

LOCAL BASES WITH STRATIFIED STRUCTURE IN I -TOPOLOGICAL VECTOR SPACES

J. X. FANG

ABSTRACT. In this paper, the concept of *local base with stratified structure* in I -topological vector spaces is introduced. We prove that every I -topological vector space has a balanced local base with stratified structure. Furthermore, a new characterization of I -topological vector spaces by means of the local base with stratified structure is given.

1. Introduction

The concept of fuzzy topological vector space was introduced rationally by Katsaras in 1981 [8]. According to the terminology of standardization in [5], it has now been renamed as I -topological vector space, where $I = [0, 1]$. In research of I -topological vector spaces, two different kind of neighborhood structures have been used, one being neighborhood introduced by Warren [11] (for short, W -neighborhood) and the other quasi-coincident neighborhood introduced by Pu and Liu [9] (for short, Q -neighborhood). In [2], the author discussed the relations between W -neighborhood (base) of a point x and Q -neighborhood (base) of a fuzzy point x_λ in I -topological spaces and gave the characterizations of I -topological vector spaces in terms of such two kind of local bases.

In this paper, we continue to investigate the local bases of I -topological vector spaces. We first introduce the concept of *local base with stratified structure* in I -topological vector spaces and prove that every I -topological vector space has a balanced local base with stratified structure, and also show that W -neighborhood bases of θ in I -topological vector spaces are all local bases with stratified structure. Next, we give a new characterization of I -topological vector spaces by means of the local base with stratified structure, which is a main theorem in the present paper. As its a deduction, we obtain a new characterization of I -topological vector spaces in terms of W -neighborhood base of θ . These results improve and simplify the corresponding results of Katsaras [8] and Fang [2]. In addition, as applications of these results, two example are given.

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2. Preliminaries

Throughout this paper, let X be a nonempty set, $\mathbb{R} = (-\infty, +\infty)$ and I^X denote a family of all fuzzy subset on X . A fuzzy set which takes the constant value $r \in I$ on X is denoted by \underline{r} . A fuzzy set on X is called a fuzzy point, denoted by x_λ , if it takes the value 0 for all $y \in X$ with $y \neq x$ and the value at x is $\lambda \in (0, 1]$. We denote the set of all fuzzy points on X by $\text{Pt}(I^X)$. Let $U \in I^X$, a fuzzy point x_λ on X is said to be quasi-coincident with U , denoted by $x_\lambda \tilde{\in} U$, if $U(x) > 1 - \lambda$.

An I -topology (i.e., fuzzy topology in Chang's sense [1]) on X is a collection of fuzzy sets of X closed under arbitrary unions and finite intersections, which contains the characteristic functions of \emptyset and X , denoted $\underline{0}$ and $\underline{1}$, respectively. The pair (X, T) is called an I -topological space if T is an I -topology on X . An I -topology T on X is said to be stratified if $\underline{r} \in T$ for each $r \in I$.

Definition 2.1. [11] Let (X, T) be an I -topological space and $x \in X$. A fuzzy set U on X is called a W -neighborhood of x if there exists $G \in T$ such that $G \subset U$ and $G(x) = U(x) > 0$.

Definition 2.2. [8] Let (X, T) be an I -topological space and $x \in X$. A family B_x of W -neighborhoods of x is called a W -neighborhood base of x if for each W -neighborhood P of x and $\alpha \in [0, P(x))$, there exists $U \in B_x$ such that $U \subset P$ and $U(x) > \alpha$.

Definition 2.3. [9] Let (X, T) be an I -topological space and x_λ be a fuzzy point on X . A fuzzy set U of X is called a Q -neighborhood of x_λ if there exists $G \in T$ such that $x_\lambda \tilde{\in} G \subset U$.

A family U_{x_λ} of Q -neighborhoods of x_λ is called a Q -neighborhood base of x_λ if for each Q -neighborhood A of x_λ there exists $U \in U_{x_\lambda}$ such that $U \subset A$.

Let X be a vector space over the field \mathbb{K} (\mathbb{R} or \mathbb{C}), $A, B \in I^X$ and $k \in \mathbb{K}$. For the definitions of $A + B$ and kA , see [7] or [6].

Definition 2.4. [8] Let X be a vector space over the field \mathbb{K} and T be a stratified I -topology on X . If the following two mappings (the addition and the scalar multiplication):

$$f : X \times X \rightarrow X, \quad (x, y) \mapsto x + y \quad \text{and} \quad g : \mathbb{K} \times X \rightarrow X, \quad (k, x) \mapsto kx,$$

are both I -continuous, where \mathbb{K} is equipped with the usual induced I -topology $\omega(J_{\mathbb{K}})$, $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product I -topologies $T \times T$ and $\omega(J_{\mathbb{K}}) \times T$, respectively. Then the pair (X, T) is called an I -topological vector space (for short, I -tvs).

Proposition 2.5. [12, 2] Let (X, T) be an I -tvs.

(1) The addition mapping f is I -continuous iff for each fuzzy point $(x, y)_\lambda$ on $X \times X$ and each Q -neighborhood W of $(x + y)_\lambda$, there exist Q -neighborhoods U of x_λ and V of y_λ such that $U + V \subset W$.

(2) The scalar multiplication mapping g is I -continuous iff for each fuzzy point $(k, x)_\lambda$ on $\mathbb{K} \times X$ and each Q -neighborhood W of kx_λ , there exist a Q -neighborhood U of x_λ and $\delta > 0$ such that $tU \subset W$ for all $t \in \mathbb{K}$ with $|t - k| < \delta$.

Proposition 2.6. [2] Let (X, T) be an I -tvs and $B \subset I^X$. Then B is a W -neighborhood base of θ iff $U(\theta) > 0$ for each $U \in B$, and for each $\lambda \in (0, 1]$,

$$U_\lambda = \{U \in B \mid U(\theta) > 1 - \lambda\}$$

is a Q -neighborhood base of θ_λ in (X, T) .

Definition 2.7. [10] Let L_1 and L_2 be two completely distributive lattices with order-reversing involution $'$, 0 and 1 are respectively their smallest and greatest elements. A mapping $\phi : L_1 \rightarrow L_2$ is called an order-homomorphism, if the following conditions hold:

(OH-1) ϕ is union-preserving, i.e., $f(\bigvee a_i) = \bigvee f(a_i)$;

(OH-2) ϕ^* is complement-preserving, i.e., for each $b \in L_2$, $\phi^*(b') = [\phi^*(b)]'$, where the mapping $\phi^* : L_2 \rightarrow L_1$ is given by $\phi^*(b) = \bigvee \{a \in L_1 \mid \phi(a) \leq b\}$.

3. The Local Bases with Stratified Structure

Definition 3.1. Let $U \subset I^X$. A family $\{U_\lambda\}_{\lambda \in (0,1]}$ of nonempty subsets of U is called the stratified structure of U if the following two equalities hold: $U = \bigcup_{\lambda \in (0,1]} U_\lambda$ and $U_\lambda = \bigcup_{\mu \in (0,\lambda)} U_\mu$.

Remark 3.2. It is easy to prove that if $\{U_\lambda\}_{\lambda \in (0,1]}$ be a family of nonempty subsets of U and $U = \bigcup_{\lambda \in (0,1]} U_\lambda$, then $\{U_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of U if and only if the following conditions are satisfied:

(1) $0 < \mu < \lambda \leq 1$ implies that $U_\mu \subseteq U_\lambda$;

(2) For each $U \in U_\lambda$ there exists $\mu_0 \in (0, \lambda)$ such that $U \in U_{\mu_0}$.

Example 3.3. Let $X = \mathbb{R}$, $U = \{(-a, a) \cap \underline{x} \mid a > 0, 0 < r \leq 1\}$. For each $\lambda \in (0, 1]$, define a subset U_λ of U by $U_\lambda = \{(-a, a) \cap \underline{x} \mid a > 0, r \in (1 - \lambda, 1]\}$. Then it is easy to verify that $\{U_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of U .

Example 3.4. Let (X, T) be an I -topological space and $x \in X$, and let $B_\lambda(x)$ be a Q -neighborhood base of x_λ for each $\lambda \in (0, 1]$. We define

$$U_\lambda(x) = \bigcup_{0 < \mu < \lambda} B_\mu(x) \quad \text{and} \quad U(x) = \bigcup_{\lambda \in (0,1]} U_\lambda(x). \quad (3.1)$$

Then $\{U_\lambda(x)\}_{\lambda \in (0,1]}$ is the stratified structure of $U(x)$.

To prove this conclusion, it suffices to show that $U_\lambda(x) = \bigcup_{0 < \mu < \lambda} U_\mu(x)$. It is evident that $\bigcup_{0 < \mu < \lambda} U_\mu(x) \subseteq U_\lambda(x)$. On the other hand, if $U \in U_\lambda(x)$, then by the first equality of (3.1), there exists $\mu \in (0, \lambda)$ such that $U \in B_\mu(x)$. Taking $\lambda_0 \in (\mu, \lambda)$, it follows that $U \in U_{\lambda_0}(x)$, and so $U \in \bigcup_{0 < \mu < \lambda} U_\mu(x)$. This shows that $U_\lambda(x) = \bigcup_{0 < \mu < \lambda} U_\mu(x)$. Therefore $\{U_\lambda(x)\}_{\lambda \in (0,1]}$ is the stratified structure of $U(x)$.

Definition 3.5. Let (X, T) be an I -tvs. A family U of fuzzy subsets on X is called a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$ in (X, T) if $\{U_\lambda\}_{\lambda \in (0,1]}$ is its stratified structure and U_λ is a Q -neighborhood base of θ_λ in (X, T) for each $\lambda \in (0, 1]$.

Example 3.6. Let $(X, \|\cdot\|, L, R)$ be a fuzzy normed space (see [4] and [13]). For a fixed $\varepsilon > 0$, the fuzzy set B_ε on X is defined by

$$B_\varepsilon(x) = \sup\{1 - \alpha : \|x\|_2^\alpha < \varepsilon\}.$$

By Theorem 3.1 in [13], we know that if $R \leq \max$, then there exists an I -topology $T_{\|\cdot\|}$ on X such that $(X, T_{\|\cdot\|})$ is an I -topological vector space and

$$U_\lambda = \{B_\varepsilon \cap \underline{r} \mid \varepsilon > 0, r \in (1 - \lambda, 1]\}$$

is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$. In addition, by using Remark 3.1, it is not difficult to prove that the family of fuzzy subsets

$$U = \{B_\varepsilon \cap \underline{r} \mid \varepsilon > 0, r \in (0, 1]\}$$

has the stratified structure $\{U_\lambda\}_{\lambda \in (0, 1]}$. Therefore U is a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0, 1]}$ in $(X, T_{\|\cdot\|})$.

Theorem 3.7. Every I -tvs has a balanced local base with stratified structure.

Proof. We know that every I -tvs (X, T) has a balanced Q -neighborhood base B_λ of θ_λ for each $\lambda \in (0, 1]$ (see [12]). Now define

$$U_\lambda = \bigcup_{\alpha \in (0, \lambda)} B_\alpha \quad \text{and} \quad U = \bigcup_{\lambda \in (0, 1)} U_\lambda.$$

By Example 3.4, we know that $\{U_\lambda\}_{\lambda \in (0, 1]}$ is the stratified structure of U .

It remains to prove that for each $\lambda \in (0, 1]$, U_λ is a Q -neighborhood base of θ_λ . It is not difficult to see that members of U_λ are all Q -neighborhoods of θ_λ . Now let W be an arbitrary Q -neighborhood of θ_λ . It is evident that there exists $\mu \in (0, \lambda)$ such that W is also Q -neighborhood of θ_μ . Since B_μ is a Q -neighborhood base of θ_μ , there exists $U \in B_\mu$ such that $U \subset W$. Note that $B_\mu \subset U_\lambda$. Hence $U \in U_\lambda$. This shows that U_λ is a Q -neighborhood base of θ_λ . Therefore U is a local base with stratified structure in (X, T) . \square

Theorem 3.8. Every W -neighborhood base B of θ in an I -tvs (X, T) is a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0, 1]}$, where $U_\lambda = \{U \in B \mid U(\theta) > 1 - \lambda\}$.

Proof. Since B is a W -neighborhood base of θ , by Proposition 2.6 we know that $U_\lambda = \{U \in B \mid U(\theta) > 1 - \lambda\}$ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$, and $B = \bigcup_{\lambda \in (0, 1]} U_\lambda$. In the following, we need only to show that $U_\lambda = \bigcup_{\mu \in (0, \lambda)} U_\mu$.

By the definition of U_λ , it is evident that $\mu < \lambda$ implies that $U_\mu \subset U_\lambda$. Hence $\bigcup_{\mu \in (0, \lambda)} U_\mu \subset U_\lambda$. On the other hand, if $U \in U_\lambda$, then $U \in B$ and $U(\theta) > 1 - \lambda$. Taking $\mu \in (1 - U(\theta), \lambda)$, it is not difficult to see that $U \in U_\mu$, and so $U \in \bigcup_{\mu \in (0, \lambda)} U_\mu$. This shows that $U_\lambda = \bigcup_{\mu \in (0, \lambda)} U_\mu$. Therefore B is a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0, 1]}$ in (X, T) . \square

As a special case of Theorem 3.8, we have the following example.

Example 3.9. Let (X, T) be an I -tvs and $U = \{G \in I^X \mid G \in T, G(\theta) > 0\}$. It is evident that U is an open W -neighborhood base of θ in (X, T) . Hence, by Theorem 3.8, we conclude that U is a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$ in (X, T) , where $U_\lambda = \{U \in T \mid U(\theta) > 1 - \lambda\}$.

4. Main Results

Theorem 4.1. Let (X, T) be an I -tvs. Then the local base U with stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$ in (X, T) has the following properties:

- (1) If $U \in U_\lambda, V \in U_\lambda$ or $V = \underline{r}$ ($1 - \lambda < r \leq 1$), then there exists $W \in U_\lambda$ such that $W \subset U \cap V$;
- (2) If $U \in U_\lambda$, then there exists $V \in U_\lambda$ such that $V + V \subset U$;
- (3) If $U \in U_\lambda$, then there exists $V \in U_\lambda$ such that $kV \subset U$ for all $|k| \leq 1$;
- (4) If $U \in U_\lambda$, then for each $x \in X$, there is an $\alpha > 0$ such that $x_\alpha \tilde{\in} \alpha U$.

Conversely, let X be a vector space over \mathbb{K} and U be a family of fuzzy subsets on X with the stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$ which satisfies the above conditions (1)–(4). Then there exists a unique I -topology T on X such that (X, T) is an I -tvs and U is a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$ in (X, T) .

Proof. (1) is evident. The proofs of (2), (3) and (4) are easy by Proposition 2.5 (see also the proof of Theorem 4.1 in [2]).

Conversely, without loss of generality, we can suppose that every member U in U satisfies $U(\theta) = \sup_{x \in X} U(x)$. Otherwise, we substitute $\tilde{U} = \bigcup_{\lambda \in (0,1]} \tilde{U}_\lambda$ for U , where $\tilde{U}_\lambda = \{U \cap \underline{U}(\theta) \mid U \in U_\lambda\}$. Obviously, $\{\tilde{U}_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of \tilde{U} and satisfies the conditions (1)–(4). We set

$$T = \{G \in I^X \mid \forall x_\mu \tilde{\in} G, \exists V \in U_\mu \text{ such that } x + V \subset G\}.$$

Then it is not difficult to verify that T is a stratified I -topology on X .

In fact, for each $r \in (0, 1]$, if $x_\lambda \tilde{\in} \underline{r}$, i.e., $r > 1 - \lambda$, then the condition (1) implies that there exists $W \in U_\lambda$ such that $W \subset \underline{r}$, and so $x + W \subset x + \underline{r} = \underline{r}$. This shows $\underline{r} \in T$. Moreover, it is easy to show that T is closed under arbitrary unions and finite intersections. Therefore T is a stratified I -topology on X .

For each $U \in U$, define a fuzzy set G_U on X as follows:

$$G_U(x) = \sup\{1 - \mu \mid x + V \subset U \text{ for some } V \in U_\mu\}. \quad (4.1)$$

We can prove the following conclusions:

- (a) $G_U \in T$; (b) $G_U \subset U$; (c) $U \in U_\lambda \Rightarrow \theta_\lambda \tilde{\in} G_U$.

(a) Let $x_\mu \tilde{\in} G_U$, i.e., $G_U(x) > 1 - \mu$. By (4.1), there exist $\nu \in (0, \mu)$ and $V \in U_\nu$ such that $x + V \subset U$. Note that $\nu < \mu$ implies $U_\nu \subset U_\mu$. Hence $V \in U_\mu$. By (2), there exists $P \in U_\mu$ such that $P + P \subset V$, and so $x + P + P \subset x + V \subset U$. Since $P \in U_\mu = \bigcup_{\alpha \in (0, \mu)} U_\alpha$, there exists $\delta \in (0, \mu)$ such that $P \in U_{\mu - \delta}$. By (1),

for each $\varepsilon > 0$, there exists $W \in U_{\mu-\delta}$ such that $W \subset P \cap \underline{(1 - \mu + \delta + \varepsilon)}$. We can prove that $x + W \subset G_U$.

In fact, if $(x + W)(z) = \alpha > 0$, then $W(z - x) = \alpha$, and so

$$\begin{aligned} (x + W + W)(w) &= (W + W)(w - x) \geq W(w - z) \wedge W(z - x) \\ &= W(w - z) \wedge \alpha = (z + W \cap \underline{\alpha})(w), \quad \forall w \in X, \end{aligned}$$

which implies that $z + W \cap \underline{\alpha} \subset x + W + W \subset x + P + P \subset U$. Note that

$$1 - \mu + \delta + \varepsilon \geq W(z - x) = \alpha \implies 1 - \alpha + \varepsilon \geq \mu - \delta \implies U_{\mu-\delta} \subset U_{1-\alpha+\varepsilon}.$$

Hence $W \in U_{1-\alpha+\varepsilon}$. Since $\alpha > 1 - (1 - \alpha + \varepsilon)$, by (1) there exists $W_0 \in U_{1-\alpha+\varepsilon}$ such that $W_0 \subset W \cap \underline{\alpha}$. So, we have $z + W_0 \subset z + W \cap \underline{\alpha} \subset U$. It follows from (4.1) that $G_U(z) \geq \alpha - \varepsilon$. By the arbitrariness of ε , we get $G_U(z) \geq \alpha$. This shows that $x + W \subset G_U$. Note that $W \in U_{\mu-\delta} \subset U_\mu$. Hence $G_U \in T$.

(b) If $x \in X$ and $G_U(x) = \alpha > 0$, then by (4.1) for each $\varepsilon \in (0, \alpha)$, there exist $\mu \in (0, 1]$ with $1 - \mu > \alpha - \varepsilon$ and $V \in U_\mu$ such that $x + V \subset U$. This implies that $U(x) \geq V(\theta) > 1 - \mu > \alpha - \varepsilon$. By the arbitrariness of ε , we get $U(x) \geq \alpha = G_U(x)$. This shows that $G_U \subset U$.

(c) Let $U \in U_\lambda$. Note that $U_\lambda = \bigcup_{\mu \in (0, \lambda)} U_\mu$. Then there exists $\mu \in (0, \lambda)$ such that $U \in U_\mu$. Since $\theta + U = U$, by (4.1) we have $G_U(\theta) \geq 1 - \mu > 1 - \lambda$, i.e., $\theta_\lambda \tilde{\in} G_U$.

From (a)–(c), it is easy to see that every member in U_λ is a Q -neighborhood of θ_λ in (X, T) . Again, by the definition of T , we conclude that U_λ is a Q -neighborhood base of θ_λ and T is a translation invariant, i.e., $G \in T$ implies that $x + G \in T$ for each $x \in X$. Therefore $U_{x_\lambda} = \{x + U \mid U \in U_\lambda\}$ is a Q -neighborhood base of x_λ .

Finally, we prove that the addition f and the scalar multiplication g are both I -continuous.

Let $(x, y)_\lambda$ be a fuzzy point on $X \times X$ and W be a Q -neighborhood of $(x + y)_\lambda$. Then there exists $U \in U_\lambda$ such that $(x + y) + U \subset W$. By (2), there exists $V \in U_\lambda$ such that $V + V \subset U$, and so we have $(x + V) + (y + V) \subset (x + y) + U \subset W$. Note that $x + V \in U_{x_\lambda}$, $y + V \in U_{y_\lambda}$. By Proposition 2.5, we infer that f is I -continuous.

Let $(k_0, x)_\lambda$ be a fuzzy point on $\mathbb{K} \times X$ and W be a Q -neighborhood of $(k_0 x)_\lambda$. Then there exists $U \in U_\lambda$ such that $k_0 x + U \subset W$. By (2), there exists $V \in U_\lambda$ such that $V + V \subset U$. By (3), there exists $V_1 \in U_\lambda$ such that $tV_1 \subset V$ for all $|t| \leq 1$. By (4), there exists a $\beta > 0$ such that $x_\lambda \tilde{\in} \beta V_1$. Put $s = (\beta V_1)(x)$, then $s > 1 - \lambda$, and so we have

$$rx_s \in r\beta V_1 \subset V \quad \text{for all } r \in \mathbb{K} \text{ with } |r| \leq 1/\beta. \quad (4.2)$$

Put $N = \lceil |k_0| \rceil + 1$, where $\lceil |k_0| \rceil$ is the largest integral part of $|k_0|$. Take $n \in \mathbb{N}$ with $2^n \geq N$. Repeatedly make use of (2), there exists $V_2 \in U_\lambda$ such that $\underbrace{V_2 + \dots + V_2}_{2^n} \subset V$. Again by (3), there exists $V_3 \in U_\lambda$ such that $tV_3 \subset V_2$ for all

$|t| \leq 1$. Put $\delta_1 = N - |k_0|$, then when $|k - k_0| \leq \delta_1$ we have

$$kV_3 \subset NV_2 \subset \underbrace{V_2 + \cdots + V_2}_{2^n} \subset V \quad (\text{because } V_2(\theta) = \sup_{x \in X} V_2(x)). \quad (4.3)$$

Let $\delta = \min\{\delta_1, 1/\beta\}$. It follows from (4.2) and (4.3) that

$$(k - k_0)x + (kV_3) \cap \underline{s} = (k - k_0)x_s + (kV_3) \cap \underline{s} \subset V + V \subset U \quad (4.4)$$

whenever $|k - k_0| \leq \delta$, which implies that

$$k(x + V_3 \cap \underline{s}) = kx + kV_3 \cap \underline{s} \subset k_0x + U \subset W. \quad (4.5)$$

whenever $|k - k_0| \leq \delta$. Note that $x + V_3 \cap \underline{s}$ is a Q -neighborhood of x_λ . By (4.5) and Proposition 2.5, we conclude that g is I -continuous. This shows that (X, T) is an I -topological vector space and U_λ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$. Note that $\{U_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of U . Therefore U is a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$ in (X, T) . \square

Remark 4.2. Theorem 4.1 is an improvement of Theorem 4.1 in [2]. Here, it is necessary to point out that in the proof of Theorem 4.1 in [2], the argument for continuity of g is incorrect.

In fact, to prove the continuity of g , it suffices to show that for each fuzzy point $(k_0, x)_\lambda$ on $\mathbb{K} \times X$ and each Q -neighborhood W of k_0x_λ , there exist $\delta > 0$ and $V_0 \in U_\lambda$ such that $k(x + V_0) \subset W$ for all $k \in \mathbb{K}$ with $|k - k_0| < \delta$. Please notice that the mentioned V_0 is irrelative to k . However, in the proof of [2] (see page 344–345), the arisen $V^{(2)}$ in (4.2) is relative to k and V_0 is relative to $V^{(2)}$ and $V^{(4)}$, $V_0 \subset V^{(2)} \cap V^{(4)}$, and so V_0 is also relative to k . Obviously, we need not such a V_0 .

Imitating the proof of Theorem 4.1, we easily correct the faults in [2].

As a consequence of Theorem 4.1, we obtain the following theorem:

Theorem 4.3. *Let (X, T) be an I -tvs and B be a W -neighborhood base of θ . The B has the following properties:*

- (i) $U \in B$ implies that $U(\theta) > 0$.
- (ii) If $U \in B$, $V \in B$ or $V = \underline{r}$ with $0 \leq \alpha < U(\theta) \wedge V(\theta)$, then there exists $W \in B$ with $W(\theta) > \alpha$ such that $W \subset U \cap V$.
- (iii) If $U \in B$ with $U(\theta) > \alpha \geq 0$, then there exists $V \in B$ with $V(\theta) > \alpha$ such that $V + V \subset U$.
- (iv) If $U \in B$ with $U(\theta) > \alpha \geq 0$, there exists $V \in B$ with $V(\theta) > \alpha$ such that $tV \subset U$ for all $t \in \mathbb{K}$ with $|t| \leq 1$.
- (v) If $U \in B$ with $U(\theta) > \alpha \geq 0$, then for any $x \in X$, there exists $t > 0$ such that $(tU)(x) > \alpha$.

Conversely, let X be a vector space over \mathbb{K} and B be a family of fuzzy sets on X satisfying the above conditions (i)-(v). Then there exists a unique I -topology T on X such that (X, T) is an I -tvs and B is a W -neighborhood base of θ .

Proof. Put $U_\lambda = \{U \in B \mid U(\theta) > 1 - \lambda\}$, $\alpha = 1 - \lambda$. Then it is easy to see that (ii)–(iv) are equivalent to (1)–(4) in Theorem 4.1. Moreover, if B is a W -neighborhood base of θ , then by Theorem 3.8 we know that B is a local base

with the stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$. Therefore the conclusion can follow from Theorem 4.1 and Proposition 2.6 immediately. \square

Remark 4.4. Theorem 4.3 is an improvement of Theorem 4.2 in [2] and Theorem 4.3 in [8]. The conditions (i)–(vii) of Theorem 4.3 in [8] have been simplified.

As applications of Theorems 4.1 and 4.3, we give the following two examples.

Example 4.5. Let $C(\mathbb{R})$ denote the set of all continuous real valued functions on \mathbb{R} . For each $\alpha \in (0, 1]$, we define a mapping $\|\cdot\|_\alpha : C(\mathbb{R}) \rightarrow [0, \infty)$ by

$$\|x\|_\alpha = \int_{-\frac{1}{\alpha}}^{\frac{1}{\alpha}} |x(t)| dt. \quad (4.6)$$

Obviously, $\|\cdot\|_\alpha$ has the following properties:

- (a) $0 < \beta < \alpha \leq 1$ implies $\|x\|_\alpha \leq \|x\|_\beta$ for all $x \in C(\mathbb{R})$;
- (b) $\|x + y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha$ for all $x, y \in C(\mathbb{R})$.

For each $\alpha \in (0, 1]$ and $r > 0$, we define a fuzzy set $U_{\alpha,r}$ on X by

$$U_{\alpha,r}(x) = \begin{cases} \alpha & \text{if } \|x\|_\alpha < r, \\ 0 & \text{otherwise} \end{cases} \quad (4.7)$$

where $\|x\|_\alpha$ is defined by (4.6). Then by Theorem 4.1, we can prove that there exists a unique I -topology T on $C(\mathbb{R})$ such that $(C(\mathbb{R}), T)$ is an I -tvs and the family of fuzzy sets $U = \{U_{\alpha,r} \mid \alpha, r \in (0, 1]\}$ is a local base with the stratified structure $\{U_\lambda\}_{\lambda \in (0,1]}$ in $(C(\mathbb{R}), T)$, where

$$U_\lambda = \{U_{\alpha,r} \mid \alpha \in (1 - \lambda, 1], r > 0\}. \quad (4.8)$$

In fact, it is easy to prove that $\{U_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of U . It remains to prove that $\{U_\lambda\}_{\lambda \in (0,1]}$ satisfies the conditions (1)–(4) in Theorem 4.1.

By (4.7) it is easy to see that $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha}$, where

$$B_{\alpha,r} = \{x \in X \mid \|x\|_\alpha < r\}. \quad (4.9)$$

(1) Let $U_{\alpha_i,r_i} \in U_\lambda$ ($i = 1, 2$). Then $U_{\alpha_i,r_i} = B_{\alpha_i,r_i} \cap \underline{\alpha_i}$, $\alpha_i > 1 - \lambda$ ($i = 1, 2$). Put $\alpha = \min\{\alpha_1, \alpha_2\}$ and $r = \min\{r_1, r_2\}$. It is obvious that $U_{\alpha,r} \in U_\lambda$ and $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha} \subset [B_{\alpha_1,r_1} \cap \underline{\alpha_1}] \cap [B_{\alpha_2,r_2} \cap \underline{\alpha_2}] = U_{\alpha_1,r_1} \cap U_{\alpha_2,r_2}$.

Let $U_{\alpha,r} \in U_\lambda$ and $\sigma \in (1 - \lambda, 1]$. Then $\beta = \min\{\alpha, \sigma\} \in (1 - \lambda, 1]$, and so $U_{\beta,r} \in U_\lambda$ and $U_{\beta,r} = B_{\beta,r} \cap \underline{\beta} \subset B_{\alpha,r} \cap \underline{\alpha} \cap \underline{\sigma} = U_{\alpha,r} \cap \underline{\sigma}$.

(2) Let $U_{\alpha,r} \in U_\lambda$. Then $U_{\alpha,\frac{r}{2}} \in U_\lambda$ and

$$U_{\alpha,\frac{r}{2}} + U_{\alpha,\frac{r}{2}} = B_{\alpha,\frac{r}{2}} \cap \underline{\alpha} + B_{\alpha,\frac{r}{2}} \cap \underline{\alpha} \subset B_{\alpha,r} \cap \underline{\alpha} = U_{\alpha,r}.$$

(3) Let $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha} \in U_\lambda$. By (4.6) and (4.9), it is not difficult to prove that $kB_{\alpha,r} \subset B_{\alpha,r}$ for all $|k| \leq 1$. Hence, we have $kU_{\alpha,r} \subset U_{\alpha,r}$ for all $|k| \leq 1$.

(4) Let $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha} \in U_\lambda$. For each $x \in X$, we take $t > \frac{\|x\|_\alpha}{r}$. It follows that $\|(1/t)x\|_\alpha < r$, i.e., $(1/t)x \in B_{\alpha,r}$. Note that $\alpha > 1 - \lambda$. Then we have $(1/t)x_\lambda \in B_{\alpha,r} \cap \underline{\alpha} = U_{\alpha,r}$, i.e., $x_\lambda \in tU_{\alpha,r}$.

Thus, by Theorem 4.1 we know that the conclusion of Example 4.5 holds.

Remark 4.6. Similarly, it is not difficult to verify that U in Example 4.5 satisfies the conditions (i)–(v) of Theorem 4.3. Therefore, by Theorem 4.3, U is a W -neighborhood base of θ in the I -tvs $(C(\mathbb{R}), T)$.

Let X be a real vector space and \tilde{f} be a fuzzy linear functional on $\text{Pt}(I^X)$, i.e., a fuzzy linear operator from $\text{Pt}(I^X)$ into $\text{Pt}(I^{\mathbb{R}})$ (See [3, Definition 4] or [14, Definition 2.3]). By [3, Theorem 4] or [14, Lemma 2.5], we know that \tilde{f} is a fuzzy linear functional on $\text{Pt}(I^X)$ if and only if there exist an ordinary linear functional on X and an order-homomorphism preserving finite meets $\phi : I \rightarrow I$ such that

$$\tilde{f}(x_\lambda) = [f(x)]_{\phi(\lambda)} \quad \text{for all } x_\lambda \in \text{Pt}(I^X).$$

In the following, for convenience, we use $(f, \phi)^\rightarrow$ instead of the fuzzy linear functional \tilde{f} . Moreover, we define the mapping $|\cdot| : \text{Pt}(I^{\mathbb{R}}) \rightarrow [0, +\infty)$ by $|y_\mu| = (1/\mu)|y|$ for all $y_\mu \in \text{Pt}(I^{\mathbb{R}})$.

Example 4.7. Let X be a real vector space and $\tilde{C}_\phi(X)$ denote the set of all fuzzy linear functionals $(f, \phi)^\rightarrow$ on $\text{Pt}(I^X)$ with the same ϕ , i.e., $\tilde{C}_\phi(X) = \{(f, \phi)^\rightarrow \mid f \in X^\#\}$, where $X^\#$ denotes the set of all ordinary linear functionals on X and ϕ is a fixed order-homomorphism preserving finite meets from I into I . For each $f \in X^\#$ and $t > 0$, we define a fuzzy set $B_{f,t}$ on X by

$$B_{f,t}(x) = \sup \{1 - \lambda : |(f, \phi)^\rightarrow(x_\lambda)| < t\} \quad \text{for all } x \in X. \quad (4.10)$$

and put $U(f_1, \dots, f_n; t) = \bigcap_{i=1}^n B_{f_i,t}$. Then by Theorem 4.3, we can prove that there exists a unique I -topology T on X such that (X, T) is an I -tvs and the family of fuzzy sets

$$B = \{U(f_1, \dots, f_n; t) \cap \underline{r} \mid f_i \in X^\#, i = 1, \dots, n, n \in \mathbb{N}; t > 0, r \in (0, 1]\}$$

is a W -neighborhood base of θ in (X, T) .

To prove this conclusion, we first prove that

$$x_\lambda \tilde{\in} B_{f,t} \iff |(f, \phi)^\rightarrow(x_\lambda)| < t, \quad \text{i.e., } |f(x)| < \phi(\lambda)t. \quad (4.11)$$

In fact, if $x_\lambda \tilde{\in} B_{f,t}$, i.e., $B_{f,t}(x) > 1 - \lambda$, then by (4.10) there exists $\lambda_0 \in (0, \lambda)$ such that $|f(x)| < \phi(\lambda_0)t$. Note that ϕ is nondecreasing. Hence we have $|f(x)| < \phi(\lambda)t$. Conversely, let $|f(x)| < \phi(\lambda)t$. Note that ϕ is union-preserving. Taking $\lambda_n \in (0, \lambda)$ with $\lambda_n \nearrow \lambda$, we have

$$|f(x)| < \phi\left(\bigvee_{n \in \mathbb{N}} \lambda_n\right)t = \bigvee_{n \in \mathbb{N}} \phi(\lambda_n)t.$$

Thus, there exists some $n_0 \in \mathbb{N}$ such that $|f(x)| < \phi(\lambda_{n_0})t$, i.e., $|(f, \phi)^\rightarrow(x_{\lambda_{n_0}})| < t$. By (4.10) we get $B_{f,t}(x) \geq 1 - \lambda_{n_0} > 1 - \lambda$, i.e., $x_\lambda \tilde{\in} B_{f,t}$. (4.11) is proved.

By using (4.10) and (4.11), it is easy to prove that $B_{f,t}$ has the following properties:

(a) $B_{f,t}(\theta) = 1$;

- (b) $0 < s < t$ implies that $B_{f,s} \subset B_{f,t}$;
 (c) $B_{f,\frac{t}{2}} + B_{f,\frac{t}{2}} \subset B_{f,t}$;
 (d) $B_{f,t}$ is balanced, i.e., $kB_{f,t} \subset B_{f,t}$ for all $|k| \leq 1$;

now we prove that \mathbf{B} satisfies the conditions (i)–(v)

(i) Let $W = U(f_1, \dots, f_n; t) \cap \underline{r} \in \mathbf{B}$. By (a), we know that $W(\theta) = r > 0$.

(ii) By (b), we have that $U(f, g; t \wedge s) \subset B_{f,t} \cap B_{g,s}$ for all $f, g \in X^\#$ and $t, s > 0$. So, if $W_1 = U(f_1, \dots, f_n; t) \cap \underline{r_1}$, $W_2 = U(g_1, \dots, g_m; s) \cap \underline{r_2} \in \mathbf{B}$ with $W_1(\theta) \wedge W_2(\theta) > \alpha \geq 0$, then there exists $W = U(f_1, \dots, f_n, g_1, \dots, g_m; t \wedge s) \cap \underline{r_1 \wedge r_2} \in \mathbf{B}$ such that $W(\theta) > \alpha$ and $W \subset W_1 \cap W_2$.

If $W_1 = U(f_1, \dots, f_n; t) \cap \underline{r_1} \in \mathbf{B}$ and $W_2 = \underline{r_2}$ ($r_2 > 0$) with $W_1(\theta) \wedge W_2(\theta) > \alpha \geq 0$, then there exists $W = U(f_1, \dots, f_n; t) \cap \underline{r_1 \wedge r_2} \in \mathbf{B}$ such that $W(\theta) > \alpha$ and $W \subset W_1 \cap W_2$.

(iii) If $W = U(f_1, \dots, f_n; t) \cap \underline{r} \in \mathbf{B}$ and $W(\theta) = r > \alpha \geq 0$, then it follows from (c) that

$$U(f_1, \dots, f_n; \frac{t}{2}) + U(f_1, \dots, f_n; \frac{t}{2}) \subset U(f_1, \dots, f_n; t),$$

and so there exists $V = U(f_1, \dots, f_n; t) \cap \underline{r} \in \mathbf{B}$ such that $V(\theta) = r > \alpha$ and $V + V \subset W$.

(iv) Let $W = U(f_1, \dots, f_n; t) \cap \underline{r} \in \mathbf{B}$ and $W(\theta) > \alpha \geq 0$. By (d), we know that each $B_{f_i,t}$ is balanced ($i = 1, \dots, n$). Hence $U(f_1, \dots, f_n; t)$ is also balanced, and so we have $kW \subset W$ for all $|k| \leq 1$.

(v) Let $W = U(f_1, \dots, f_n; t) \cap \underline{r} \in \mathbf{B}$ and $W(\theta) > \alpha \geq 0$. For any $x \in X$, taking $\beta > \frac{\max_{1 \leq i \leq n} |f_i(x)|}{t\phi(1-\alpha)}$, then we have $|f_i(x/\beta)| < \phi(1-\alpha)t$, $i = 1, 2, \dots, n$. It follows from (4.11) that $(x/\beta)_{1-\alpha} \tilde{\in} \bigcap_{i=1}^n B_{f_i,t} = U(f_1, \dots, f_n; t)$. Note that $r = W(\theta) > \alpha$. Hence $W(x/\beta) = [U(f_1, \dots, f_n; t) \cap \underline{r}](x/\beta) > \alpha$ i.e., $(\beta W)(x) > \alpha$.

Thus, By Theorem 4.3, we have the conclusion of Example 4.7.

REFERENCES

- [1] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.
- [2] J. X. Fang, *On local bases of fuzzy topological vector spaces*, Fuzzy Sets and Systems, **87** (1997), 341-347.
- [3] J. X. Fang, *Fuzzy linear order-homomorphism and its structures*, J. Fuzzy Math., **4(1)** (1996), 93-102.
- [4] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, **48** (1992), 239-248.
- [5] U. Höhle and S. E. Rodabaugh (Eds.), *Mathematics of fuzzy sets: logic, topology and measure theory*, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht, **3** (1999).
- [6] S. Q. Jing and C. H. Yan, *Fuzzy bounded sets and totally fuzzy bounded sets in I-topological vector spaces*, Iranian Journal of Fuzzy Systems, **4(1)** (2009).
- [7] A. K. Katsaras and D. B. Liu, *Fuzzy vector spaces and fuzzy topological vector spaces*, J. Math. Anal. Appl., **58** (1997), 135-146.
- [8] A. K. Katsaras, *Fuzzy topological vector spaces I*, Fuzzy Sets and Systems, **6** (1981), 85-95.
- [9] P. M. Pu and Y. M. Liu, *Fuzzy topology I, neighborhood structures of a fuzzy points and Moore-Smith convergence*, J. Math. Anal. Appl., **76** (1980), 571-599.
- [10] G. J. Wang, *Order-homomorphism of fuzzes*, Fuzzy Sets and Systems, **12** (1982), 281-288.

- [11] R. H. Warren, *Neighborhoods bases and continuity in fuzzy topological spaces*, Rocky Mountain J. Math., **8** (1978), 459-470.
- [12] C. X. Wu and J. X. Fang, *Redefine of fuzzy topological vector space*, Since Exploration, in Chinese, **2(4)** (1982), 113-116.
- [13] G. H. Xu and J. X. Fang, *A new I -vector topology generated by a fuzzy norm*, Fuzzy Sets and Systems, **158** (2007), 2375-2385.
- [14] C. H. Yan and J. X. Fang, *L -fuzzy bilinear operator and its continuity*, Iranian Journal of Fuzzy Systems, **4(1)** (2007), 65-73.

JIN-XUAN FANG, SCHOOL OF MATHEMATICAL SCIENCE, NANJING NORMAL UNIVERSITY, NANJING, JIANGSU 210097, P. R. CHINA

E-mail address: jxfang@njnu.edu.cn

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