M-FUZZIFYING DERIVED OPERATORS AND DIFFERENCE DERIVED OPERATORS

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ABSTRACT. This paper presents characterizations of M-fuzzifying matroids by means of two kinds of fuzzy operators, called M-fuzzifying derived operators and M-fuzzifying difference derived operators.

1. Introduction

As an abstract generalization of a matrix and a graph, matroid theory can be applied in some combinatorial optimization problems. After a development period of more than 30 years, fuzzy algebras and fuzzy graphs have become rather diverse in their topics as well as their methods. Fuzzy graphs can present one possible approach to the fuzzification of matroids as matroids are often applied to graph problems. A matroid is a pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a subfamily of 2^E satisfying three axioms. Several natural ways of extending a matroid to fuzzy setting are:

(i) Use fuzzy sets instead of sets. In this case, $\mathcal{I}$ should be a subfamily of $[0,1]^E$ satisfying a set of axioms. This setting has been considered by Goetschel et al (see [3, 4, 5, 6, 7, 10, 11, 12]).

(ii) Use a mapping from 2^E to $[0,1]$ instead of a family of sets. This setting is considered in [16] and is called an M-fuzzifying matroid, which is defined as a pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a mapping from 2^E to M satisfying three axioms.

(iii) Combining (i) and (ii), use a mapping from $[0,1]^E$ to $[0,1]$ instead of a family of sets. This setting is considered in [17].

This paper focuses on M-fuzzifying matroids in the sense of [16]. We find that, the fuzzy vector spaces and fuzzy graphs can be regarded as cases of M-fuzzifying matroids. For explicit, given a fuzzy vector space (V, μ) and any finite subset E of V, we can construct a $[0,1]$-fuzzifying matroid $(E, \widetilde{\mathcal{I}}_\mu)$ and also given a graph $G = (V, E)$ and its fuzzy subgraph (χ_V, μ), we can obtain a $[0,1]$-fuzzifying matroid $(E, \widetilde{\mathcal{I}}_\mu)$. These two examples indicate that M-fuzzifying matroids may be applied in some optimization problems in future.

For the theoretical aspect, as has been shown in [16] (resp., [19]), M-fuzzifying matroids can be characterized by M-fuzzifying rank functions (resp., M-fuzzifying

closure operators), which shows the reasonableness of the definition of M-fuzzifying matroids.

This paper is a successor of [16] and [19]. We shall show that M-fuzzifying matroids can also be characterized by another two kinds of fuzzy operators, called M-fuzzifying derived operators and M-fuzzifying difference derived operators.

2. Preliminaries

We will use the following notation in establishing the results of this paper.

If E is a finite set, $\mathcal{A} \subseteq 2^E$, then

$$Opp(\mathcal{A}) = \{X \subseteq E : X \notin \mathcal{A}\},$$
$$Min(\mathcal{A}) = \{X \in \mathcal{A} : \forall Y \in \mathcal{A}, \text{if } Y \subseteq X, \text{then } X = Y\}.$$

Throughout this paper, $(M, \bigvee, \bigwedge, ')$ denotes a completely distributive lattice with an order-reversing involution $'$. The smallest element and the largest element in M are denoted by $\perp_M$ and $\top_M$ (or $\perp$ and $\top$) respectively.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [2]. $\{a \in M : a \prec b\}$ is called the greatest minimal family of b in the sense of [18], denoted by $\beta(b)$. In a completely distributive lattice M, there exists $\beta(b)$ for each $b \in M$, and $b = \bigvee \beta(b)$ (see [18]). Note that $\beta(\perp) = \emptyset$.

For a completely distributive lattice M, $A \in M^X$ and $a \in M$, we can define

$$A_{[a]} = \{x \in X : A(x) \geqslant a\}, \qquad A_{(a)} = \{x \in X : a \in \beta(A(x))\}.$$

Some properties of these cut sets can be found in [8, 14, 15].

For a set E and $X \subseteq E$, $e \in E$, we often abbreviate $X \cup \{e\}$ and $X - \{e\}$ to $X \cup e$ and $X - e$, respectively.

Definition 2.1. [13] Let E be a finite set and $\mathcal{I} \subseteq 2^E$ be a set with the following properties:

(I1) $\emptyset \in \mathcal{I}$;

(I2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;

(I3) For any $A, B \in \mathcal{I}$ which satisfy $|A| < |B|$, there exists an $e \in B - A$ such that $A \cup e \in \mathcal{I}$, where $|A|, |B|$ denote the cardinality of A, B.

Then $\mathcal{I}$ is called a system of matroid independent sets on E and $(E, \mathcal{I})$ is called a matroid.

Remark 2.2. [13] Let $(E, \mathcal{I})$ be a matroid. $A \in \mathcal{I}$ is called an independent set. $A \in Opp(\mathcal{I}) = 2^E - \mathcal{I}$ is called a dependent set. The minimal dependent set is called a circuit. Let $\mathcal{C}_{\mathcal{I}}$ denote the family of all circuits, then $\mathcal{C}_{\mathcal{I}} = Min(Opp(\mathcal{I})) = Min \circ Opp(\mathcal{I})$.

In [16, 17], the notion of M-fuzzifying matroids was presented.

Definition 2.3. [17] Let E be a finite set and $\mathcal{I} : 2^E \to M$ be a mapping with the following properties:

(FI1) $\mathcal{I}(\emptyset) = \top$;

(FI2) For any $A, B \in 2^E$, $A \subseteq B \Rightarrow \mathcal{I}(A) \geqslant \mathcal{I}(B)$;

(FI3) If $A, B \in 2^E$ and $|A| < |B|$, then $\bigvee_{e \in B - A} \mathcal{I}(A \cup e) \geqslant \mathcal{I}(A) \wedge \mathcal{I}(B)$.

Then $\mathcal{I}$ is called an M-fuzzy family of independent sets on E and the pair $(E, \mathcal{I})$ is called an M-fuzzifying matroid.

By the following example, we find that M-fuzzifying matroids can be applied to fuzzy algebras and fuzzy graphs.

Example 2.4. [17] (1) Let (V, μ) be a fuzzy vector space and E be a finite subset of V. Define a mapping $\widetilde{\mathcal{I}}_\mu : 2^E \to [0, 1]$ by

$$\widetilde{\mathcal{I}}_\mu(A) = \bigvee \{a \in [0, 1) : A \subseteq \mu_{(a)} \text{ and } A \text{ is linearly independent}\}.$$

Then $(E, \widetilde{\mathcal{I}}_\mu)$ is a $[0, 1]$-fuzzifying matroid.

(2) Let $G = (V, E)$ be a graph and (σ, μ) be a fuzzy subgraph of G, where $\sigma = \chi_V$. Define a mapping $\widetilde{\mathcal{I}}_\mu : 2^E \to [0, 1]$ by

$$\widetilde{\mathcal{I}}_\mu(A) = \bigvee \{a \in [0, 1) : A \subseteq \mu_{(a)} \text{ and } A \text{ does not contain any cycle of } (V, \mu_{(a)})\}.$$

Then $(E, \widetilde{\mathcal{I}}_\mu)$ is a $[0, 1]$-fuzzifying matroid.

In the sequel, we always suppose that M satisfies the condition that $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for any $a, b \in M$.

Theorem 2.5. [17] *Let E be a finite set and $\mathcal{I} : 2^E \to M$ be a mapping, then $(E, \mathcal{I})$ is an M-fuzzifying matroid if and only if for each $a \in \beta(\top)$, $(E, \mathcal{I}_{(a)})$ is a matroid.*

It is well known that closure operators play an important role in matroids and it is a very good way to characterize matroids. In order to present characterizations of M-fuzzifying matroids by means of fuzzy closure operators, Wang and Shi generalize the notion of closure operators of matroids to M-fuzzy setting, namely M-fuzzifying closure operators, as follows:

Definition 2.6. [19] Let E be a finite set. A mapping $cl : 2^E \times \beta(\top) \to 2^E$ is called an M-fuzzifying closure operator on E, if $\forall A, B \in 2^E, \forall x, y \in E$ and $\forall a \in \beta(\top)$, it satisfies

(FCL1) $A \subseteq cl(A, a)$;
(FCL2) $cl(A, a) \subseteq cl(B, a)$ if $A \subseteq B$;
(FCL3) $cl(cl(A, a), a) = cl(A, a)$;
(FCL4) $y \in cl(A \cup x, a) - cl(A, a) \Rightarrow x \in cl(A \cup y, a)$;
(FCL5) $A \cap \left(\bigcap_{x \in A} cl(A - x, a) \right) = \emptyset \Leftrightarrow \exists b \in \beta(\top)$ such that $a \in \beta(b)$ and

$A \cap \left(\bigcap_{x \in A} cl(A - x, b) \right) = \emptyset.$

Definition 2.6 actually relies on the cut matroid $(E, \mathcal{I}_{(a)})$ of the M-fuzzifying matroid $(E, \mathcal{I})$, i.e., $cl(A, a)$ can be viewed as the closure of A in matroid $(E, \mathcal{I}_{(a)})$.

Theorem 2.7. [19] *Let* $(E, \mathcal{I})$ *be an* M*-fuzzifying matroid. Define a mapping* $cl_{\mathcal{I}} : 2^E \times \beta(\top) \to 2^E$ *by*

$$cl_{\mathcal{I}}(A, a) = \{x \in E : R_{(a)}(A) = R_{(a)}(A \cup x)\},$$

where $R_{(a)}$ *is the rank function of* $(E, \mathcal{I}_{(a)})$. *Then* $cl_{\mathcal{I}}$ *is an* M*-fuzzifying closure operator on* E.

Theorem 2.8. [19] *Let* $(E, \mathcal{I})$ *be an* M*-fuzzifying matroid. Then for any* $a \in \beta(\top)$, $A \in \mathcal{I}_{(a)}$ *if and only if* $A \cap \left(\bigcap\limits_{x \in A} cl_{\mathcal{I}}(A - x, a) \right) = \emptyset$.

Theorem 2.9. [19] *Let* $cl : 2^E \times \beta(\top) \to 2^E$ *be an* M*-fuzzifying closure operator. Define a mapping* $\mathcal{I}_{cl} : 2^E \to M$ *by*

$$\mathcal{I}_{cl}(A) = \bigvee \left\{ a \in \beta(\top) : A \cap \left(\bigcap\limits_{x \in A} cl(A - x, a) \right) = \emptyset \right\}.$$

Then $(E, \mathcal{I}_{cl})$ *is an* M*-fuzzifying matroid, and* $cl = cl_{\mathcal{I}_{cl}}$.

Theorem 2.10. [19] *For an* M*-fuzzifying matroid* $(E, \mathcal{I})$, *it follows that* $\mathcal{I}_{cl_{\mathcal{I}}} = \mathcal{I}$.

By Theorem 2.9 and Theorem 2.10, an M-fuzzifying matroid can be completely characterized by an M-fuzzifying closure operator.

3. M-fuzzifying Derived Operators

In this section, in order to present the characterizations of M-fuzzifying matroids, we shall give the notion of M-fuzzifying derived operators.

In a vector space V, $E \subseteq V$ is a finite subset. Let d be the function from 2^E into 2^E defined, for all $A \subseteq E$, by

$$d(A) = \{x \in E : x \text{ is the linear combination of } A - x\}.$$

Then it is easy to prove that d satisfies the following conditions:

(D1) $d(A) \subseteq d(B)$ if $A \subseteq B$;
(D2) $x \in d(A) \Rightarrow x \in d(A - x)$;
(D3) $d(A \cup d(A)) \subseteq A \cup d(A)$;
(D4) $y \in d(A) - d(A - x) \Rightarrow x \in d((A - x) \cup y)$.

Based on the above fact, we give the definition of an M-fuzzifying derived operator as follows:

Definition 3.1. Let E be a finite set. A mapping $d : 2^E \times \beta(\top) \to 2^E$ is called an M-fuzzifying derived operator on E, if $\forall A, B \in 2^E$, $\forall x, y \in E$ and $\forall a \in \beta(\top)$, it satisfies

(FD1) $d(A, a) \subseteq d(B, a)$ if $A \subseteq B$;
(FD2) $x \in d(A, a) \Rightarrow x \in d(A - x, a)$;
(FD3) $d(A \cup d(A, a), a) \subseteq A \cup d(A, a)$;
(FD4) $y \in d(A, a) - d(A - x, a) \Rightarrow x \in d((A - x) \cup y, a)$;

(FD5) $A \cap \left(\bigcap_{x \in A} [d(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow \exists b \in \beta(\top)$ such that $a \in \beta(b)$

and $A \cap \left(\bigcap_{x \in A} [d(A - x, b) \cup (A - x)] \right) = \emptyset$.

Example 3.2. Let $E = \{x, y, z\}$ and $M = 2^{\{1,2,3\}} \cup \{\bot\}$. Then $\beta(\top_M) = \beta(\{1, 2, 3\}) = \{\bot, \emptyset, \{1\}, \{2\}, \{3\}\}$. Define a mapping $d : 2^E \times \beta(\top_M) \to 2^E$ by for any $A \in 2^E$,

$$d(A, \bot) = d(A, \emptyset) = \emptyset; \quad d(A, \{2\}) = \begin{cases} \emptyset, & A \in \{\emptyset, \{z\}\}, \\ \{y\}, & A \in \{\{x\}, \{x, z\}\}, \\ \{x\}, & A \in \{\{y\}, \{y, z\}\}, \\ \{x, y\}, & A \in \{\{x, y\}, E\}; \end{cases}$$

$$d(A, \{1\}) = \begin{cases} \emptyset, & A = \{x, z\}, \\ \{z\}, & \text{otherwise}; \end{cases} \quad d(A, \{3\}) = \begin{cases} \emptyset, & A \in \{\emptyset, \{y\}\}, \\ \{z\}, & A \in \{\{x\}, \{x, y\}\}, \\ \{x\}, & A \in \{\{z\}, \{y, z\}\}, \\ \{x, z\}, & A \in \{\{x, z\}, E\}. \end{cases}$$

Then we can check that d satisfies (FD1)–(FD5), hence it is an M-fuzzifying derived operator on E.

Theorem 3.3. *Let* $cl : 2^E \times \beta(\top) \to 2^E$ *be an M-fuzzifying closure operator. Define a mapping* $d_{cl} : 2^E \times \beta(\top) \to 2^E$ *by*

$$d_{cl}(A, a) = \{x \in E : x \in cl(A - x, a)\}.$$

Then d_{cl} is an M-fuzzifying derived operator.

Proof. Let $A, B \in 2^E$, $a \in \beta(\top)$. We need to prove d_{cl} satisfies (FD1)-(FD5).

(FD1) Suppose that $A \subseteq B$, $x \in d_{cl}(A, a)$. By the definition of d_{cl} and (FCL2), $x \in cl(A - x, a) \subseteq cl(B - x, a) = d_{cl}(B, a)$. This means $d_{cl}(A, a) \subseteq d_{cl}(B, a)$.

(FD2) Let $x \in d_{cl}(A, a)$. Then $x \in cl(A - x, a) = cl((A - x) - x, a)$, hence $x \in d_{cl}(A - x, a)$ by the definition of d_{cl}, i.e. (FD2) holds.

(FD3) In order to prove $d_{cl}(A \cup d_{cl}(A, a), a) \subseteq A \cup d_{cl}(A, a)$, let $x \in d_{cl}(A \cup d_{cl}(A, a), a)$. If $x \in A$, then $x \in A \cup d_{cl}(A, a)$; if $x \notin A$, then

$$\begin{aligned} x \quad & \in cl(A \cup d_{cl}(A, a) - x, a) \\ & \subseteq cl(A \cup d_{cl}(A, a), a) \\ & \subseteq cl(cl(A, a), a) \quad (\text{notice } A \cup d_{cl}(A, a) \subseteq cl(A, a)) \\ & = cl(A, a) = cl(A - x, a), \end{aligned}$$

thus $x \in d_{cl}(A, a) \subseteq A \cup d_{cl}(A, a)$.

(FD4) Let $x \in E$, $y \in d_{cl}(A, a) - d_{cl}(A - x, a)$. Then $y \in cl(A - y, a)$ and $y \notin cl((A - x) - y, a)$. By (FCL4), $x \in cl(((A - x) - y) \cup y, a) = cl((A - x) \cup y - x, a)$, thus $x \in d_{cl}((A - x) \cup y, a)$ by the definition of d_{cl}.

(FD5) Let $x \in E$. By the definition of d_{cl}, $d_{cl}(A - x, a) \cup (A - x) = cl(A - x, a)$, then $A \cap \left(\bigcap_{x \in A} [d_{cl}(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow A \cap \left(\bigcap_{x \in A} cl(A - x, a) \right) = \emptyset$ for

every $a \in \beta(\top)$. Therefore, $A \cap \left(\bigcap\limits_{x \in A} [d_{cl}(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow \exists b \in \beta(\top)$ such that $a \in \beta(b)$ and $A \cap \left(\bigcap\limits_{x \in A} [d_{cl}(A - x, b) \cup (A - x)] \right) = \emptyset$ by (FCL5).

$\square$

By Proposition 1.4.10 in [13], we can easily obtain the following lemma.

Lemma 3.4. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid. Then for any $A \in 2^E$ and $a \in \beta(\top)$,*

$$cl_{\mathcal{I}}(A, a) = A \cup \{x \in E : \exists C \in Min \circ Opp(\mathcal{I}_{(a)}), x \in C \subseteq A \cup x\}.$$

By Theorem 3.3 and Lemma 3.4, the following corollary is obvious.

Corollary 3.5. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid. Define a mapping $d_{\mathcal{I}} :$ $2^E \times \beta(\top) \to 2^E$ by*

$$d_{\mathcal{I}}(A, a) = \{x \in E : \exists C \in Min \circ Opp(\mathcal{I}_{(a)}), x \in C \subseteq (A - x) \cup x\}.$$

Then $d_{\mathcal{I}}$ is an M-fuzzifying derived operator.

Theorem 3.6. *Let $d : 2^E \times \beta(\top) \to 2^E$ be an M-fuzzifying derived operator. Define a mapping $cl_d : 2^E \times \beta(\top) \to 2^E$ by $cl_d(A, a) = A \cup d(A, a)$. Then cl_d is an M-fuzzifying closure operator.*

Proof. Let $A, B \in 2^E$, $a \in \beta(\top)$. We need to show that cl_d satisfies (FCL1)-(FCL5).
(FCL1) Obviously, $A \subseteq cl_d(A, a)$ by the definition of cl_d.
(FCL2) Suppose that $A \subseteq B$ and $x \in cl_d(A, a)$. If $x \in A$, then $x \in B \subseteq cl_d(B, a)$ since $A \subseteq B$; if $x \notin A$, then $x \in d(A, a) \subseteq d(B, a) \subseteq cl_d(B, a)$ by (FD1) and the definition of cl_d.
(FCL3) By the definition of cl_d and (FD3), we have that

$$\begin{aligned} cl_d(cl_d(A, a), a) \quad &= cl_d(A, a) \cup d(cl_d(A, a), a) \\ &= (A \cup d(A, a)) \cup d(A \cup d(A, a), a) \\ &= A \cup d(A, a) = cl_d(A, a). \end{aligned}$$

(FCL4) Let $x, y \in E$ and $y \in cl_d(A \cup x, a) - cl_d(A, a)$. If $y = x$, then $x \in cl_d(A \cup y, a)$; if $y \neq x$, then $y \in d(A \cup x, a) - d(A, a)$ by the definition of cl_d and $y \notin cl_d(A, a)$, thus $x \in d(A \cup y, a) \subseteq cl_d(A \cup y, a)$ by (FD4) and the definition of cl_d.
(FCL5) Since $A \cap \left(\bigcap\limits_{x \in A} [d(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow A \cap \left(\bigcap\limits_{x \in A} cl_d(A - x, a) \right) = \emptyset$ for every $a \in \beta(\top)$, $A \cap \left(\bigcap\limits_{x \in A} cl_d(A - x, a) \right) = \emptyset \Leftrightarrow \exists b \in \beta(\top)$ such that $a \in \beta(b)$ and $A \cap \left(\bigcap\limits_{x \in A} cl_d(A - x, b) \right) = \emptyset$ by (FD5). $\square$

By Theorem 2.9 and Theorem 3.6, the following corollary is obvious.

Corollary 3.7. *Let $d : 2^E \times \beta(\top) \to 2^E$ be an M-fuzzifying derived operator. Define a mapping $\mathcal{I}_d : 2^E \to M$ by*

$$\mathcal{I}_d(A) = \bigvee \left\{ a \in \beta(\top) : A \cap \left(\bigcap_{x \in A} [d(A - x, a) \cup (A - x)] \right) = \emptyset \right\}.$$

Then $(E, \mathcal{I}_d)$ is an M-fuzzifying matroid.

Theorem 3.8. *For an M-fuzzifying derived operator d on E, it follows that $d = d_{cl_d}$.*

Proof. In order to prove that $d = d_{cl_d}$, we need to prove that $\forall A \in 2^E$ and $\forall a \in \beta(\top)$, $d(A, a) = d_{cl_d}(A, a)$. If $x \in d_{cl_d}(A, a)$, then $x \in cl_d(A - x, a) = (A - x) \cup d(A - x, a)$ by Theorem 3.3 and Theorem 3.6, thus $x \in d(A - x, a) \subseteq d(A, a)$ by (FD1). This implies that $d_{cl_d}(A, a) \subseteq d(A, a)$. If $x \in d(A, a)$, then $x \in d(A - x, a)$ by (FD2). Thus $x \in cl_d(A - x, a) = d_{cl_d}(A, a)$ by Theorem 3.3 and Theorem 3.6. This implies that $d(A, a) \subseteq d_{cl_d}(A, a)$. Therefore, $d(A, a) = d_{cl_d}(A, a)$. $\square$

Theorem 3.9. *For an M-fuzzifying closure operator cl on E, it follows that $cl = cl_{d_{cl}}$.*

Proof. In order to prove that $cl = cl_{d_{cl}}$, we need to prove that $\forall A \in 2^E$ and $\forall a \in \beta(\top)$, $cl(A, a) = cl_{d_{cl}}(A, a)$. By Theorem 3.3, Theorem 3.6, (FCL1) and (FCL2), we know that

$$cl_{d_{cl}}(A, a) = A \cup d_{cl}(A, a) = A \cup \{x \in E : x \in cl(A - x, a)\} \subseteq cl(A, a).$$

Conversely, let $x \in cl(A, a)$. If $x \in A$, then $x \in cl_{d_{cl}}(A, a)$; if $x \notin A$, then $x \in cl(A, a) = cl(A - x, a) = d_{cl}(A, a) \subseteq cl_{d_{cl}}(A, a)$. This implies that $cl(A, a) \subseteq cl_{d_{cl}}(A, a)$. Therefore, $cl(A, a) = cl_{d_{cl}}(A, a)$. $\square$

By Theorem 3.8 and Theorem 3.9 we can obtain the following result:

Theorem 3.10. *There is a one-to-one correspondence between M-fuzzifying matroids and M-fuzzifying derived operators. That is, an M-fuzzifying matroid can be completely characterized by an M-fuzzifying derived operator.*

4. *M*-fuzzifying Difference Derived Operators

In this section, in order to present the characterizations of M-fuzzifying matroids, we shall give the notion of M-fuzzifying difference derived operators.

In a vector space V, $E \subseteq V$ is a finite subset. Let δ be the function from 2^E into 2^E defined, for all $A \subseteq E$, by

$$\delta(A) = \{x \in E - A : x \text{ is the linear combination of } A\}.$$

Then it is easy to prove that d satisfies the following conditions:
  (DD1) $A \cap \delta(A) = \emptyset$;
  (DD2) $\delta(A) \subseteq B \cup \delta(B)$ if $A \subseteq B$;
  (DD3) $\delta(A \cup \delta(A)) = \emptyset$;
  (DD4) $y \in \delta(A \cup x) - \delta(A) \Rightarrow x \in \delta(A \cup y)$.

Based on the above fact, we give the definition of an M-fuzzifying difference derived operator as follows:

Definition 4.1. Let E be a finite set. A mapping $\delta : 2^E \times \beta(\top) \to 2^E$ is called an M-fuzzifying difference derived operator on E, if $\forall A, B \in 2^E$, $\forall x, y \in E$ and $\forall a \in \beta(\top)$, it satisfies

(FDD1) $A \cap \delta(A, a) = \emptyset$;
(FDD2) $\delta(A, a) \subseteq B \cup \delta(B, a)$ if $A \subseteq B$;
(FDD3) $\delta(A \cup \delta(A, a), a) = \emptyset$;
(FDD4) $y \in \delta(A \cup x, a) - \delta(A, a) \Rightarrow x \in \delta(A \cup y, a)$;

(FDD5) $A \cap \left(\bigcap_{x \in A} [\delta(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow \exists b \in \beta(\top)$ such that $a \in \beta(b)$

and $A \cap \left(\bigcap_{x \in A} [\delta(A - x, b) \cup (A - x)] \right) = \emptyset$.

Example 4.2. Let $E = \{x, y, z\}$ and $M = 2^{\{1,2,3\}} \cup \{\bot\}$. Then $\beta(\top_M) = \beta(\{1, 2, 3\}) = \{\bot, \emptyset, \{1\}, \{2\}, \{3\}\}$. Define a mapping $\delta : 2^E \times \beta(\top_M) \to M$ by for any $A \in 2^E$,

$$\delta(A, \bot) = d(A, \emptyset) = \emptyset;$$

$$\delta(A, \{1\}) = \left\{ \begin{array}{ll} \{z\}, & A \in \{\emptyset, \{x\}, \{y\}, \{x, y\}\}, \\ \emptyset, & A \in \{\{z\}, \{x, z\}, \{y, z\}, E\}; \end{array} \right.$$

$$\delta(A, \{2\}) = \left\{ \begin{array}{ll} \emptyset, & A \in \{\emptyset, \{z\}, \{x, y\}, E\}, \\ \{y\}, & A \in \{\{x\}, \{x, z\}\}, \\ \{x\}, & A \in \{\{y\}, \{y, z\}\}; \end{array} \right.$$

$$\delta(A, \{3\}) = \left\{ \begin{array}{ll} \emptyset, & A \in \{\emptyset, \{y\}, \{x, z\}, E\}, \\ \{z\}, & A \in \{\{x\}, \{x, y\}\}, \\ \{x\}, & A \in \{\{z\}, \{y, z\}\}. \end{array} \right.$$

We can check that δ satisfies (FDD1)–(FDD5), hence it is an M-fuzzifying difference derived operator on E.

Theorem 4.3. *Let cl be an M-fuzzifying closure operator. Define a mapping $\delta_{cl} : 2^E \times \beta(\top) \to 2^E$ by $\delta_{cl}(A, a) = cl(A, a) - A$. Then δ_{cl} is an M-fuzzifying difference derived operator.*

Proof. Let $A, B \in 2^E$, $a \in \beta(\top)$. We need to show that δ_{cl} satisfies (FDD1)-(FDD5).

(FDD1) Obviously, $A \cap \delta_{cl}(A, a) = \emptyset$.

(FDD2) Suppose that $A \subseteq B$. By the definition of δ_{cl} and (FCL2), $\delta_{cl}(A, a) \subseteq cl(A, a) \subseteq cl(B, a) = B \cup \delta_{cl}(B, a)$.

(FDD3) By the definition of δ_{cl} and (FCL3),

$$\begin{aligned} \delta_{cl}(A \cup \delta_{cl}(A, a), a) &= \delta_{cl}(cl(A, a), a) \\ &= cl(cl(A, a), a) - cl(A, a) = \emptyset. \end{aligned}$$

(FDD4) Let $x \in E$ and $y \in \delta_{cl}(A \cup x, a) - \delta_{cl}(A, a)$. Then $y \in cl(A \cup x, a) - cl(A, a)$, this implies that $x \in cl(A \cup y, a)$ by (FCL4). By $x \notin A \cup y$, we obtain that $x \in \delta_{cl}(A \cup y, a)$.

(FDD5) Let $x \in E$. By the definition of δ_{cl}, we know that for every $a \in \beta(\top)$,

$$A \cap \left(\bigcap_{x \in A} [\delta_{cl}(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow A \cap \left(\bigcap_{x \in A} cl(A - x, a) \right) = \emptyset.$$

Therefore by (FCL5), we have that $A \cap \left(\bigcap_{x \in A} [\delta_{cl}(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow \exists b \in$

$\beta(\top)$ such that $a \in \beta(b)$ and $A \cap \left(\bigcap_{x \in A} [\delta_{cl}(A - x, b) \cup (A - x)] \right) = \emptyset$. $\square$

By Lemma 3.4 and Theorem 4.3, the following corollary is obvious.

Corollary 4.4. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid. Define a mapping $\delta_{\mathcal{I}}$: $2^E \times \beta(\top) \to 2^E$ by*

$$\delta_{\mathcal{I}}(A, a) = \{x \in E - A : \exists C \in Min \circ Opp(\mathcal{I}_{(a)}), x \in C \subseteq A \cup x\}.$$

Then $\delta_{\mathcal{I}}$ is an M-fuzzifying difference derived operator.

For an M-fuzzifying matroid $(E, \mathcal{I})$, as we know, $\delta_{\mathcal{I}}(A, a) = d_{\mathcal{I}}(A, a) - A$ for any $A \in 2^E, a \in \beta(\top)$. Hence $\delta_{\mathcal{I}} \leqslant d_{\mathcal{I}}$. But $\delta_{\mathcal{I}} = d_{\mathcal{I}}$ is not true generally, i.e., it is possible that there exist $A \in 2^E, a \in \beta(\top)$ such that $\delta_{\mathcal{I}}(A, a) \neq d_{\mathcal{I}}(A, a)$. This can be seen from the following example.

Example 4.5. Let $E = \{x, y, z\}, M = 2^{\{1,2,3\}} \cup \{\bot\}$. Define an M-fuzzy family of independent sets $\mathcal{I} : 2^E \to M$ on E by

$$\mathcal{I}(A) = \begin{cases} E, & A \in \{\emptyset, \{x\}, \{y\}\}; \\ \{2, 3\}, & A \in \{\{z\}, \{y, z\}\}; \\ \{1, 3\}, & A = \{x, y\}; \\ \{2\}, & A = \{x, z\}; \\ \emptyset, & A \in \{E\}. \end{cases}$$

Then

$d_{\mathcal{I}}(\{z\}, \{1\}) = \{x \in E : \exists C \in Min \circ Opp(\mathcal{I}_{(\{1\})}), x \in C \subseteq (\{z\} - x) \cup x\} = \{z\};$

$\delta_{\mathcal{I}}(\{z\}, \{1\}) = \{x \in E - \{z\} : \exists C \in Min \circ Opp(\mathcal{I}_{(\{1\})}), x \in C \subseteq \{z\} \cup x\} = \emptyset.$

Obviously, $d_{\mathcal{I}}(\{z\}, \{1\}) \neq \delta_{\mathcal{I}}(\{z\}, \{1\})$.

Theorem 4.6. *Let $\delta : 2^E \times \beta(\top) \to 2^E$ be an M-fuzzifying difference derived operator. Define a mapping $cl_\delta : 2^E \times \beta(\top) \to 2^E$ by $cl_\delta(A, a) = A \cup \delta(A, a)$. Then cl_δ is an M-fuzzifying closure operator.*

Proof. Let $A, B \in 2^E$ and $a \in \beta(\top)$. We need to show that cl_δ satisfies (FCL1)–(FCL5).

(FCL1) Obviously, $A \subseteq cl_\delta(A, a)$ by the definition of cl_δ.

(FCL2) Suppose that $A \subseteq B$. Then $cl_\delta(A, a) = A \cup \delta(A, a) \subseteq B \cup \delta(B, a) = cl_\delta(B, a)$ by the definition of cl_δ and (FDD2).

(FCL3) By the definition of cl_δ and (FDD3), $cl_\delta(cl_\delta(A, a), a) = cl_\delta(A, a) \cup \delta(A \cup \delta(A, a), a) = cl_\delta(A, a)$.

(FCL4) Let $x \in E$ and $y \in cl_\delta(A \cup x, a) - cl_\delta(A, a)$. Then $y = x$ or $y \in \delta(A \cup x, a) - \delta(A, a)$ by the definition of cl_δ. If $y = x$, then $x \in \delta(A \cup y, a)$; if $y \in \delta(A \cup x, a) - \delta(A, a)$, then by (FDD4), $x \in \delta(A \cup y, a) \subseteq cl_\delta(A \cup y, a)$.

(FCL5) By the definition of cl_δ we know that $A \cap \left(\bigcap\limits_{x \in A} [\delta(A - x, a) \cup (A - x)] \right) = \emptyset \Leftrightarrow A \cap \left(\bigcap\limits_{x \in A} cl_\delta(A - x, a) \right) = \emptyset$ for every $a \in \beta(\top)$. This implies that $A \cap \left(\bigcap\limits_{x \in A} cl_\delta(A - x, a) \right) = \emptyset \Leftrightarrow \exists b \in \beta(\top)$ such that $a \in \beta(b)$ and $A \cap \left(\bigcap\limits_{x \in A} cl_\delta(A - x, b) \right) = \emptyset$ by (FDD5). $\qquad \square$

By Theorem 2.7 and Theorem 4.6, the following corollary is obvious.

Corollary 4.7. *Let* $\delta : 2^E \times \beta(\top) \to 2^E$ *be an M-fuzzifying difference derived operator. Define a mapping* $\mathcal{I}_\delta : 2^E \to M$ *by*

$$\mathcal{I}_\delta(A) = \bigvee \left\{ a \in \beta(\top) : A \cap \left(\bigcap\limits_{x \in A} [\delta(A - x, a) \cup (A - x)] \right) = \emptyset \right\}.$$

Then $(E, \mathcal{I}_\delta)$ *is an M-fuzzifying matroid.*

Theorem 4.8. *For an M-fuzzifying derived operator* δ *on* E, *it follows that* $\delta = \delta_{cl_\delta}$.

Proof. Let $A \in 2^E$ and $a \in \beta(\top)$. By Theorem 4.3, Theorem 4.6 and (FDD1), we have that $\delta_{cl_\delta}(A, a) = cl_\delta(A, a) - A = A \cup \delta(A, a) - A = \delta(A, a)$. This implies that $\delta = \delta_{cl_\delta}$. $\qquad \square$

Theorem 4.9. *For an M-fuzzifying closure operator* cl *on* E, *it follows that* $cl = cl_{\delta_{cl}}$.

Proof. Let $A \in 2^E$ and $a \in \beta(\top)$. By Theorem 4.4, Theorem 4.7 and (FCL1), we have that $cl_{\delta_{cl}}(A, a) = A \cup \delta_{cl}(A, a) = A \cup (cl(A, a) - A) = cl(A, a)$. This implies that $cl = cl_{\delta_{cl}}$. $\qquad \square$

By Theorem 4.8 and Theorem 4.9 we can obtain the following result:

Theorem 4.10. *There is a one-to-one correspondence between M-fuzzifying matroids and M-fuzzifying difference derived operators. That is, an M-fuzzifying matroid can be completely characterized by an M-fuzzifying difference derived operator.*

5. Conclusion

The notions of M-fuzzifying derived operators and M-fuzzifying difference derived operators are presented, by which we can characterize M-fuzzifying matroids. That is, there is a one-to-one correspondence between M-fuzzifying matroids and M-fuzzifying derived operators, and there is also a one-to-one correspondence between M-fuzzifying matroids and M-fuzzifying difference derived operators. Until now, the M-fuzzifying matroids can be completely characterized by M-fuzzifying rank functions, M-fuzzifying closure operators, M-fuzzifying derived operators or

M-fuzzifying difference derived operators. All of these indicate the reasonableness of definition of M-fuzzifying matroids. At the practical aspect, with weighted graphs as examples, we may say that the M-fuzzifying matroids could be applied in some combinatorial optimization problems in future.

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