ORDERED SEMIGROUPS CHARACTERIZED BY THEIR INTUITIONISTIC FUZZY BI-IDEALS

A. KHAN, Y. B. JUN AND M. SHABIR

ABSTRACT. Fuzzy bi-ideals play an important role in the study of ordered semigroup structures. The purpose of this paper is to initiate and study the intiuitionistic fuzzy bi-ideals in ordered semigroups and investigate the basic theorem of intuitionistic fuzzy bi-ideals. To provide the characterizations of regular ordered semigroups in terms of intuitionistic fuzzy bi-ideals and to discuss the relationships of left(resp. right and completely regular) ordered semigroups in terms intuitionistic fuzzy bi-ideals.

1. Introduction

The theory of fuzzy sets proposed by Zadeh [27] in 1965, has achieved a great success in various fields. Also several higher order fuzzy sets, introduced by Atanassov (see [1], [2] and [3]) have been found to be highly useful to deal with vagueness. In [4], Borzooei and Jun introduced the concept of an intuitionistic fuzzy hyper BCK-ideal of a hyper BCK-algebra and established some related properties using this notion. Rafi and Noorani used this notion and introduced the concept of intuitionistic fuzzy contraction mappings and proved a fixed point theorem in intuitionistic fuzzy metric spaces (see [22]). In [11], Hur et al. introduced some intuitionistic fuzzy congruences. Hosseini et al. introduced intuitionistic fuzzy metric and normed spaces and proved several theorems about completeness, compactness and weak convergence in these spaces [10]. Gau and Buehre in [9], presented the concept of vague sets. But, Burillo and Bustince in [5], have shown that the notion of vague sets coincides with that of intuitionistic fuzzy sets. Szmidt and Kacprzyk (see [26]) proposed a non-probabilistic type entropy measures for intuitionistic fuzzy sets. De et al. [7] studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of intuitionistic fuzzy set theory. Dengfeng and Chunfian [8] introduced the concept of the degree of similarity between intuitionistic fuzzy sets, which may be finite or continuous, and gave corresponding proofs of these similarity measure and discussed applications of the similarity measures between intuitionistic fuzzy sets to pattern recognition problems. Intuitionistic fuzzy sets have many applications in mathematics, Davvaz et al. [6], applied this concept in H_v-modules. They introduced the notion of an intuitionistic fuzzy H_v-submodule

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of an H_v-module and studied the related properties. In [19], [20], Kim and Jun introduced the concept of intuitionistic fuzzy (interior) ideals of semigroups. In [24], Shabir and Khan gave the concept of an intuitionistic fuzzy interior ideal of ordered semigroups and characterized different classes of ordered semigroups in terms of intuitionistic fuzzy interior ideals. They also gave the concept of an intuitionistic fuzzy generalized bi-ideal in [25] and discussed different calsses of ordered semigroups in terms of intituitionistic fuzzy generalized bi-ideals. Fuzzy bi-ideals in ordered semigroups were introduced by Kehayopulu and Tsingelis in [15] and studied the basic properties of ordered semigroups in terms of fuzzy bi-ideals.

In this paper, we characterize regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of intuitionistic fuzzy bi-ideals. In this respect, we prove that: An ordered semigroup S is regular, left and right simple if and only if every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S is a constant mapping. We also prove that S is completely regular if and only if for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$ for every $a \in S$. We prove that an ordered semigroup S is a semilattice of left and right simple semigroups if and only if for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have $\mu_A(a) = \mu_A(a^2)$, $\gamma_A(a) = \gamma_A(a^2)$ and $\mu_A(ab) = \mu_A(ba)$, $\gamma_A(ab) = \gamma_A(ba)$ for every $a, b \in S$. Next we characterize regular ordered semigroups in terms of intuitionistic fuzzy bi-ideals of S. We prove that an ordered semigroup S is a regular ordered semigroup if and only if for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have $A = A \circ 1_\sim \circ A$.

2. **Preliminaries**

An ordered semigroup is an ordered set S at the same time a semigroup such that

$$(\forall a, b, x \in S)\ (a \leq b \Longrightarrow xa \leq xb \text{ and } ax \leq bx).$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $A \subseteq S$, we denote

$$(A] := \{t \in S | t \leq h \text{ for some } h \in A\}.$$

For $A, B \subseteq S$ we denote, $AB := \{ab | a \in A, b \in B\}$.

For subsets A and B of an ordered semigroup S we have $A \subseteq (A], (A](B] \subseteq (AB], ((A]] = (A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is called a *right* (resp. *left*) *ideal* of S if: (1) $AS \subseteq A$ (resp. $SA \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$ [9]. If A is both right and left ideal of S, then it is called a *two-sided ideal* or simply an *ideal* of S.

Let $(S, \cdot, \leq)$ be an ordered semigroup, $\emptyset \neq A \subseteq S$. Then A is called a *subsemigroup* of S if $A^2 \subseteq A$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *bi-ideal* of S if [15]:

(i) $A^2 \subseteq A$,

(ii) $ASA \subseteq A$,

(iii) If $a \in A$ and $S \ni b \leq a$ then $b \in A$.

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the set $(A \cup A^2 \cup ASA]$ is the bi-ideal of S generated by A. In particular, if $A = \{x\}(x \in S)$, then we write $(x \cup x^2 \cup xSx]$, instead of $(\{x\} \cup \{x^2\} \cup \{x\}S\{x\}][15]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. By a *fuzzy subset* μ of S, we mean a *mapping* $\mu : S \longrightarrow [0, 1]$.

Let μ and λ be fuzzy subsets of an ordered semigroup S, then the fuzzy subsets $\mu \cap \lambda$ and $\mu \cup \lambda$ of S are defined as follows:

$$(\forall x \in S)((\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x))$$
$$(\forall x \in S)((\mu \cup \lambda)(x) = \mu(x) \vee \lambda(x)).$$

A fuzzy subset μ of S is called a *fuzzy subsemigroup* of S if [15]:

$$(\forall x, y \in S)(\mu(xy) \geq \min\{\mu(x), \mu(y)\}).$$

A fuzzy subsemigroup μ of S is called a *fuzzy bi-ideal* of S if [15]:

(1) $(\forall x, y \in S)(x \leq y \Longrightarrow \mu(x) \geq \mu(y))$ and

(2) $(\forall x, y, z \in S)(\mu(xyz) \geq \min\ \{\mu(x), \mu(z)\})$.

3. Intuitionistic Fuzzy Bi-ideals

As an important generalization of the notion of fuzzy sets in S, Atanassov [1], introduced the concept of an intuitionistic fuzzy set (IFS for short) defined on a non-empty set S as objects having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\}$,

where the functions $\mu_A : S \longrightarrow [0, 1]$ and $\gamma_A : S \longrightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in S$ to the set A, respectively and $0 \leq \mu(x) + \gamma(x) \leq 1$, for each $x \in S$. For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \gamma_A \rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\}$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. An IFS $A = \langle \mu_A, \gamma_A \rangle$ in S is called an *intuitionistic fuzzy subsemigroup* of S (cf. [12]) if:

(1) $(\forall x, y \in S)(\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y))$,

(2) $(\forall x, y \in S)(\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y))$.

Definition 3.1. An *intuitionistic fuzzy subsemigroup* $A = \langle \mu_A, \gamma_A \rangle$ of S is called an *intuitionistic fuzzy bi-ideal* of S (cf. [12]) if:

(1) $(\forall x, y \in S)(x \leq y \Longrightarrow \mu_A(x) \geq \mu_A(y), \gamma_A(x) \leq \gamma_A(y))$,

(2) $(\forall x, y, z \in S)(\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z))$,

(3) $(\forall x, y, z \in S)(\gamma_A(xyz) \leq \gamma_A(x) \vee \gamma_A(z))$.

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the *intuitionistic characteristic function* $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is defined as

$$\mu_{\chi_A} : S \longrightarrow [0, 1] | x \longrightarrow \mu_{\chi_A}(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A \end{cases}$$

and

$$\gamma_{\chi_A} : S \longrightarrow [0, 1] | x \longrightarrow \gamma_{\chi_A}(x) := \begin{cases} 0 \text{ if } x \in A, \\ 1 \text{ if } x \notin A \end{cases} .$$

We denote by $IF(S)$ the set of all intuitionistic fuzzy sets in an ordered semigroup S. For *IFSs* $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ of S we define the order relation "$\subseteq$ " as follows:

$A \subseteq B$ if and only if $\mu_A \preceq \mu_B, \gamma_A \succeq \gamma_B$ if and only if $\mu_A(x) \leq \mu_B(x)$, $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in S$.

Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be any two $IFSs$ in an ordered semigroup S. Then (1) $A = B \iff A \subseteq B$ and $B \subseteq A$

(2) $A^c = \langle \gamma_A, \mu_A \rangle$

(3) $A \cap B = \langle \mu_{A \wedge B}, \gamma_{A \vee B} \rangle$ and

(4) $0_\sim = \langle 0, 1 \rangle, 1_\sim = \langle 1, 0 \rangle$.

Lemma 3.2. *(cf. [12]). Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is a bi-ideal of S if and only if the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy bi-ideal of S.*

An ordered semigroup $(S, \cdot, \leq)$ is called *regular*, if for every $a \in S$ there exists $x \in S$, such that $a \leq axa$ [13].

Equivalent definitions:

(1) $(\forall a \in S)(a \in (aSa])$.

(2) $(\forall A \subseteq S)(A \subseteq (ASA])$.

An ordered semigroup $(S, \cdot, \leq)$ is *left* (resp. *right*) *regular* (cf. [17]), if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2 x$).

Equivalent definitions:

(1) $(\forall a \in S)(a \in (Sa^2]$(resp. $a \in (a^2 S]$)).

(2) $(\forall A \subseteq S)(A \subseteq (SA^2]$(resp. $A \subseteq (A^2 S]$)).

An ordered semigroup S is called *completely regular* if it is regular, left regular and right regular (see [15]).

Lemma 3.3. *(cf. [15]) An ordered semigroup S is completely regular if and only if $A \subseteq (A^2 S A^2]$ for every $A \subseteq S$. Equivalently, if $a \in (a^2 S a^2]$ for every $a \in S$.*

Theorem 3.4. *An ordered semigroup $(S, \cdot, \leq)$ is completely regular if and only if for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have*

$$\mu_A(a) = \mu_A(a^2) \text{ and } \gamma_A(a) = \gamma_A(a^2) \text{ for every } a \in S.$$

Proof. $\Longrightarrow$. Assume that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S. Let $a \in S$. Since S is completely regular, then $a \in (a^2 S a^2]$. That is $a \leq a^2 x a^2$ for some $x \in S$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S, we have

$$\begin{aligned} \mu_A(a) &\geq \mu_A(a^2 x a^2) \geq \mu_A(a^2) \wedge \mu_A(a^2) \\ &= \mu_A(a^2) \geq \mu_A(a) \wedge \mu_A(a) = \mu_A(a), \end{aligned}$$

and

$$\begin{aligned} \gamma_A(a) &\leq \gamma_A(a^2 x a^2) \leq \gamma_A(a^2) \vee \gamma_A(a^2) \\ &= \gamma_A(a^2) \leq \gamma_A(a) \vee \gamma_A(a) = \gamma_A(a). \end{aligned}$$

Hence $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$.

$\Longleftarrow$. Let $A(a^2)$ be a bi-ideal of S generated by a^2, i.e., $A(a^2) = (a^2 \cup a^4 \cup a^2Sa^2]$. By Lemma 1, the intuitionistic characteristic function $\chi_{A(a^2)} = \langle \mu_{\chi_{A(a^2)}}, \gamma_{\chi_{A(a^2)}} \rangle$ of $A(a^2)$ defined by:

$$\mu_{\chi_{A(a^2)}} : S \longrightarrow [0,1] | x \longrightarrow \mu_{\chi_{A(a^2)}}(x) := \begin{cases} 1 \text{ if } x \in A(a^2) \\ 0 \text{ if } x \notin A(a^2), \end{cases}$$

$$\gamma_{\chi_{A(a^2)}} : S \longrightarrow [0,1] | x \longrightarrow \gamma_{\chi_{A(a^2)}}(x) := \begin{cases} 0 \text{ if } x \in A(a^2) \\ 1 \text{ if } x \notin A(a^2), \end{cases}$$

is an intuitionistic fuzzy bi-ideal of S. Then by hypothesis, we have

$$\mu_{\chi_{A(a^2)}}(a) = \mu_{\chi_{A(a^2)}}(a^2) \text{ and } \gamma_{\chi_{A(a^2)}}(a) = \gamma_{\chi_{A(a^2)}}(a^2).$$

Since $a^2 \in A(a^2)$, $\mu_{\chi_{A(a^2)}}(a^2) = 1$ and $\gamma_{\chi_{A(a^2)}}(a^2) = 0$. Then $\mu_{\chi_{A(a^2)}}(a) = 1$ and $\gamma_{\chi_{A(a^2)}}(a) = 0$ and hence $a \in A(a^2) = (a^2 \cup a^4 \cup a^2Sa^2]$. Thus

$$a \leq a^2 \text{ or } a \leq a^4 \text{ or } a \leq a^2xa^2 \text{ for some } x \in S.$$

If $a \leq a^2$ then

$$a \leq a^2 = aa \leq a^2a^2 = aa^2a \leq a^2a^2a^2 \in a^2Sa^2.$$

Similarly, in other cases we get $a \leq a^2va^2$ for some $v \in S$. Consequently, $a \in (a^2Sa^2]$ and by Lemma 2, S is completely regular. □

The intuitionisitc fuzzy set "$0_\sim$" (resp. "$1_\sim$") is the least (resp. the greatest) element in the set $(IF(S), \subseteq)$ (that is, $0_\sim \subseteq A$ and $A \subseteq 1_\sim$ for every $A \in IF(S)$). The IFS "$0_\sim$" is the zero element of $(IF(S), \circ, \subseteq)$ (that is, $A \circ 0_\sim = 0_\sim \circ A = 0_\sim$ and $0_\sim \subseteq A$ for every $A \in IF(S)$).

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup F of S is called a *filter* of S if:

(1) $(\forall a, b \in S)(ab \in F \Longrightarrow a \in F$ and $b \in F)$.

(2) $(\forall c \in S)(c \geq a \in F \Longrightarrow c \in F)$.

For $x \in S$, we denote by $N(x)$ the filter of S generated by x (that is the least filter with respect to inclusion relation containing x). $\mathcal{N}$ denotes the equivalence relation on S defined by $\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\}$. Let S be an ordered semigroup. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for each $a, b \in S$ (see [15]). If σ is a semilattice congruence on S then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$ (see [15]). An ordered semigroup S is called a *semilattice of left and right simple semigroups* if there exists a semilattice congruence σ on S such that the *σ-class* $(x)_\sigma$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$.

Equivalent definition:

There exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that

(1) $S_\alpha \cap S_\beta = \emptyset \ \ \forall \alpha, \beta \in Y, \quad \alpha \neq \beta$,

(2) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ $\forall \alpha, \beta \in Y$.

Lemma 3.5. (*cf.* [15]). *An ordered semigroup* $(S, \cdot, \leq)$ *is a semilattice of left and right simple semigroups if and only if for all bi-ideals* A, B *of* S, *we have*

$$(A^2] = A \text{ and } (AB] = (BA].$$

Theorem 3.6. *Let* S *be an ordered semigroup. Then the following are equivalent:*

(1) S is regular, left and right simple.

(2) Every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S is a constant mapping.

Proof. (1)$\Longrightarrow$(2). Assume that S is regular, left and right simple. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-ideal of S. We consider the set

$$E_\Omega := \{e \in S | e^2 \geq e\}$$

Then $E_\Omega \neq \emptyset$. In fact, Since S is regular, let $a \in S$. Then there exists $x \in S$ such that $a \leq axa$. Then

$$(ax)^2 = (axa)x \geq ax,$$

and so $ax \in E_\Omega$.

(1) Let $t \in E_\Omega$, we prove that $A = \langle \mu_A, \gamma_A \rangle$ is a constant mapping on E_Ω. That is, $\mu_A(e) = \mu_A(t)$ and $\gamma_A(e) = \gamma_A(t)$ for every $e \in E_\Omega$. In fact, Since S is left and right simple, and $t \in S$, we have $S = (St]$ and $S = (tS]$, since $e \in S$, we have $e \in (tS]$ and $e \in (St]$. Then $e \leq ts$ and $e \leq zt$ for some $s, z \in S$. Then

$$e^2 \leq t(sz)t$$

Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S, we have

$$\begin{aligned} \mu_A(e^2) &\geq \mu_A(t(sz)t) \geq \min\{\mu_A(t), \mu_A(t)\} = \mu_A(t), \\ \gamma_A(e^2) &\leq \gamma_A(t(sz)t) \leq \max\{\gamma_A(t), \gamma_A(t)\} = \gamma_A(t). \end{aligned}$$

On the other hand, since $e \in E_\Omega$, $e^2 \geq e$ and we have $\mu_A(e) \geq \mu_A(e^2), \gamma_A(e) \leq \gamma_A(e^2)$. Thus $\mu_A(e) \leq \mu_A(t), \gamma_A(e) \leq \gamma_A(t)$. In a similar way we can prove that $\mu_A(t) \leq \mu_A(e), \gamma_A(t) \leq \gamma_A(e)$. Thus $\mu_A(t) = \mu_A(e), \gamma_A(t) = \gamma_A(e)$.

(2) Now, we prove that $A = \langle \mu_A, \gamma_A \rangle$ is a constant mapping on S. That is, $\mu_A(a) = \mu_A(t)$ and $\gamma_A(a) = \gamma_A(t)$ for every $a \in S$. In fact, Since S is regular there exists $x \in S$ such that $a \leq axa$. Then

$$(ax)^2 = (axa)x \geq ax \text{ and } (xa)^2 = x(axa) \geq xa$$

and hence ax and $xa \in E_\Omega$. Thus by (1) we have $\mu_A(ax) = \mu_A(t), \gamma_A(ax) = \gamma_A(t)$ and $\mu_A(xa) = \mu_A(t), \gamma_A(xa) = \gamma_A(t)$. Since

$$(ax)a(xa) = (axa)xa \geq axa \geq a,$$

$$\begin{aligned} \mu_A(a) &\geq \mu_A((ax)a(xa)) \geq \min\{\mu_A(ax), \mu_A(ax)\} = \mu_A(ax) = \mu_A(t), \\ \gamma_A(a) &\leq \gamma_A((ax)a(xa)) \leq \max\{\gamma_A(ax), \gamma_A(ax)\} = \gamma_A(ax) = \gamma_A(t). \end{aligned}$$

Since S is left and right simple, we have $(Sa] = S$ and $(aS] = S$. Since $t \in S$ we have $t \in (Sa]$ and $t \in (aS]$. Then $t \leq s_1 a$ and $t \leq a z_1$ for some $s_1, z_1 \in S$. Then $t^2 \leq a(z_1 s_1)a$, since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S, we have

$$\begin{aligned} \mu_A(t^2) &\leq \mu_A(a(z_1 s_1)a) \leq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a), \\ \gamma_A(t^2) &\leq \gamma_A(a(z_1 s_1)a) \leq \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a). \end{aligned}$$

Since $t \in E_\Omega$, we have $t^2 \geq t$, then $\mu_A(t) \geq \mu_A(t^2)$, $\gamma_A(t^2) \geq \gamma_A(t)$. Hence $\mu_A(t) \geq \mu_A(a)$ and $\gamma_A(t) \leq \gamma_A(a)$. Therefore, $\mu_A(t) = \mu_A(a)$ and $\gamma_A(t) = \gamma_A(a)$.

(2) $\Longrightarrow$ (1). Let $a \in S$. Then $(Sa]$ is a bi-ideal of S. In fact: (i) $(Sa](Sa] \subseteq (Sa]$, (ii) $(Sa]S(Sa] \subseteq (Sa]$ and (iii) If $a \in (Sa]$ such that $S \ni b \leq a \in (Sa]$, then $b \in ((Sa]] = (Sa]$. By Lemma 1, the intuitionistic characteristic mapping $\chi_{(Sa]} = \langle \mu_{\chi_{(Sa]}}, \gamma_{\chi_{(Sa]}} \rangle$

$$\mu_{\chi_{(Sa]}} \ : \ S \longrightarrow [0,1] | x \longrightarrow \mu_{\chi_{(Sa]}}(x) := \begin{cases} 1 & \text{if } x \in (Sa] \\ 0 & \text{if } x \notin (Sa], \end{cases}$$

$$\gamma_{\chi_{(Sa]}} \ : \ S \longrightarrow [0,1] | x \longrightarrow \gamma_{\chi_{(Sa]}}(x) := \begin{cases} 0 & \text{if } x \in (Sa] \\ 1 & \text{if } x \notin (Sa], \end{cases}$$

is an intuitionistic fuzzy bi-ideal of S. By hypothesis, $\chi_{(Sa]} = \langle \mu_{\chi_{(Sa]}}, \gamma_{\chi_{(Sa]}} \rangle$ is a constant mapping, that is, for every $x \in S$ there exists $c \in \{0,1\}$ such that

$$\mu_{\chi_{(Sa]}}(x) = c \text{ and } \gamma_{\chi_{(Sa]}}(x) = c \qquad (*)$$

Let $(Sa] \subset S$. Let $t \in S$ be such that $t \notin (Sa]$. Then $\mu_{\chi_{(Sa]}}(t) = 0$ and $\gamma_{\chi_{(Sa]}}(t) = 1$. Since $a^2 \in (Sa]$, we have $\mu_{\chi_{(Sa]}}(a^2) = 1$ and $\gamma_{\chi_{(Sa]}}(a^2) = 0$, which is a contradiction. Thus $S = (Sa]$. By symmetry we can prove that $(aS] = S$. Thus S is left and right simple. Since $a \in S = (Sa] = (aS]$, we have $a \in (aS] = (a(Sa]] = (aSa]$ and S is regular. □

Lemma 3.7. (*cf.* [15]) *Let $(S, \cdot, \leq)$ be an ordered semigroup. Let $B(x)$ and $B(y)$ be the bi-ideals of S generated by the elements x, y of S, respectively. Then*

$$B(x)SB(y) \subseteq (xSy].$$

Theorem 3.8. *An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of left and right simple semigroups if and only if for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have*

$$\begin{aligned} \mu_A(a) &= \mu_A(a^2), \gamma_A(a) = \gamma_A(a^2) \\ \text{and } \mu_A(ab) &= \mu_A(ba), \gamma_A(ab) = \gamma_A(ba) \text{ for all } a, b \in S. \end{aligned}$$

Proof. $\Longrightarrow$. (*A*) Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-ideal of S and let S be a semilattice of left and right simple subsemigroups. Then by hypothesis there exists a semilattice and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that:

(1) $S_\alpha \cap S_\beta = \emptyset \ \ \forall \alpha, \beta \in Y, \quad \alpha \neq \beta$,

(2) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}\ \forall \alpha, \beta \in Y$. Let $a \in S$. Since $a \in S = \bigcup_{\alpha \in Y} S_\alpha$, there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left and right simple then,

$$S_\alpha = (aS_\alpha a] = \{t \in S | t \leq axa \text{ for some } x \in S_\alpha\},$$

then $a \leq axa$ for some $x \in S_\alpha$. Since $x \in S_\alpha$ we have $x \leq aya$ for some $y \in S_\alpha$. Then we have $a \leq axa \leq a(aya)a = a^2ya^2$, and so $a \in (a^2Sa^2]$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S by Theorem 1, we have $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$.

(B). Let $a, b \in S$. Then by (A) we have

$$\mu_A(ab) = \mu_A((ab)^2) = \mu_A((ab)^4) \text{ and } \gamma_A(ab) = \gamma_A((ab)^2) = \gamma_A((ab)^4).$$

On the other hand,

$$\begin{aligned}
(ab)^4 &= (aba)(babab) \in B(aba)B(babab) \subseteq (B(aba)B(babab)] \\
&= (B(babab)B(aba)] \text{ (by Lemma 3)} \\
&= (B(babab)(B(aba)^2]] \text{ (by Lemma 3)} \\
&= (B(babab)(B(aba)B(aba)]] = ((Bbabab)](B(aba)B(aba)]] \\
&= (B(babab)B(aba)B(aba)]] \\
&\subseteq (B(babab)SB(aba))]] \\
&= (((babab)S(aba)]](\text{ by by Lemma 4)} \\
&= ((babab)S(aba)].
\end{aligned}$$

Then $(ab)^4 \leq (babab)z(aba)$ for some $z \in S$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S, we have

$$\begin{aligned}
\mu_A((ab)^4) &\geq \gamma_A(ba(babza)ba)) \geq \min\{\mu_A(ba), \mu_A(ba)\} = \mu_A(ba), \\
\gamma_A((ab)^4) &\leq \gamma_A(ba(babza)ba)) \leq \max\{\gamma_A(ba), \gamma_A(ba)\} = \gamma_A(ba).
\end{aligned}$$

Hence we have $\mu_A(ab) \geq \mu_A(ba)$ and $\gamma_A(ab) \leq \gamma_A(ba)$. By symmetry we can prove that $\mu_A(ba) \geq \mu_A(ab)$ and $\gamma_A(ba) \leq \gamma_A(ab)$. Therefore $\mu_A(ab) = \mu_A(ba)$ and $\gamma_A(ab) = \gamma_A(ba)$.

$\Longleftarrow$. Assume that

$$\mu_A(a) = \mu_A(a^2), \gamma_A(a) = \gamma_A(a^2)$$

and

$$\mu_A(ab) = \mu_A(ba), \gamma_A(ab) = \gamma_A(ba),$$

for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S. By condition (1) and Theorem 1, it follows that S is completely regular. Let A be a bi-ideal of S and let $a \in A$. Since $a \in S$ and S is completely regular then by Lemma 2, we have

$$a \leq a^2xa^2 = a(axa)a \in A(ASA)A \subseteq AAA \subseteq AA = A^2,$$

then $A \subseteq A^2$ and we have $(A] \subseteq (A^2] \Longrightarrow A \subseteq (A^2]$. On the other hand, since A is a subsemigroup of S, we have $A^2 \subseteq A$ then $(A^2] \subseteq (A] = A$.

Let A and B be bi-ideals of S and $x \in (BA]$, then $x \leq ba$ for some $a \in A$ and $b \in B$. We consider the bi-ideal $B(ab)$ generated by ab. That is, the set

$B(ab) = (ab \cup abab \cup abSab]$. By Lemma 1, the intuitionistic characteristic mapping $\chi_{B(ab)} = \langle \mu_{\chi_{B(ab)}}, \gamma_{\chi_{B(ab)}} \rangle$, of $B(ab)$ defined by

$$\mu_{\chi_{B(ab)}} \quad : \quad S \longrightarrow [0,1] | x \longrightarrow \mu_{\chi_{B(ab)}}(x) := \begin{cases} 1 \text{ if } x \in B(ab) \\ 0 \text{ if } x \notin B(ab), \end{cases}$$

$$\gamma_{\chi_{B(ab)}} \quad : \quad S \longrightarrow [0,1] | x \longrightarrow \gamma_{\chi_{B(ab)}}(x) := \begin{cases} 0 \text{ if } x \in B(ab) \\ 1 \text{ if } x \notin B(ab), \end{cases}$$

is an intuitionistic fuzzy bi-ideal of S. By hypothesis

$$\mu_{\chi_{B(ab)}}(ab) = \mu_{\chi_{B(ab)}}(ba) \text{ and } \gamma_{\chi_{B(ab)}}(ab) = \gamma_{\chi_{B(ab)}}(ba).$$

Since $ab \in B(ab)$, we have $\mu_{\chi_{B(ab)}}(ab) = 1$ and $\gamma_{\chi_{B(ab)}}(ab) = 0$. Thus $\mu_{\chi_{B(ab)}}(ba) = 1$ and $\gamma_{\chi_{B(ab)}}(ba) = 0$ and $ba \in B(ab) = (ab \cup abab \cup abSab]$ and we have $ba \leq ab$ or $ba \leq abab$ or $ba \leq abxab$ for some $x \in S$. If $ba \leq ab$, then $x \leq ab \in AB$ and $x \in (AB]$. If $ba \leq abab$, then $x \leq abab \in ABAB \subseteq (AB)^2 \subseteq (AB]^2 \subseteq (AB]$ and $x \in ((AB]] = (AB]$. If $ba \leq abxab$, then $x \leq abxab \in (AB)S(AB) \subseteq (AB]S(AB] \subseteq (AB]$ and $x \in ((AB]] = (AB]$. Hence in any case we have $(BA] \subseteq (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. Thus $(AB] = (BA]$ and by Lemma 3, S is a semilattice of left and right simple subsemigroups. □

Let $X \subseteq S$, and $a \in S$, then we define

$$X_a := \{(y,z) \in S \times S | a \leq yz\} \ [13].$$

For any two intuitionistic fuzzy sets $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ in an ordered semigroup S, the intuitionistic fuzzy product $A \circ B = \langle \mu_{A \circ B}, \gamma_{A \circ B} \rangle$ of A and B is an intuitionistic fuzzy set in S defined as follows:

$$\mu_{A \circ B} : S \longrightarrow [0,1] | x \longrightarrow \mu_{A \circ B}(x) := \bigvee_{(y,z) \in X_a} \min\{\mu_A(y), \mu_B(z)\},$$

$$\gamma_{A \circ B} : S \longrightarrow [0,1] | x \longrightarrow \gamma_{A \circ B}(x) := \bigwedge_{(y,z) \in X_a} \max\{\gamma_A(y), \gamma_B(z)\},$$

if $X_a \neq \emptyset$. Otherwise, we define $A \circ B = 0_\sim$, i.e. $\mu_{A \circ B}(x) = 0$ and $\gamma_{A \circ B}(x) = 1$.

Clearly, if A, B, and C are any *IFSs* in S such that $A \subseteq B$ then $A \circ C \subseteq B \circ C$ and $C \circ A \subseteq C \circ B$.

For a non-empty family of $IFSs$ $\{A_i\}_{i \in I}$ of an ordered semigroup S. We have $\bigcup_{i \in I} A_i = \left(\bigvee_{i \in I} \mu_{A_i}, \bigwedge_{i \in I} \gamma_{A_i} \right)$ and $\bigcap_{i \in I} A_i = \left(\bigwedge_{i \in I} \mu_i, \bigvee_{i \in I} \gamma_i \right)$ where

$$\bigvee_{i \in I} \mu_{A_i} : S \longrightarrow [0,1] \, | a \longrightarrow \left(\bigvee_{i \in I} \mu_{A_i} \right)(a) := \sup_{i \in I}\{\mu_{A_i}(a) | a \in S\},$$

$$\bigwedge_{i \in I} \mu_{A_i} : S \longrightarrow [0,1] \, | a \longrightarrow \left(\bigwedge_{i \in I} \mu_{A_i} \right)(a) := \inf_{i \in I} \{\mu_{A_i}(a) \, | a \in S\}.$$

Lemma 3.9. (*cf.* [25, *Proposition* 3.4]) *Let* $(S,\cdot,\leq)$ *be an ordered semigroup and* $A,B\subseteq S$. *Then*

$$\chi_A\circ\chi_B=\chi_{(AB]},\ \textit{i.e. }\ \mu_{\chi_A\circ\chi_B}=\mu_{\chi_{(AB]}}\ \textit{and}\ \gamma_{\chi_A\circ\chi_B}=\gamma_{\chi_{(AB]}}.$$

Lemma 3.10. *Let* S *be an ordered semigroup and* $A=\langle\mu_A,\gamma_A\rangle$ *an intuitionistic fuzzy bi-ideal of* S. *Then*

$$A\circ 1_\sim\circ A\subseteq A.$$

Proof. Let $a\in S$. If $X_a=\emptyset$ then $\mu_{A\circ 1_\sim\circ A}(a)=0\leq\mu_A(a)$ and $\gamma_{A\circ 1_\sim\circ A}(a)=1\geq\gamma_A$. If $X_a\neq\emptyset$, then

$$\begin{aligned}\mu_{A\circ 1_\sim\circ A}(a)\ :&=\bigvee_{(y,z)\in X_a}\min\{\mu_{A\circ 1_\sim}(y),\mu_A(z)\}\\&=\bigvee_{(y,z)\in X_a}\min\{\bigvee_{(y_1,z_1)\in X_y}\min\{\mu_A(y_1),1(z_1)\},\mu_A(z)\}\\&=\bigvee_{(y,z)\in X_a}\bigvee_{(y_1,z_1)\in X_y}\min\{\min\{\mu_A(y_1),1\},\mu_A(z)\}\\&=\bigvee_{(y,z)\in X_a}\bigvee_{(y_1,z_1)\in X_y}\min\{\mu_A(y_1),\mu_A(z)\}.\end{aligned}$$

$$\begin{aligned}\gamma_{A\circ 1_\sim\circ A}(a)\ :&=\bigwedge_{(y,z)\in X_a}\max\{\gamma_{A\circ 1_\sim}(y),\gamma_A(z)\}\\&=\bigwedge_{(y,z)\in X_a}\max\{\bigwedge_{(y_1,z_1)\in X_y}\max\{\gamma_A(y_1),0(z_1)\},\gamma_A(z)\}\\&=\bigwedge_{(y,z)\in X_a}\bigwedge_{(y_1,z_1)\in X_y}\max\{\max\{\gamma_A(y_1),0\},\gamma_A(z)\}\\&=\bigwedge_{(y,z)\in X_a}\bigwedge_{(y_1,z_1)\in X_y}\max\{\gamma_A(y_1),\gamma_A(z)\}.\end{aligned}$$

Since $a\leq yz$ and $y\leq y_1z_1$, we have $a\leq(y_1z_1)z=y_1z_1z$. Since $A=\langle\mu_A,\gamma_A\rangle$ is an intuitionistic fuzzy bi-ideal of S, we have

$$\begin{aligned}\mu_A(a)&\geq\mu_A(y_1z_1z)\geq\min\{\mu_A(y_1),\mu_A(z)\},\\\gamma_A(a)&\leq\gamma_A(y_1z_1z)\leq\max\{\gamma_A(y_1),\gamma_A(z)\}.\end{aligned}$$

Then

$$\begin{aligned}\bigvee_{(y,z)\in A_a}\bigvee_{(y_1,z_1)\in A_y}\min\{\mu_A(y_1),\mu_A(z)\}&\leq\bigvee_{(y,z)\in A_a}\bigvee_{(y_1,z_1)\in A_y}\mu_A(a)\\&=\mu_A(a),\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{(y,z)\in X_a}\bigwedge_{(y_1,z_1)\in X_y}\max\{\gamma_A(y_1),\gamma_A(z)\}&\geq\bigwedge_{(y,z)\in X_a}\bigwedge_{(y_1,z_1)\in X_y}\gamma_A(a)\\&=\gamma_A(a).\end{aligned}$$

Hence $\mu_{A\circ 1_\sim \circ A}(a) \leq \mu_A(a)$ and $\gamma_{A\circ 1_\sim \circ A}(a) \geq \gamma_A(a)$. Therefore, $(A \circ 1_\sim \circ A)(a) \subseteq A(a)$. □

Lemma 3.11. (*cf.* [23]). *Let S be an ordered semigroup. Then the following are equivalent:*

(1) S is regular.
(2) $B = (BSB]$ for every bi-ideal B of S.
(3) $B(a) = (B(a)SB(a)]$ for every $a \in S$.

Theorem 3.12. *An ordered semigroup is regular if and only if and only if for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have*

$$A = A \circ 1_\sim \circ A.$$

Proof. $\Longrightarrow$. Let S be regular. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-ideal of S and $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa = a(xa)$. Then $(a, xa) \in X_a$ and we have

$$\begin{aligned}
\mu_{A\circ 1_\sim \circ A}(a) &= \bigvee_{(y,z)\in X_a} \min\{\mu_A(y), \mu_{1_\sim \circ A}(z)\} \\
&\geq \min\{\mu_A(a), \mu_{1_\sim \circ A}(xa)\} \\
&= \min\{\mu_A(a), \bigvee_{(y_1,z_1)\in X_{xa}} \min\{1(y_1), \mu_A(z_1)\}\} \\
&\geq \min\{\mu_A(a), \min\{1(x), \mu_A(axa)\}\} \\
&= \min\{\mu_A(a), \min\{1, \mu_A(axa)\}\} \\
&= \min\{\mu_A(a), \mu_A(axa)\} \\
&\geq \min\{\mu_A(a), \mu_A(a)\} (\mu_A(axa) \geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a)) \\
&= \mu_A(a),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{A\circ 1_\sim \circ A}(a) &= \bigwedge_{(y,z)\in X_a} \max\{\gamma_A(y), \gamma_{1_\sim \circ A}(z)\} \\
&\leq \max\{\gamma_A(a), \gamma_{1_\sim \circ A}(xa)\} \\
&= \max\{\gamma_A(a), \bigwedge_{(y_1,z_1)\in X_{xa}} \max\{1(y_1), \mu_A(z_1)\}\} \\
&\leq \max\{\gamma_A(a), \max\{1(x), \gamma_A(axa)\}\} \\
&= \min\{\gamma_A(a), \max\{1, \gamma_A(axa)\}\} \\
&= \max\{\gamma_A(a), \gamma_A(axa)\} \\
&\leq \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a) \\
(\gamma_A(axa) &\leq \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a)).
\end{aligned}$$

Hence $\mu_{A\circ 1_\sim \circ A}(a) \geq \mu_A(a)$ and $\gamma_{A\circ 1_\sim \circ A}(a) \leq \gamma_A(a)$. On the other hand, by Lemma 6, we have $\mu_{A\circ 1_\sim \circ A}(a) \leq \mu_A(a)$ and $\gamma_{A\circ 1_\sim \circ A}(a) \geq \gamma_A(a)$. Therefore, $A(a) = (A \circ 1_\sim \circ A)(a)$.

$\Longleftarrow$. Assume that $A = A \circ 1_\sim \circ A$ for every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$. Then S is regular. In fact, by Lemma 7, it is enough to prove that

$$B(a) = (B(a)SB(a)] \text{ for every } a \in S.$$

Let $y \in B(a)$. Then $y \in (B(a)SB(a)]$. Indeed, since $B(a)$ is a bi-ideal of S generated by $a(a \in S)$. By Lemma 1, the intuitionistic characteristic function $\chi_{B(a)} = \langle \mu_{\chi_{B(a)}}, \gamma_{\chi_{B(a)}} \rangle$ of $B(a)$ defined by

$$\mu_{\chi_{B(a)}} : S \longrightarrow [0,1] | x \longrightarrow \mu_{\chi_{B(a)}}(x) := \begin{cases} 1 \text{ if } x \in B(a) \\ 0 \text{ if } x \notin B(a) \end{cases}$$

and

$$\gamma_{\chi_{B(a)}} : S \longrightarrow [0,1] | x \longrightarrow \gamma_{\chi_{B(a)}}(x) := \begin{cases} 0 \quad \text{if } x \in B(a) \\ 1 \text{ if } x \notin B(a), \end{cases}$$

is an intuitionistic fuzzy bi-ideal of S. By hypothesis

$$\chi_{B(a)}(y) = (\chi_{B(a)} \circ 1_\sim \circ \chi_{B(a)})(y).$$

Since $y \in B(a)$, we have $\mu_{\chi_{B(a)}}(y) = 1$ and $\gamma_{\chi_{B(a)}}(y) = 0$ and we have

$$\begin{aligned} (\chi_{B(a)} \circ 1_\sim \circ \chi_{B(a)})(y) &= \langle \mu_{\chi_{B(a)}}(y), \gamma_{\chi_{B(a)}}(y) \rangle \\ &= \langle 1, 0 \rangle = 1_\sim. \end{aligned}$$

By Lemma 5, $\chi_{B(a)} \circ 1_\sim \circ \chi_{B(a)} = \chi_{(B(a)SB(a)]}$, then $\chi_{(B(a)SB(a)]}(y) = 1_\sim \Longrightarrow y \in (B(a)SB(a)]$. Hence $B(a) \subseteq (B(a)SB(a)]$. On the other hand, $(B(a)SB(a)] \subseteq (B(a)] = B(a)$. □

Lemma 3.13. *Let S be an ordered semigroup. Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be intuitionistic fuzzy bi-ideals of S. Then so is $A \circ B = \langle \mu_{A\circ B}, \gamma_{A\circ B} \rangle$.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be intuitionistic fuzzy bi-ideals of S. Let $x, y, z \in S$. Then

$$\begin{aligned} \mu_{A\circ B}(x) \wedge \mu_{A\circ B}(z) &= \left[\bigvee_{(p,q)\in X_x} \{\mu_A(p) \vee \mu_B(q)\} \wedge \bigvee_{(p_1,q_1)\in X_z} \{\mu_A(p_1) \wedge \mu_B(q_1)\} \right] \\ &= \bigvee_{(p,q)\in X_x} \bigvee_{(p_1,q_1)\in X_z} [\{\mu_A(p) \wedge \mu_B(q)\} \wedge \{\mu_A(p_1) \wedge \mu_B(q_1)\}] \\ &= \bigvee_{(p,q)\in X_x} \bigvee_{(p_1,q_1)\in X_z} [\{\mu_A(p) \wedge \mu_A(p_1) \wedge \mu_B(q) \wedge \mu_B(q_1)\}] \\ &\leq \bigvee_{(p,q)\in X_x} \bigvee_{(p_1,q_1)\in X_z} [\{\mu_A(p) \wedge \mu_A(p_1) \wedge \mu_A(q_1)\}]. \end{aligned}$$

Hence

$$\mu_{A\circ B}(x) \wedge \mu_{A\circ B}(z) \leq \bigvee_{(p,q)\in X_x} \bigvee_{(p_1,q_1)\in X_z} [\{\mu_A(p) \wedge \mu_A(p_1) \wedge \mu_B(q_1)\}]$$

By a similar way we have

$$\gamma_{A\circ B}(x) \vee \gamma_{A\circ B}(z) \geq \bigwedge_{(p,q)\in X_x} \bigwedge_{(p_1,q_1)\in X_z} [\{\gamma_A(p) \vee \gamma_A(p_1) \vee \gamma_A(q_1)\}]$$

Since $x \leq pq$ and $z \leq p_1q_1$, we have $xyz \leq (pq)y(p_1q_1) = p(qyp_1)q_1$ and $(p(qy)p_1, q_1) \in X_{xyz}$. Thus

$$\begin{aligned} \mu_A(p(qyp_1)q_1) &\geq \{\mu_A(p) \wedge \mu_A(q_1)\} \\ \gamma_A(p(qy)p_1) &\leq \{\gamma_A(p) \vee \gamma_A(p_1)\}. \end{aligned}$$

Thus

$$\begin{aligned} \mu_{A\circ B}(x) \wedge \mu_{A\circ B}(z) &\leq \bigvee_{(p,q)\in X_x} \bigvee_{(p_1,q_1)\in X_z} [\{\mu_A(p) \wedge \mu_A(p_1) \wedge \mu_B(q_1)\}] \\ &\leq \bigvee_{(p(qy)p_1,q_1)\in X_{xyz}} [\{\mu_A(p(qyp_1)q_1) \wedge \mu_B(q_1)\}] \\ &= \bigvee_{(p(qy)p_1,q_1)\in X_{xyz}} \mu_A(xyz) = \mu_A(xyz) \end{aligned}$$

and

$$\begin{aligned} \gamma_{A\circ B}(x) \vee \gamma_{A\circ B}(z) &\geq \bigwedge_{(p,q)\in X_x} \bigwedge_{(p_1,q_1)\in X_z} [\{\gamma_A(p) \vee \gamma_A(p_1) \vee \gamma_B(q_1)\}] \\ &\geq \bigwedge_{(p(qy)p_1,q_1)\in X_{xyz}} [\{\gamma_A(p(qyp_1)q_1) \vee \gamma_B(q_1)\}] \\ &= \bigwedge_{(p(qy)p_1,q_1)\in X_{xyz}} \gamma_A(xyz) = \gamma_A(xyz). \end{aligned}$$

Hence $\mu_A(xyz) \geq \mu_A(x) \vee \mu_A(z)$ and $\gamma_A(xyz) \leq \gamma_{A\circ B}(x) \vee \gamma_{A\circ B}(z)$.

Let $x, y \in S$ be such that $x \leq y$. Then $\mu_{A\circ B}(x) \geq \mu_{A\circ B}(y)$ and $\gamma_{A\circ B}(x) \leq \gamma_{A\circ B}(y)$. In fact, if $(p,q) \in A_y$ then $y \leq pq$ and so $x \leq y \leq pq \Longrightarrow (p,q) \in X_x \Longrightarrow X_y \subseteq X_x$. If $X_x = \emptyset$ then $X_y = \emptyset$ and we have $\mu_{A\circ B}(y) = 0 = \mu_{A\circ B}(x)$ and $\gamma_{A\circ B}(y) = 1 = \gamma_{A\circ B}(x)$. If $X_x \neq \emptyset$ then $X_y \neq \emptyset$ and hence

$$\begin{aligned} \mu_{A\circ B}(y) &= \bigvee_{(p,q)\in X_y} \{\mu_A(p) \wedge \mu_B(z)\} \\ &\leq \bigvee_{(p,q)\in X_x} \{f(c) \wedge g(d)\} \\ &= \mu_{A\circ B}(x), \end{aligned}$$

and

$$\begin{aligned}\gamma_{A\circ B}(y) &= \bigwedge_{(p,q)\in X_y} \{\gamma_A(p)\vee\gamma_B(z)\} \\ &\geq \bigwedge_{(p,q)\in_x} \{\gamma_A(c)\vee\gamma_A(d)\} \\ &= \gamma_{A\circ B}(x).\end{aligned}$$

Thus $\mu_{A\circ B}(x) \geq \mu_{A\circ B}(y)$ and $\gamma_{A\circ B}(x) \leq \gamma_{A\circ B}(y)$. Consequently, $A\circ B = \langle\mu_{A\circ B},\gamma_{A\circ B}\rangle$ is an intuitionistic fuzzy bi-ideal of S. □

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Asghar Khan*, Department of Mathematics, COMSATS Institute of IT Abbottabad, Pakistan

E-mail address: azhar4set@yahoo.com

Young Bae Jun, Department of Mathematics Educations and RINS, Gyengsang National University, Chinju 660-701, Korea

E-mail address: skywine@gmail.com

Muhammad Shabir, Department of Mathematics Quaid-i-Azam University, Islamabad, Pakistan

E-mail address: mshabirbhatti@yahoo.co.uk

*Corresponding author