y,x)$(symmetric),
$(R3)$ $R(x,z) \geq min\{R(x,y), R(y,z)\}$ (transitive),
for all $x,y,z \in X$.

3. **Fuzzy Convex Subalgebras of *CRL***

Throughout this section, L will denote a commutative residuated lattice.

Lemma 3.1. *If R is a fuzzy equivalence relation on L, then*
(1) $R(e,e) = R(x,x)$*, for all* $x \in L$*,*
(2) $R(e,e) \geq R(x,y)$*, for all* $x,y \in L$*.*

Proof. (1) It will be followed by Definition 2.14 part (R1).
(2) Let $x,y \in L$. Then $R(e,e) = R(x,x) = \sup\{R(y,z)$: for all $y,z \in L\} \geq R(x,y)$. □

Definition 3.2. A fuzzy equivalence relation θ on L is called a *fuzzy congruence relation on* L if
(C1) $\theta(y : x, w : z) \geq min\{\theta(x,z), \theta(y,w)\}$,
(C2) $\theta(x * y, z * w) \geq min\{\theta(x,z), \theta(y,w)\}$,
(C3) $\theta(x \wedge y, z \wedge w) \geq min\{\theta(x,z), \theta(y,w)\}$,
(C4) $\theta(x \vee y, z \vee w) \geq min\{\theta(x,z), \theta(y,w)\}$,
for all $x,y,z,w \in L$.

Example 3.3. Let $L = \{0, a, b, e, 1\}$ with $0 < a < b < e < 1$. We define

$*$	0	a	b	e	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	1
e	0	a	b	e	1
1	0	a	1	1	1

$:$	0	a	b	e	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	a	e	e	1
e	0	a	b	e	1
1	0	a	a	a	1

and so L become a commutative residuated lattice. Define
$\theta\ (0,a) = \theta(0,1) = \theta(0,b) = \theta(0,e) = x$,
$\theta(1,a) = \theta(a,b) = \theta(e,1) = \theta(a,e) = \theta(b,1) = y$,
$\theta(b,e) = z$,
such that $0 \leq x \leq y \leq z \leq \theta(e,e) \leq 1$. Then θ is a fuzzy congruence relation on L.

Theorem 3.4. *Let θ be a fuzzy equivalence relation on L. Then θ is a fuzzy congruence relation on L iff*
$(C1)'$ $\theta(z : x, z : y) \geq \theta(x, y)$,
$(C1)''$ $\theta(x : z, y : z) \geq \theta(x, y)$,
$(C2)'$ $\theta(x * z, y * z) \geq \theta(x, y)$,
$(C3)'$ $\theta(x \wedge z, y \wedge z) \geq \theta(x, y)$,
$(C4)'$ $\theta(x \vee z, y \vee z) \geq \theta(x, y)$,
for all $x, y, z \in L$.

Proof. Let θ be a fuzzy congruence relation on L and let $x, y, z \in L$. Then

$$\theta(z : x, z : y) \geq min\{\theta(x, y), \theta(z, z)\} = \theta(x, y)$$

by Lemma 3.1 and $(C1)$. Similarly, we can show that the other conditions of Definition 3.2 hold.

Conversely, let θ be a fuzzy equivalence relation on L which satisfies conditions $(C1)'$ - $(C4)'$ and let $x, y, z, w \in L$. Then

$$\begin{aligned}\theta(y : x, w : z) &\geq min\{\theta(y : x, y : z), \theta(y : z, w : z)\} \\ &= min\{\theta(x, z), \theta(y, w)\}\end{aligned}$$

by $(R3)$, $(C1)'$ and $(C1)''$. Similarly, one can show that the other conditions hold. Hence θ is a fuzzy congruence relation on L. □

Theorem 3.5. *Let θ be a fuzzy relation on L. θ is a fuzzy congruence relation on L iff for all $t \in [0, 1]$, θ_t is either empty or a congruence relation on L.*

Proof. The proof is routine. □

Notation: The set of all fuzzy congruence relations on L is denoted by $FCon(L)$.

Definition 3.6. A fuzzy subset S of L is called a *fuzzy subalgebra* of L iff
(1) $S(e) \geq S(x)$,
(2) $S(y : x) \geq min\{S(x), S(y)\}$,
(3) $S(x * y) \geq min\{S(x), S(y)\}$,
(4) $S(x \wedge y) \geq min\{S(x), S(y)\}$,
(5) $S(x \vee y) \geq min\{S(x), S(y)\}$,
for all $x, y \in L$.

Example 3.7. Consider the commutative residuated lattice which is defined in Example 3.3 and define

$$0 < S(0) < S(b) < S(a) = S(1) < S(e) < 1.$$

Then S is a fuzzy subalgebra of L.

Definition 3.8. A fuzzy subalgebra S of L is said to be a *fuzzy convex subalgebra* of L if for $a \in S_\alpha$, $b \in S_\beta$ and $a \leq c \leq b$, there exists a γ between α and β such that $c \in S_\gamma$.

Example 3.9. [6] Let $L = \{0, a, b, e, 1\}$ with $0 < a, b < e < 1$ and elements a, b are incomparable. We define

$*$	0	a	b	e	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
e	0	a	b	e	1
1	0	a	b	1	1

$:$	0	a	b	e	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
e	0	a	b	e	1
1	0	a	b	1	1

and so L is a commutative residuated lattice. Define fuzzy set S_1 of L by $S_1(0) = S_1(a) = x$, $S_1(b) = y$ and $S_1(1) = z$, where $0 \le x \le y \le z \le S_1(e) \le 1$. Also, define fuzzy set S_2 of L by $S_2(0) = S_2(b) = t$, $S_2(a) = s$ and $S_2(1) = w$, where $0 \le t \le s \le w \le S_2(e) \le 1$. It is easy to show that S_1 and S_2 are fuzzy convex subalgebras of L.

Remark 3.10. We notice that each fuzzy subalgebra of L may not be a fuzzy convex subalgebra of L. Consider the fuzzy subalgebra S of L in Example 3.7 which is not a fuzzy convex subalgebra of L.

Theorem 3.11. *Let S be a fuzzy subset of L. S is a fuzzy (convex) subalgebra of L iff for all $t \in [0,1]$, S_t is either empty or a (convex) subalgebra of L.*

Proof. It is easy to show that S is a fuzzy subalgebra of L iff for all $t \in [0,1]$, S_t is either empty or a fuzzy subalgebra of L.
Suppose that S is a fuzzy convex subalgebra of L and for $t \in [0,1]$, S_t is not empty. Let $a, b \in S_t$ and $a \le c \le b$. Then there exists $t \le \gamma \le t$ such that $c \in S_\gamma$, i.e., $c \in S_t$.

Conversely, suppose that for all $t \in [0,1]$, S_t is either empty or a convex subalgebra of L. It is easy to show that S is a fuzzy subalgebra of L. Let $a \in S_\alpha$, $b \in S_\beta$, $a \le c \le b$. Without loss of generality, assume that $\alpha \le \beta$. Then $b \in S_\beta \subseteq S_\alpha$. Since S_α is a convex subalgebra of L, $c \in S_\alpha$, i.e., $S(c) \ge \alpha$. Consider the following cases:
Case 1: $S(c) \le \beta$. Put $\gamma = S(c)$.
Case 2: $S(c) > \beta$. Put $\gamma = \beta$. Therefore $S(c) > \gamma = \beta$ and $\alpha \le \gamma \le \beta$.
Hence $c \in S_\gamma$ where $\alpha \le \gamma \le \beta$. □

Corollary 3.12. *Let S be a non-empty subset of L. S is a convex subalgebra of L iff χ_S is a fuzzy convex subalgebra of L, where χ_S is the characteristic function of S.*

Theorem 3.13. *Let $\{S_\alpha\}_{\alpha \in I}$ be an arbitrary family of fuzzy convex subalgebras of L. Then $\bigcap\limits_{\alpha \in \Gamma} S_\alpha$ is a fuzzy convex subalgebra of L.*

Proof. The proof is routine. □

Remark 3.14. Let $\{S_\alpha\}_{\alpha \in I}$ be an arbitrary family of fuzzy convex subalgebras of L. Then $\bigcup\limits_{\alpha \in \Gamma} S_\alpha$ may not be a fuzzy convex subalgebra of L. See the following example:

Example 3.15. Consider the fuzzy convex subalgebras S_1 and S_2 in Example 3.9 and suppose that $0 < x < t < s < y < w < z < 1$. Then we get

$(S_1 \cup S_2)(0) = t$, $(S_1 \cup S_2)(a) = s$, $(S_1 \cup S_2)(b) = y$ and $(S_1 \cup S_2)(1) = z$.
We have $(S_1 \cup S_2)(a * b) = (S_1 \cup S_2)(0) = t$, $min\{(S_1 \cup S_2)(a), (S_1 \cup S_2)(b)\} = y$ and $t < y$. Hence $S_1 \cup S_2$ is not a fuzzy convex subalgebra of L.

Notation: The set of all fuzzy convex subalgebras of L is denoted by $FSub_c(L)$.

Clearly $FSub_c(L)$ is a lattice, because if $S_1, S_2 \in FSub_c(L)$, then $S_1 \vee S_2$ (i.e, the intersection of all fuzzy convex subalgebras containing $S_1 \cup S_2$) is the least upper bound of S_1 and S_2. Also $S_1 \cap S_2 \in FSub_c(L)$ is the greatest lower bound of S_1 and S_2. Since we can replace the set $\{S_1, S_2\}$ by an arbitrary family of fuzzy convex subalgebras, the lattice $(FSub_c(L), \vee, \cap)$ is a complete lattice.

Lemma 3.16. *Let S be a fuzzy convex subalgebra of L. Then*

$$min\{S(x \wedge e), S((y \wedge e) : (x \wedge e))\} \leq S(y \wedge e),$$

for all $x, y \in L$.

Proof. Let $S(x \wedge e) = \alpha$ and $S((y \wedge e) : (x \wedge e)) = \beta$. Since

$$(x \wedge e) * ((y \wedge e) : (x \wedge e)) \leq y \wedge e \leq (y \wedge e) : (x \wedge e)$$

and S is a fuzzy subalgebra of L, we have
$\min(\alpha, \beta) = min\{S(x \wedge e), S((y \wedge e) : (x \wedge e))\} \leq S((x \wedge e) * [(y \wedge e) : (x \wedge e)])$.
Hence $(x \wedge e) * ((y \wedge e) : (x \wedge e)) \in S_{\min(\alpha,\beta)}$. Since S is a fuzzy convex subalgebra of L, there exists γ between $\min(\alpha, \beta)$ and β such that $y \wedge e \in S_\gamma$, i.e., $S(y \wedge e) \geq \gamma \geq \min(\alpha, \beta) = min\{S(x \wedge e), S((y \wedge e) : (x \wedge e))\}$. □

Theorem 3.17. *Let S be a fuzzy convex subalgebra of L. For any $x, y, z \in L$, the following properties hold:*
(1) *if $x \leq y$, then $S(x \wedge e) \leq S(y \wedge e)$,*
(2) *if $S((y \wedge e) : (x \wedge e)) = S(e)$, then $S(x \wedge e) \leq S(y \wedge e)$,*
(3) *$S((y : x) \wedge e) \leq S([(y * z) : (x * z)] \wedge e)$,*
(5) *$S((y : x) \wedge e) \leq S([(z : x) : (z : y)] \wedge e)$,*
(6) *$S((y : x) \wedge e) \leq S([(y : z) : (x : z)] \wedge e)$,*
(7) *$S((y : x) \wedge e) \leq S([(y \wedge z) : (x \wedge z)] \wedge e)$,*
(8) *$S((y : x) \wedge e) \leq S([(y \vee z) : (x \vee z)] \wedge e)$,*
(9) *$S((y : x) \wedge e) \leq S([y : (x \vee y)] \wedge e)$.*

Proof. (1) Since $x \leq y$, we have $x \wedge e \leq y \wedge e \leq e$. But S is a fuzzy convex subalgebra of L, hence there exists γ between $S(x \wedge e)$ and $S(e)$ such that $y \wedge e \in S_\gamma$, i.e., $S(x \wedge e) \leq S(y \wedge e)$.
(2) It is clear by Lemma 3.14.
(3) By Theorem 2.4 part (8) and (10)

$$\begin{aligned}((y : x) \wedge e) * ((z : y) \wedge e) &\leq [((y : x) \wedge e) * (z : y)] \wedge [((y : x) \wedge e) * e] \\ &\leq [(y : x) * (z : y)] \wedge (z : y) \wedge e \\ &\leq (z : x) \wedge e.\end{aligned}$$

Hence by part (1) and Definition 3.6 part (3), we get that

$$min\{S((y:x)\wedge e), S((z:y)\wedge e)\} \leq S(((y:x)\wedge e)*((z:y)\wedge e)) \\ \leq S((z:x)\wedge e).$$

(4) Since $(y:x)\wedge e \leq [(y*z):(x*z)]\wedge e$ by Theorem 2.4 part (3), we have

$$S((y:x)\wedge e) \leq S([(y*z):(x*z)]\wedge e),$$

by part (1).
(5) We have $((y:x)\wedge e)*(z:y) \leq ((y:x)*(z:y))\wedge(z:y) \leq (z:x)$. So $(y:x)\wedge e \leq ((z:x):(z:y))\wedge e$. Hence

$$S((y:x)\wedge e) \leq S([(z:x):(z:y)]\wedge e),$$

by part(1).
(6) The proof is similar to the part(5).
(7) Since $(y\wedge z)*((x:y)\wedge e) \leq [y*((x:y)\wedge e)]\wedge[z*((x:y)\wedge e)] \leq x\wedge z$, then $((x:y)\wedge e) \leq (x\wedge z):(y\wedge z)$. So we have$((x:y)\wedge e) \leq ((x\wedge z):(y\wedge z))\wedge e$. By part(1), we conclude that $S((y:x)\wedge e) \leq S([(y\wedge z):x\wedge z]\wedge e)$.
(8) We have $(y\vee z)*((x:y)\wedge e) = [y*((x:y)\wedge e)]\vee[z*((x:y)\wedge e)] \leq x\vee z$ by Theorem 2.4 part (7). Then $((x:y)\wedge e) \leq ((x\vee z):(y\vee z))\wedge e$. Hence $S((y:x)\wedge e) \leq S([(y\vee z):(x\vee z)]\wedge e)$, by part(1).
(9) Use Theorem 2.4 part(2) and (4), then $x*((y:x)\wedge e) \leq x*(y:x) \leq y$ and $y*((y:x)\wedge e) \leq y*e \leq y$.
Hence

$$x \leq y:((x:y)\wedge e) \quad \text{and} \quad y \leq y:((x:y)\wedge e).$$

Therefore $y\vee x \leq y:((y:x)\wedge e)$. So $(y:x)\wedge e \leq y:(y\vee x)$. Hence

$$S((y:x)\wedge e) \leq S([y:(x\vee y)]\wedge e).$$

by part (1). □

Definition 3.18. Let S be a fuzzy convex subalgebra of L. Fuzzy relation θ_S on L which is defined by

$$\theta_S(x,y) = min\{S((y:x)\wedge e), S((x:y)\wedge e)\}$$

is called the *fuzzy relation induced* by S.

Theorem 3.19. *Let S be a fuzzy convex subalgebra of L. Then θ_S is a fuzzy congruence relation on L.*

Proof. The proof follows from Theorem 3.17. □

Theorem 3.20. *Let S be a fuzzy convex subalgebra of L and θ_S be a fuzzy congruence relation induced by S and $t \in Im\theta$. Then $(\theta_S)_t$ is the congruence on L induced by S_t, i.e., $(\theta_S)_t = \theta_{S_t}$, where*

$$\theta_{S_t} = \{(x,y) : (y:x)\wedge e \in S_t, (x:y)\wedge e \in S_t\}.$$

Proof. Let $(x,y) \in (\theta_S)_t$. Therefore $\theta_S(x,y) \geq t$. So we have

$$min\{S((x:y)\wedge e), S((y:x)\wedge e)\} \geq t.$$

Thus $(x : y) \wedge e \in S_t$ and $(y : x) \wedge e \in S_t$. Hence $(x, y) \in \theta_{S_t}$. Thus $(\theta_S)_t \subseteq \theta_{S_t}$. By reversing the above arguments we get, $\theta_{S_t} \subseteq (\theta_S)_t$. Hence$(\theta_S)_t = \theta_{S_t}$. □

Definition 3.21. Let θ be a fuzzy congruence relation on L. Then the fuzzy subset S_θ which is defined by

$$S_\theta(x) = \theta(x, e)$$

is called the *fuzzy subset induced* by θ.

Theorem 3.22. *Let θ be a fuzzy congruence relation on L. Then S_θ is a fuzzy convex subalgebra of L.*

Proof. For all $x, y \in L$

$$\begin{aligned} S_\theta(y : x) = \theta(e, y : x) = \theta(e : e, y : x) & \\ \geq min\{\theta(e, x), \theta(e, y)\} &= min\{S_\theta(x), S_\theta(y)\}. \end{aligned}$$

The proof of the other conditions of Definition 3.6 is similar. Hence S_θ is a fuzzy subalgebra of L.
Let $a \in_\alpha S_\theta$, $b \in_\beta S_\theta$ and $a \leq c \leq b$. Then $S_\theta(a) = \theta(e, a) \geq \alpha$ and $S_\theta(b) = \theta(e, b) \geq \beta$.
If $\beta \geq \alpha$, then

$$\begin{aligned} \theta(a, c) = \theta(a \wedge b \wedge c, b \wedge c) &\geq min\{\theta(a \wedge b,\, b), \theta(c, c)\} \\ &\geq min\{\theta(a,\, b), \theta(b, b)\} \\ &\geq min\{\theta(a,\, e), \theta(e, b)\} = min\{\alpha, \beta\}. \end{aligned}$$

Hence

$$\theta(e, c) \geq min\{\theta(e, a), \theta(a, c)\} \geq \alpha.$$

If $\alpha \geq \beta$, then

$$\begin{aligned} \theta(b, c) = \theta(a \vee b \vee c, a \vee c) &\geq min\{\theta(a \vee b, a), \theta(c, c)\} \\ &\geq min\{\theta(b,\, a), \theta(a, a)\} \\ &\geq min\{\theta(a,\, e), \theta(e, b)\} = min\{\alpha, \beta\}. \end{aligned}$$

Hence

$$\theta(e, c) \geq min\{\theta(e,\, b), \theta(b, c)\} \geq \beta.$$

Therefore $S_\theta(c) = \theta(e, c) \geq min\{\alpha, \beta\} = \gamma$, where γ is between α and β. Thus S_θ is a fuzzy convex subalgebra of L. □

Theorem 3.23. *Let θ be a fuzzy congruence relation on L and let S_θ be a fuzzy convex subalgebra induced by θ. Let $t \in Im\theta$.Then$(S_\theta)_t$ is the convex subalgebra induced by θ_t, i.e., $(S_\theta)_t = S_{\theta_t}$ where $S_{\theta_t} = \{a \in L : (a, e) \in \theta_t\}$.*

Proof. Let $a \in (S_\theta)_t$. Then $S_\theta(a) \geq t$ and we have $\theta(e, a) \geq t$. Hence $a \in S_{\theta_t}$ and $(S_\theta)_t \subseteq S_{\theta_t}$. By reversing the above arguments, we get $S_{\theta_t} \subseteq (S_\theta)_t$. □

Theorem 3.24. *Let S be a fuzzy convex subalgebra of L. Then $S_{\theta_S} = S$.*

Proof. Let $x \in L$. Since S is a fuzzy subalgebra, we have

$$\begin{aligned} S_{\theta_S}(x) = \theta_S(e,x) &= min\{S((x:e)\wedge e), S((e:x)\wedge e)\} \\ &= min\{S(e\wedge x), S((e:x)\wedge e)\} \geq S(x) \end{aligned}$$

by Definition 3.6.
Conversely, we will show that

$$S(x) \geq min\{S((e:x)\wedge e), S(x\wedge e)\}.$$

Suppose that $x \in L$. Put $h = (e:x)\wedge(x\wedge e)$. Clearly $h \leq x \leq e:h$. Let $S(h) = \alpha$ and $S(e:h) = \beta$. Then $\alpha, \beta \geq min\{S(x\wedge e), S((e:x)\wedge e)\}$. Since S is a fuzzy convex subalgebra of L, there exists a γ between α and β such that $S(x) \geq \gamma \geq min\{\alpha,\beta\} \geq min\{S(x\wedge e), S((e:x)\wedge e)\}$. Hence $S(x) \geq S_{\theta_S}(x)$. □

Lemma 3.25. *Let θ be a fuzzy congruence relation on L. Then*

$$\theta(x\vee y, y) \geq S_\theta((y:x)\wedge e),$$

for all $x, y \in L$.

Proof. We have

$$\begin{aligned} \theta(x\vee y, y) &= \theta(x\vee y, y\vee(x*((y:x)\wedge e))) \\ &\geq min\{\theta(y,y), \theta(x, x*((y:x)\wedge e))\} \\ &\geq min\{\theta(x,x), \theta(e,(y:x)\wedge e)\} \\ &\geq S_\theta((y:x)\wedge e). \end{aligned}$$

□

Theorem 3.26. *Let θ be a fuzzy congruence relation on L. Then $\theta_{S_\theta} = \theta$.*

Proof. Let $x, y \in L$. Then

$$\begin{aligned} \theta_{S_\theta}(x,y) &= min\{S_\theta((y:x)\wedge e), S_\theta((x:y)\wedge e)\} \\ &= min\{\theta(e,(y:x)\wedge e), \theta(e,(x:y)\wedge e)\} \\ &= min\{\theta(e\wedge(x:x),(y:x)\wedge e), \theta(e\wedge(y:y),(x:y)\wedge e)\} \\ &\geq min\{\theta(e,e), \theta(y,y), \theta(x,x), \theta(x,y)\} = \theta(x,y). \end{aligned}$$

Conversely, we have

$$\begin{aligned} \theta(x,y) &\geq min\{\theta(x, x\vee y), \theta(x\vee y, y)\} \\ &\geq min\{S_\theta((y:x)\wedge e), S_\theta((x:y)\wedge e)\} \\ &= \theta_{S_\theta}(x,y) \end{aligned}$$

□

Theorem 3.27. *(Correspondence theorem) There is a bijection between the set of all fuzzy convex subalgebras of L and the set of all fuzzy congruence relations on L.*

Proof. Define the function ψ as follows:

$$\begin{aligned} \psi : FCon(L) &\to FSub_c(L) \\ \theta &\mapsto S_\theta \end{aligned}$$

Then by Theorem 3.24 and 3.26, ψ is a bijection. □

Definition 3.28. Let θ be a fuzzy congruence relation on L and $x \in L$. Define the fuzzy set $[\theta]_x$ by $[\theta]_x(y) = \theta(x, y)$. The fuzzy set $[\theta]_x$ is called a *fuzzy congruence class* of x by θ in L.

Theorem 3.29. *If S is a fuzzy convex subalgebra of L, then*
(1) $[\theta_S]_x = [\theta_S]_y$ if and only if $S((y : x) \wedge e) = S((x : y) \wedge e) = S(e)$,
(2) $[\theta_S]_x = [\theta_S]_e$ if and only if $S(x) = S(e)$.

Proof. (1) If $[\theta_S]_x = [\theta_S]_y$, then $[\theta_S]_x(x) = [\theta_S]_y(x)$. So we have

$$S((x : x) \wedge e) = S(e) = min\{S((y : x) \wedge e), S((x : y) \wedge e)\}.$$

It follows that $S((y : x) \wedge e) = S((x : y) \wedge e) = S(e)$.

Conversely, suppose that$S((y : x) \wedge e) = S((x : y) \wedge e) = S(e)$. By Theorem 3.6 part (3) and Theorem 2.4 part (9) and (10), we can show that

$$min\{S((y : x) \wedge e), S((z : y) \wedge e)\} \leq S((z : x) \wedge e),$$
$$min\{S((x : y) \wedge e), S((z : x) \wedge e)\} \leq S((z : y) \wedge e).$$

By using assumption, we have

$$S((z : x) \wedge e) \leq S((z : y) \wedge e) \qquad \text{and} \qquad S((z : y) \wedge e) \leq S((z : x) \wedge e).$$

Therefore $S((z : x)\wedge e) = S((z : y)\wedge e)$. Similarly, we can show that $S((x : z)\wedge e) = S((y : z) \wedge e)$. Thus $[\theta_S]_x(z) = [\theta_S]_y(z)$ for all $z \in L$. Hence$[\theta_S]_x = [\theta_S]_y$.
(2) It follows from part (1) and Theorem 2.4 part(1). □

Theorem 3.30. *Let S be a fuzzy convex subalgebra of L. Define*

$$x \equiv_S y \qquad \text{if and only if} \qquad [\theta_S]_x = [\theta_S]_y.$$

Then $\equiv_S$ is a congruence relation on L.

Proof. The proof follows from Theorem 3.17. □

Definition 3.31. Let S be a fuzzy convex subalgebra of L, θ_S be the fuzzy congruence relation induced by S. The set of all fuzzy congruence class is denoted by $\frac{L}{\theta_S}$. On this set, we define

$$[\theta_S]_x \vee [\theta_S]_y = [\theta_S]_{x\vee y} \quad , \quad [\theta_S]_x \wedge [\theta_S]_y = [\theta_S]_{x\wedge y}$$
$$[\theta_S]_x * [\theta_S]_y = [\theta_S]_{x*y} \quad , \quad [\theta_S]_y : [\theta_S]_x = [\theta_S]_{y:x},$$

for all $x, y \in L$. Then $\frac{L}{\theta_S}$ is called the *fuzzy quotient algebra* respect to the fuzzy convex subalgebra S.

Theorem 3.32. *Let S be a fuzzy convex subalgebra of L. Then $\frac{L}{\theta_S} = (\frac{L}{\theta_S}, \wedge, \vee, *, :, [\theta_S]_e)$ is a commutative residuated lattice.*

Proof. By Theorem 3.28, we have $[\theta_S]_x = [\theta_S]_y$ and $[\theta_S]_z = [\theta_S]_w$ if and only if $x \equiv_S y$ and $z \equiv_S w$. Since $\equiv_S$ is the congruence relation on L by Theorem 3.28, all the above operations are well defined. It is easy to show that $(\frac{L}{\theta_S}, \wedge, \vee)$ is a lattice, $*$ is commutative, associative and has $[\theta_S]_e$ as an identity. The operation $\vee$ defines a relation $\leq$ on $\frac{L}{\theta_S}$ by

$$[\theta_S]_x \leq [\theta_S]_y \qquad \text{if and only if} \qquad [\theta_S]_{x\vee y} = [\theta_S]_y \qquad \text{for all } x, y \in L.$$

This relation is a partial order on $\frac{L}{\theta_S}$. Using Theorem 3.27, we see that

$$[\theta_S]_x \leq [\theta_S]_y \quad \text{if and only if} \quad (y:x) \wedge e \in S_{S(e)} \quad \text{for all } x, y \in L \quad (1).$$

Now, we will show that $[\theta_S]_z \leq [\theta_S]_y : [\theta_S]_x$ if and only if $[\theta_S]_z * [\theta_S]_x \leq [\theta_S]_y$ for all $x, y, z \in L$. We have

$$\begin{aligned}
[\theta_S]_z \leq [\theta_S]_y : [\theta_S]_x \quad &\Longleftrightarrow \quad [\theta_S]_z \leq [\theta_S]_{y:x} && \text{by Definition 3.31} \\
&\Longleftrightarrow \quad ((y:x):z) \wedge e \in S_{S(e)} && \text{by (1)} \\
&\Longleftrightarrow \quad (y:(z*x)) \wedge e \in S_{S(e)} && \text{by Theorem 2.4(5)} \\
&\Longleftrightarrow \quad [\theta_S]_{z*x} \leq [\theta_S]_y && \text{by (1)} \\
&\Longleftrightarrow \quad [\theta_S]_z * [\theta_S]_x \leq [\theta_S]_y && \text{by Definition 3.21.}
\end{aligned}$$

This completes the proof. □

Theorem 3.33. *Let S be a fuzzy convex subalgebra of L and $\frac{L}{\theta_S}$ be the corresponding quotient algebra. Then the map $h : L \to \frac{L}{\theta_S}$ defined by $h(x) = [\theta_S]_x$ for all $x \in L$ is a surjective homomorphism and $\ker(h) = S_{S(e)}$, where $\ker(h) = \{x \in L : h(x) = [\theta_S]_e\}$. Moreover, $\frac{L}{\theta_S}$ is isomorphic to the commutative residuated lattice $\frac{L}{\equiv_S}$.*

Proof. It follows from Definition 3.31 and Theorem 3.32, that h is surjective homomorphism. Now, we show that $\ker(h) = S_{S(e)}$.
$x \in \ker(h)$ if and only if $[\theta_S]_x = h(x) = [\theta_S]_e$ if and only if $S(x) = S(e)$ (By Theorem 3.27 part (2)) if and only if $x \in S_{S(e)}$.
By part (1) and (2), $\frac{L}{\theta_S}$ is isomorphic to the commutative residuated lattice $\frac{L}{\equiv_S}$. □

4. Fuzzy Convex Subalgebras of *ICRL*

In this section, we consider the class of integral commutative residuated lattice. Suppose that L is an integral commutative residuated lattice.

Theorem 4.1. *Let S be a fuzzy convex subalgebra of L. For any $x, y, z \in L$, the following hold:*
(1) *if $x \leq y$, then $S(x) \leq S(y)$,*
(2) *if $S(y:x) = S(1)$, then $S(x) \leq S(y)$,*
(3) *$S(x*y) = \min\{S(x), S(y)\}$,*
(4) *$S(x \wedge y) = \min\{S(x), S(y)\}$.*

Proof. (1) If $x \leq y$, then $S(x) = S(x \wedge 1) \leq S(y \wedge 1) = S(y)$ by Theorem 3.17 part(1).
(2) It is clear by Theorem 3.17 part(2).
(3) Since S is a fuzzy subalgebra of L, we have $S(x*y) \geq \min\{S(x), S(y)\}$. On the other hand, we have $x*y \leq x, y$ by Theorem 2.8 part(2). Hence $S(x*y) \leq S(x), S(y)$ by part (1). Thus $S(x*y) \leq \min\{S(x), S(y)\}$.
(4) The proof is similar to the part (3). □

Definition 4.2. Let f be a fuzzy set of L. f is called a *fuzzy filter* of L iff
$(fF1)$ $f(1) \geq f(x)$,
$(fF2)$ $f(y) \geq min\{f(x), f(y:x)\}$,
for all $x, y \in L$.

Theorem 4.3. *A fuzzy set f of L is a fuzzy filter iff*
$(fF1)'$ *if $x \leq y$, then $f(x) \leq f(y)$,*
$(fF2)'$ $min\{f(x), f(y)\} \leq f(x*y)$,
for all $x, y \in L$.

Proof. See ([14]). □

Example 4.4. Let $L = \{0, a, d, 1\}$ with $0 < a, d < 1$ and elements a, d are incomparable. We define

$*$	0	a	d	1
0	0	0	0	0
a	0	a	0	a
d	0	0	d	d
1	0	a	d	1

$:$	0	a	d	1
0	1	1	1	1
a	d	1	d	1
d	a	a	1	1
1	0	a	d	1

and so L is an integral commutative residuated lattice. Define fuzzy set f_1 in L by $f_1(0) = f_1(a) = x$, $f_1(d) = y$ and $f_1(1) = z$ where $0 \leq x \leq y \leq z \leq 1$. Also, define fuzzy set f_2 in L by $f_2(0) = f_2(d) = t$, $f_2(a) = s$ and $f_2(1) = w$ where $0 \leq t \leq s \leq w \leq 1$. It is easy to show thatf_1 and f_2 are fuzzy filters of L.

Theorem 4.5. *Let S be a fuzzy set of L. S is a fuzzy convex subalgebra of L iff S is a fuzzy filter of L.*

Proof. Let S be a fuzzy convex subalgebra of L. Then
(1) if $x \leq y$, then $S(x) \leq S(y)$,
(2) $S(x*y) = \min\{S(x), S(y)\}$,
by Theorem 5.1 part(1)and (3). Hence S is a fuzzy filter of L by Theorem 5.3.

Conversely, let S be a fuzzy filter of L. First, we show that S is a fuzzy subalgebra of L.
(1) $S(1) \geq S(x)$, by $(fF1)$,
(2) $S(y:x) \geq S(y) \geq min\{S(x), S(y)\}$, by Theorem 2.8 part(4) and $(fF1)'$,
(3) $S(x*y) \geq min\{S(x), S(y)\}$, by $(fF2)'$,
(4) $S(x \wedge y) \geq S(x*(y:x)) \geq min\{S(x), S(y)\}$ by Theorem 2.8 part(4) and$(fF1)'$,
(5) $S(x \vee y) \geq min\{S(x), S(y)\}$, by $(fF1)'$.
for all $x, y \in L$. Now, we show that S is a fuzzy convex subalgebra of L. Suppose that $a \in S_\alpha$, $b \in S_\beta$ and $a \leq c \leq b$. Then

$min\{\alpha, \beta\} \leq min\{S(a), S(b)\} \leq S(b*(a:b)) \leq S(b*(c:b)) \leq S(c)$.

Define $\gamma = min\{\alpha, \beta\}$. Hence there exists a γ between α and β such that $c \in S_\gamma$. □

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