ABOUT THE FUZZY GRADE OF THE DIRECT PRODUCT OF TWO HYPERGROUPOIDS

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ABSTRACT. The aim of this paper is the study of the sequence of join spaces and fuzzy subsets associated with a hypergroupoid. In this paper we give some properties of the membership function $\widetilde{\mu}_{\otimes}$ corresponding to the direct product of two hypergroupoids and we determine the fuzzy grade of the hypergroupoid $\langle H \times H, \otimes \rangle$ in a particular case.

1. Introduction

The hyperstructure theory, which is a generalization of the ordinary algebraic structure theory, has been introduced by F.Marty in 1934 [21] at the 8^{th} Congress of the Scandinavian Mathematicians. The hyperstructures are now used extensively from both theoretical point of view and their numerous applications; some of them, especially those from the last decades, are presented by Corsini and Leoreanu [11] and they are connected with: geometry, hypergraphs, binary relations, fuzzy sets and rough sets, automata, codes, cryptography, artificial intelligence and probabilities.

Real situations are very often not crisp and deterministic and they cannot be described precisely. The vagueness concerning the description of the semantic meaning of the events, phenomena or statements is called *fuzziness*. Since the notion of fuzzy set has been introduced by L.A. Zadeh in 1965 [25], there have been attempts to extend useful mathematical notions to this domain, replacing the crisp sets by the fuzzy sets. The first association between fuzzy sets and algebraic structures has been done by A. Rosenfeld in 1971 [23], when he applied the concept of fuzzy set to the theory of groups and studied fuzzy subgroups of a group. Later, many researchers investigated connections between fuzzy sets and hyperstructures: P. Corsini, B. Davvaz, V. Leoreanu-Fotea, R. Ameri, M.M. Zahedi, J. Zhan, Y.B. Yun, Ath. Kehagias and K. Serafimidis, H. Hedayati, M. Bakhshi, R.A. Borzooei, M. Ştefănescu, I. Cristea and many others (see for example [1, 2, 3], [5]-[10], [12]-[20], [24], [26]). In this paper we deal with those introduced by P. Corsini in 1994 and 2003 (see [5, 6]), which lead to a sequence of join spaces associated with a hypergroupoid. Recently, new results about this connection have been obtained (see [14, 15, 24]); here, we remember some of them: a sufficient condition such that two consecutive join spaces of the sequence are not isomorphic, that is the sequence

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does not end, has been determined; for any natural number $n \geq 2$, there is a hypergroup such that the length of its associated sequence is equal to n; various properties of the hypergroups in connection with the reduced hypergroups have been determined.

Here we analyze similar properties for the direct product of hypergroups and we give some new results for the theory of the fuzzy subhypergroups.

2. Preliminaries

We recall some basic definitions for the sake of completeness (see [4]).

Let H denote a nonempty set and $\mathcal{P}^*(H)$ the set of all nonempty subsets of H.

Definition 2.1. A set H endowed with a mapping $\circ : H^2 \longrightarrow \mathcal{P}^*(H)$, named *hyperoperation*, is called a *hypergroupoid*.

The image of the pair $(a,b) \in H^2$ is denoted by $a \circ b$ and it is called the *hyperproduct* of a and b.

If A and B are nonempty subsets of H, then $A \circ B = \bigcup\limits_{\substack{a\in A \\ b\in B}} a \circ b$.

Definition 2.2. A *semihypergroup* is a hypergroupoid $\langle H, \circ\rangle$ with the property: for any $(a,b,c) \in H^3, (a \circ b) \circ c = a \circ (b \circ c)$. A *quasihypergroup* is a hypergroupoid $\langle H, \circ\rangle$ which satisfies the *reproductive law*: for any $a \in H, H \circ a = a \circ H = H$.
A *hypergroup* is a semihypergroup which is also a quasihypergroup. A hypergroup is called a *total hypergroup*, if, for any $(x,y) \in H^2$, $x \circ y = H$.

For each pair $(a,b) \in H^2$, we denote:
$a/b = \{x \mid a \in x \circ b\}$ and $b\backslash a = \{y \mid a \in b \circ y\}$.

Definition 2.3. A commutative hypergroupoid $\langle H, \circ\rangle$ is called a *join space* if, for any $(a,b,c,d) \in H^4$ such that $a/b \cap c/d \neq \emptyset$ it results $a \circ d \cap b \circ c \neq \emptyset$.

The notion of join space has been introduced and studied for the first time by W.Prenowitz. Later, together with J.Jantosciak, he has reconstructed, from the algebraic point of view, several branches of geometry: the projective, the descriptive and the spherical geometry (see [22]).

Definition 2.4. Let ρ be an equivalence relation on a hypergroupoid $\langle H, \circ\rangle$. We say that ρ is *regular to the right* if $a\rho b$ implies that, for any $u \in H$, for any $x \in a \circ u$, there exists $y \in b \circ u$ such that $x\rho y$ and for any $\bar{y} \in b \circ u$, there exists $\bar{x} \in a \circ u$ such that $\bar{x}\rho\bar{y}$. Similarly, the *regularity to the left* can be defined. We say that ρ is *regular* if it is regular to the right and to the left.

3. Some Properties of the Fuzzy Subset $\widetilde{\mu}_{\otimes}$ Associated with the Direct Product $\langle H_1 \times H_2, \otimes\rangle$

First we remember the construction of the sequence of join spaces and membership functions associated with a hypergroupoid. P. Corsini, I. Cristea, M. Ştefănescu have determined several properties of the sequence in general and also for particular hypergroups: i.p.s. hypergroups, complete hypergroups, 1-hypergroups ([7, 8, 9], [12, 13]). In this section we give some properties of the fuzzy subsets $\widetilde{\mu}_{\otimes}$ and $\widetilde{\mu}_1 \times \widetilde{\mu}_2$ in relation with the direct product of hypergroupoids. More specifically, we establish, in Theorem 3.6, a relation between the corresponding membership functions associated with two distinct hypergroupoids and the membership function associated with their direct product.

For any hypergroupoid $\langle H, \circ \rangle$, Corsini [6] defined a fuzzy subset $\widetilde{\mu}$ of H in the following way.

Definition 3.1. For any $(x, y) \in H^2$ and any $u \in H$, we consider:

$$\widetilde{\mu}(u) = q(u)^{-1} \sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}, \tag{1}$$

where $Q(u) = \{(a, b) \in H^2 \mid u \in a \circ b\}$, $q(u) = |Q(u)|$ and, for any $(x, y) \in H^2$, the set $x \circ y$ is finite.

In other words, $\widetilde{\mu}(u)$ is the average value of the reciprocals of the cardinalities of $x \circ y$ for all $x \circ y$ containing u.

Theorem 3.2. *With the hypergroupoid H endowed with the fuzzy subset $\widetilde{\mu}$, we associate the join space $\langle {}^1H, \circ_1 \rangle$, with ${}^1H = H$, as it follows (see [5]): for any $(x, y) \in H^2$,*

$$x \circ_1 y = \{z \in H \mid \widetilde{\mu}(x) \wedge \widetilde{\mu}(y) \leq \widetilde{\mu}(z) \leq \widetilde{\mu}(x) \vee \widetilde{\mu}(y)\}.$$

Using the same procedure as in (1), from ${}^1H$ we obtain a membership function ${}^1\widetilde{\mu}$ and the associated join space ${}^2H$ and so on. A sequence of fuzzy subsets and join spaces $(\langle {}^rH, \circ_r \rangle, {}^r\widetilde{\mu})_{r\geq 1}$ is determined in this way. We denote ${}^0\widetilde{\mu} = \widetilde{\mu}$, ${}^0H = H$.

The procedure is iterative: every hyperproduct "$\circ_i$" is defined on the set H, which is denoted by ${}^iH$ at the step i, to indicate the current number of iteration. From $\langle {}^iH, \circ_i \rangle$, we obtain as in (1) the membership function ${}^{i+1}\widetilde{\mu}$.

This construction is important for at least two reasons: it provides examples of hypergroup structures on a given set and it gives the possibility of studying fuzzy sets in an algebraic approach. On the other hand, the construction could start either from a fuzzy subset or from a hypergroup structure on a non-empty set H. The construction could be repeated, getting a sequence of hypergroups and fuzzy subsets on H. If two consecutive hypergroups are isomorphic, then the construction stops.

The length of this sequence has been called by Corsini and Cristea [7] the (strong) fuzzy grade of the hypergroupoid H. More precisely, they gave the following definition:

Definition 3.3. A hypergroupoid H has the *fuzzy grade* m, $m \in \mathbb{N}^*$, and we write $f.g.(H) = m$ if, for any i, $0 \leq i < m$, the join spaces ${}^iH$ and ${}^{i+1}H$ associated with H are not isomorphic and for any s, $s > m$, ${}^sH$ is isomorphic with ${}^mH$. We say that the hypergroupoid H has the *strong fuzzy grade* m and we write $s.f.g.(H) = m$ if $f.g.(H) = m$ and for all $s, s > m$, ${}^sH = {}^mH$.

For the sake of illustration we consider the following examples.

Example 3.4. Consider the hypergroupoid $H = \{a, b, c, d\}$ with the following hyperoperation:

$\circ$	a	b	c	d
a	a	$\{a,b\}$	$\{a,b,c\}$	H
b	$\{a,b\}$	b	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	c	$\{c,d\}$
d	H	$\{b,c,d\}$	$\{c,d\}$	d

then

$$\widetilde{\mu}(a) = \widetilde{\mu}(d) = 19/42, \quad \widetilde{\mu}(b) = \widetilde{\mu}(c) = 29/66$$

and so we have

$\circ_1$	a	b	c	d
a	$\{a,d\}$	H	H	$\{a,d\}$
b	H	$\{b,c\}$	$\{b,c\}$	H
c	H	$\{b,c\}$	$\{b,c\}$	H
d	$\{a,d\}$	H	H	$\{a,d\}$

then

$$\widetilde{\mu}_1(a) = \widetilde{\mu}_1(b) = \widetilde{\mu}_1(c) = \widetilde{\mu}_1(d) = 4/12,$$

and so we obtain

$\circ_2$	a	b	c	d
a	H	H	H	H
b	H	H	H	H
c	H	H	H	H
d	H	H	H	H

Thus, any associated join space ${}^sH$, $s \geq 3$ is identical with ${}^2H$ and we conclude that $s.f.g.(H) = 2$.

Example 3.5. Consider $H = \{a, b, c, d\}$ endowed with the fuzzy subset μ defined as it follows:

$$\mu(a) = 0.23, \quad \mu(b) = 0.45, \quad \mu(c) = \mu(d) = 0.78$$

then the associated join space $\langle H, \circ \rangle$ is represented by the table

$\circ$	a	b	c	d
a	a	$\{a,b\}$	H	H
b	$\{a,b\}$	b	$\{b,c,d\}$	$\{b,c,d\}$
c	H	$\{b,c,d\}$	$\{c,d\}$	$\{c,d\}$
d	H	$\{b,c,d\}$	$\{c,d\}$	$\{c,d\}$

Then we calculate

$$\widetilde{\mu}(a) = 3/7, \quad \widetilde{\mu}(b) = 13/33, \quad \widetilde{\mu}(c) = \widetilde{\mu}(d) = 13/36$$

and it follows that the join space $\langle {}^1H, \circ_1 \rangle = \langle H, \circ \rangle$; thus the sequence associated with H contains only one join space, so $s.f.g.(H) = 1$.

Let $\langle H_1, \circ_1 \rangle$ and $\langle H_2, \circ_2 \rangle$ be two hypergroupoids and, for any $i \in \{1,2\}$, let $\widetilde{\mu}_i$ be their associated membership function defined as in (1).

On the direct product $H_1 \times H_2$ we define the hyperproduct

$$(a_1, a_2) \otimes (b_1, b_2) = \{(x, y) \mid x \in a_1 \circ_1 b_1, y \in a_2 \circ_2 b_2\}$$

and we associate, as in (1), with the new hypergroupoid $\langle H_1 \times H_2, \otimes \rangle$ the corresponding fuzzy subset, denoted by $\widetilde{\mu}_{\otimes} : H_1 \times H_2 \longrightarrow [0,1]$.
Here we give a relation, useful in the third section, between the membership functions $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_{\otimes}$.

Theorem 3.6. *If $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_{\otimes}$ are the membership functions associated like in (1) with H_1, H_2 and $H_1 \times H_2$ respectively, then, for any $(x, y) \in H_1 \times H_2$, the following relation holds:*

$$\widetilde{\mu}_{\otimes}(x, y) = \widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y).$$

Proof. Using the notations from (1), we write

$$\widetilde{\mu}_1(x) = \frac{A_1(x)}{q_1(x)}, \text{ for any } x \in H_1 \text{ and } \widetilde{\mu}_2(y) = \frac{A_2(y)}{q_2(y)}, \text{ for any } y \in H_2.$$

For an arbitrary pair $(x, y) \in H_1 \times H_2$ we find:

$$\begin{aligned} q(x,y) &= |\{((a_1,a_2),(b_1,b_2)) \mid (x,y) \in (a_1,a_2) \otimes (b_1,b_2)\}| = \\ &= |\{((a_1,a_2),(b_1,b_2)) \mid x \in a_1 \circ_1 b_1, y \in a_2 \circ_2 b_2\}| = \\ &= |\{(a_1,b_1) \mid x \in a_1 \circ_1 b_1\}| \cdot |\{(a_2,b_2) \mid y \in a_2 \circ_2 b_2\}| = \\ &= q_1(x) \cdot q_2(y). \end{aligned} \tag{2}$$

Then we obtain:

$$\begin{aligned} |(a_1,a_2) \otimes (b_1,b_2)| &= |\{(x,y) \mid x \in a_1 \circ_1 b_1, y \in a_2 \circ_2 b_2\}| = \\ &= |\{x \in H_1 \mid x \in a_1 \circ_1 b_1\}| \cdot |\{y \in H_2 \mid y \in a_2 \circ_2 b_2\}| = \\ &= |a_1 \circ_1 b_1| \cdot |a_2 \circ_2 b_2| \end{aligned}$$

and therefore

$$\begin{aligned} A(x,y) &= \sum_{(x,y)\in(a_1,a_2)\otimes(b_1,b_2)} \frac{1}{|(a_1,a_2)\otimes(b_1,b_2)|} = \\ &= \sum_{\substack{(x,y)\in H_1\times H_2\\ x\in a_1\circ_1 b_1\\ y\in a_2\circ_2 b_2}} \left(\frac{1}{|a_1\circ_1 b_1|}\cdot\frac{1}{|a_2\circ_2 b_2|}\right) = \\ &= \sum_{x\in a_1\circ_1 b_1}\ \sum_{y\in a_2\circ_2 b_2} \frac{1}{|a_1\circ_1 b_1|}\cdot\frac{1}{|a_2\circ_2 b_2|} = \\ &= \left(\sum_{x\in a_1\circ_1 b_1}\frac{1}{|a_1\circ_1 b_1|}\right)\cdot\left(\sum_{y\in a_2\circ_2 b_2}\frac{1}{|a_2\circ_2 b_2|}\right) = \\ &= A_1(x)\cdot A_2(y). \end{aligned} \tag{3}$$

By consequence, $\widetilde{\mu}_\otimes(x,y) = \widetilde{\mu}_1(x)\cdot\widetilde{\mu}_2(y)$. □

Definition 3.7. (see [17])

(i) Let μ and λ be two fuzzy subsets of a nonempty set X. We say that μ *is contained in* λ (and we write $\mu\subseteq\lambda$) if, for any $x\in X$, $\mu(x)\le\lambda(x)$.

(ii) Let μ and λ be two fuzzy subsets of two nonempty sets X and Y respectively. We define on $X\times Y$ the fuzzy subset $\mu\times\lambda$ by

$$(\mu\times\lambda)(x,y) = \mu(x)\wedge\lambda(y), \forall(x,y)\in X\times Y.$$

(iii) Let μ and λ be two fuzzy subsets of a nonempty sets X. A fuzzy relation R of $X\times X$ (i.e. $R: X\times X\longrightarrow[0,1]$) is said to be *a fuzzy relation from* λ *to* μ if $R\subseteq\lambda\times\mu$.

Corollary 3.8. *If* $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ *and* $\widetilde{\mu}_\otimes$ *are the membership functions associated like in* (1) *with* H_1, H_2 *and* $H_1\times H_2$ *respectively, then we find* $\widetilde{\mu}_\otimes\subseteq\widetilde{\mu}_1\times\widetilde{\mu}_2$.

Proof. For any $(x_1,x_2)\in X_1\times X_2$ we obtain

$$\widetilde{\mu}_\otimes(x_1,x_2) = \widetilde{\mu}_1(x_1)\cdot\widetilde{\mu}_2(x_2)\le\widetilde{\mu}_1(x_1)\wedge\widetilde{\mu}_2(x_2) = (\widetilde{\mu}_1\times\widetilde{\mu}_2)(x_1,x_2).$$

□

Now, the following result is obvious

Corollary 3.9. *If* $\langle H_1,\circ_1\rangle = \langle H_2,\circ_2\rangle$ *and we denote them by* $\langle H,\circ\rangle$*, then the fuzzy relation* $\widetilde{\mu}_\otimes$ *of* $H\times H$ *is a fuzzy relation on* $\widetilde{\mu}_1=\widetilde{\mu}_2$ *denoted by* $\widetilde{\mu}$.

Definition 3.10. (see [16]) Let $\langle H,\circ\rangle$ be a hypergroup and let μ be a fuzzy subset of H. Then μ is called a *fuzzy subhypergroup* of H if the following axioms hold:

(i) $min\{\mu(x),\mu(y)\}\le \underset{\alpha\in x\circ y}{inf}\{\mu(\alpha)\}$, for all $x,y\in H$;

(ii) for all $x,a\in H$, there exists $y\in H$ such that:
$x\in a\circ y$ and $min\{\mu(a),\mu(x)\}\le\mu(y)$;

(iii) for all $x, a \in H$, there exists $z \in H$ such that:
$x \in z \circ a$ and $min\{\mu(a), \mu(x)\} \leq \mu(z)$.

Proposition 3.11. *Let H be a non-empty set and μ a fuzzy subset of H. Then μ is a fuzzy subhypergroup of $\langle H, \circ\rangle$, where, for any $x, y \in H$,*

$$x \circ y = y \circ x = \{z \in H \mid \mu(x) \wedge \mu(y) \leq \mu(z) \leq \mu(x) \vee \mu(y)\}. \tag{4}$$

Proof. The proof is the same as that of Proposition 2 [16]. □

Note. For this hyperproduct, we consider, from now on, that $\alpha \in x \circ y$ if and only if $\mu(x) \leq \mu(\alpha) \leq \mu(y)$ (that is, we suppose $\mu(x) \leq \mu(y)$; if $\mu(y) \leq \mu(x)$, we write $\alpha \in y \circ x$ if and only if $\mu(y) \leq \mu(\alpha) \leq \mu(x)$).

Proposition 3.12. *Let $\langle H, \circ\rangle$ be a hypergroupoid and $(\langle {}^rH, \circ_r\rangle, \widetilde{\mu}_r)_{r\geq 1}$ be the sequence of join spaces and fuzzy subsets associated with H. Then $\widetilde{\mu}$ is a fuzzy subhypergroup of the join space $\langle {}^1H, \circ_1\rangle$. Moreover, for any $i \geq 1$, $\widetilde{\mu}_i$ is a fuzzy subhypergroup of the join space $\langle {}^{i+1}H, \circ_{i+1}\rangle$.*

Proof. The proof is straightforward, by considering Proposition 3.11. □

Proposition 3.13. ([17], Lemma 3.10) *Let $\langle H_1, \circ_1\rangle$ and $\langle H_2, \circ_2\rangle$ be two hypergroups. If μ_1 is a fuzzy subhypergroup of H_1 and μ_2 is a fuzzy subhypergroup of H_2, then $\mu_1 \times \mu_2$ is a fuzzy subhypergroup of the hypergroup $\langle H_1 \times H_2, \otimes\rangle$.*

Example 3.14. Set $H_1 = \{a, b, c\}$ and $\mu_1 : H_1 \longrightarrow [0, 1]$, $\mu_1(a) = 0.3$, $\mu_1(b) = 0.5$, $\mu_1(c) = 0.7$. Then, by Proposition 3.11, μ_1 is a fuzzy subhypergroup of the hypergroup $\langle H_1, \circ_1\rangle$, where the hyperoperation "$\circ_1$" is defined as in Theorem 3.2, i.e. H_1 has the table

$\circ_1$	a	b	c
a	a	$\{a,b\}$	H
b	$\{a,b\}$	b	$\{b,c\}$
c	H	$\{b,c\}$	c

Similarly set $H_2 = \{x, y\}$ and $\mu_2 : H_2 \longrightarrow [0, 1]$, $\mu_2(x) = 0.7$, $\mu_2(y) = 0.8$. Then, by Proposition 3.11, μ_2 is a fuzzy subhypegroup of the hypergroup $\langle H_2, \circ_2\rangle$ which has the table

$\circ_2$	x	y
x	x	$\{x,y\}$
y	$\{x,y\}$	y

Then the direct product $\langle H_1 \times H_2, \otimes\rangle$ is represented by the commutative table

$\otimes$	(a,x)	(a,y)	(b,x)	(b,y)	(c,x)	(c,y)
(a,x)	(a,x)	(a,x) (a,y)	(a,x) (b,x)	$(a,x),(a,y)$ $(b,x),(b,y)$	(a,x) (b,x) (c,x)	$H_1 \times H_2$
(a,y)		(a,y)	$(a,x),(a,y)$ $(b,x),(b,y)$	$(a,y),(b,y)$	$H_1 \times H_2$	$(a,y),(b,y)$ (c,y)
(b,x)			(b,x)	$(b,x),(b,y)$	(b,x) (c,x)	$(b,x),(b,y)$ $(c,x),(c,y)$
(b,y)				(b,y)	$(b,x),(b,y)$ $(c,x),(c,y)$	$(b,y),(c,y)$
(c,x)					(c,x)	$(c,x),(c,y)$
(c,y)						(c,y)

Then it results

$$\begin{aligned}
\mu_1 \times \mu_2(a,x) &= 0.3 \wedge 0.7 = 0.3;\\
\mu_1 \times \mu_2(a,y) &= 0.3 \wedge 0.8 = 0.3;\\
\mu_1 \times \mu_2(b,x) &= 0.5 \wedge 0.7 = 0.5;\\
\mu_1 \times \mu_2(b,y) &= 0.5 \wedge 0.8 = 0.5;\\
\mu_1 \times \mu_2(c,x) &= 0.7 \wedge 0.7 = 0.7;\\
\mu_1 \times \mu_2(c,y) &= 0.7 \wedge 0.8 = 0.7.
\end{aligned}$$

Now by the previous proposition it follows that $\mu_1 \times \mu_2$ is a fuzzy subhypergroup of the hypergroup $\langle H_1 \times H_2, \otimes\rangle$. Moreover, by simple computations, we may verify that $\mu_1 \times \mu_2$ satisfies the conditions of Definition 3.10.

Remark 3.15. With our notations, using the previous proposition, if $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are fuzzy subhypergroups of the hypergroups H_1 and H_2 respectively, then the membership function $\widetilde{\mu}_1 \times \widetilde{\mu}_2$ is a fuzzy subhypergroup of $H_1 \times H_2$. We don't have the same property for the membership function $\widetilde{\mu}_\otimes$. Indeed, let us verify the first condition from Definition 3.10:
for any $(\alpha_1, \alpha_2) \in (x_1, x_2) \otimes (y_1, y_2)$, $\widetilde{\mu}_\otimes(x_1, x_2) \wedge \widetilde{\mu}_\otimes(y_1, y_2) \leq \widetilde{\mu}_\otimes(\alpha_1, \alpha_2)$.
Since $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are fuzzy subhypergroups of the hypergroups H_1 and H_2 respectively, for $\alpha_1 \in x_1 \circ_1 y_1$ we obtain $\widetilde{\mu}_1(\alpha_1) \geq \widetilde{\mu}_1(x_1) \wedge \widetilde{\mu}_1(y_1)$ and similarly, for $\alpha_2 \in x_2 \circ_2 y_2$, $\widetilde{\mu}_2(\alpha_2) \geq \widetilde{\mu}_2(x_2) \wedge \widetilde{\mu}_2(y_2)$. From these relations we cannot conclude, in general, that $\widetilde{\mu}_1(\alpha_1) \cdot \widetilde{\mu}_2(\alpha_2) \geq \widetilde{\mu}_1(x_1) \cdot \widetilde{\mu}_2(x_2) \wedge \widetilde{\mu}_1(y_1) \cdot \widetilde{\mu}_2(y_2)$.

But, as an immediate consequence of Proposition 3.12, the membership functions $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ are fuzzy subhypergroups of the hypergroups $\langle {}^1H_1, \bar{\circ}_1\rangle$ and $\langle {}^1H_2, \bar{\circ}_2\rangle$, the first join spaces from the sequences of join spaces associated with the hypergroups H_1 and H_2, that is, for any $i \in \{1, 2\}$ and $x_i, y_i \in H_i$:

$$x_i \bar{\circ}_i y_i = \{z_i \in H_i \mid \widetilde{\mu}_i(x) \wedge \widetilde{\mu}_i(y) \leq \widetilde{\mu}_i(z_i) \leq \widetilde{\mu}_i(x) \vee \widetilde{\mu}_i(y)\}.$$

Proposition 3.16. *If $\widetilde{\mu}_\otimes$ is the membership functions associated like in* (1) *with $H_1 \times H_2$, then $\widetilde{\mu}_\otimes$ is a fuzzy subhypergroup of the direct product of join spaces ${}^1H_1 \times {}^1H_2$.*

Proof. i) For any $(\alpha_1, \alpha_2) \in (x_1, x_2) \otimes (y_1, y_2)$, we shall prove

$$\widetilde{\mu}_{\otimes}(x_1, x_2) \wedge \widetilde{\mu}_{\otimes}(y_1, y_2) \leq \widetilde{\mu}_{\otimes}(\alpha_1, \alpha_2). \tag{5}$$

Since $\alpha_1 \in x_1 \bar{\circ}_1 y_1$ means $\widetilde{\mu}_1(x_1) \leq \widetilde{\mu}_1(\alpha_1) \leq \widetilde{\mu}_1(y_1)$ and similarly $\alpha_2 \in x_2 \bar{\circ}_2 y_2$ means $\widetilde{\mu}_2(x_2) \leq \widetilde{\mu}_2(\alpha_2) \leq \widetilde{\mu}_2(y_2)$ it follows that $\widetilde{\mu}_1(x_1) \cdot \widetilde{\mu}_2(x_2) \leq \widetilde{\mu}_1(\alpha_1) \cdot \widetilde{\mu}_2(\alpha_2) \leq \widetilde{\mu}_1(y_1) \cdot \widetilde{\mu}_2(y_2)$ and therefore the relation (5) is satisfied.

ii) It remains to verify that, for any $(x_1, x_2), (a_1, a_2) \in {}^1H_1 \times {}^1H_2$, there exists $(y_1, y_2) \in {}^1H_1 \times {}^1H_2$ such that $(x_1, x_2) \in (a_1, a_2) \otimes (y_1, y_2)$ and

$$\widetilde{\mu}_{\otimes}(a_1, a_2) \wedge \widetilde{\mu}_{\otimes}(x_1, x_2) \leq \widetilde{\mu}_{\otimes}(y_1, y_2). \tag{6}$$

Since $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ are fuzzy subhypergroups of the hypergroups $\langle {}^1H_1, \bar{\circ}_1\rangle$ and $\langle {}^1H_2, \bar{\circ}_2\rangle$, there exists $(y_1, y_2) \in {}^1H_1 \times {}^1H_2$ such that $x_1 \in a_1 \bar{\circ}_1 y_1$, $x_2 \in a_2 \bar{\circ}_2 y_2$ and $\widetilde{\mu}_1(a_1) \wedge \widetilde{\mu}_1(x_1) \leq \widetilde{\mu}_1(y_1)$, $\widetilde{\mu}_2(a_2) \wedge \widetilde{\mu}_2(x_2) \leq \widetilde{\mu}_2(y_2)$. From the definitions of the hyperproducts "$\bar{\circ}_1$", "$\bar{\circ}_2$", we obtain $\widetilde{\mu}_1(a_1) \leq \widetilde{\mu}_1(x_1) \leq \widetilde{\mu}_1(y_1)$ and $\widetilde{\mu}_2(a_2) \leq \widetilde{\mu}_2(x_2) \leq \widetilde{\mu}_2(y_2)$. It follows that $\widetilde{\mu}_1(a_1) \cdot \widetilde{\mu}_2(a_2) \leq \widetilde{\mu}_1(x_1) \cdot \widetilde{\mu}_2(x_2) \leq \widetilde{\mu}_1(y_1) \cdot \widetilde{\mu}_2(y_2)$ and therefore $\widetilde{\mu}_1(a_1) \cdot \widetilde{\mu}_2(a_2) \wedge \widetilde{\mu}_1(x_1) \cdot \widetilde{\mu}_2(x_2) \leq \widetilde{\mu}_1(y_1) \cdot \widetilde{\mu}_2(y_2)$; thus the relation (6) is satisfied. □

4. Fuzzy Grade of the Hypergroupoid $\langle H \times H, \otimes\rangle$, when $|H/R_{\widetilde{\mu}}| = 2$

Let $\langle H, \circ\rangle$ be a hypergroupoid and $\widetilde{\mu}$ be the fuzzy subset associated with it by the relation (1); it leads to the regular equivalence:

$$xR_{\widetilde{\mu}}y \iff \widetilde{\mu}(x) = \widetilde{\mu}(y).$$

This relation determines on H a partition $H = \bigcup_{i=1}^{r} C_i$, $|C_i| = k_i$ with $\sum_{i=1}^{r} k_i = n = |H|$, where $x, y \in C_i \iff \widetilde{\mu}(x) = \widetilde{\mu}(y)$ and we do the following convention:

$$\text{for } \ u \in C_i, v \in C_j, \ \text{ with } \ i < j, \text{we have } \ \widetilde{\mu}(u) < \widetilde{\mu}(v). \tag{7}$$

In this manner we associate with H the ordered r-tuple $(k_1, k_2, ..., k_r)$.

In the following, we determine the fuzzy grade of a hypergroupoid H in some cases, when the r-tuple $(k_1, k_2, ..., k_r)$ associated with H has particular forms. It is enough to know the form of this r-tuple in order to establish the fuzzy grade of a hypergroupoid.

Proposition 4.1. *Let us suppose that we associate with the hypergroupoid $\langle H, \circ\rangle$ the pair (k, l). Then we associate with $\langle H \times H, \otimes\rangle$ the triple $(k^2, 2kl, l^2)$.*

Proof. We denote $H/R_{\widetilde{\mu}} = \{C_1, C_2\}$, with $C_1 = \{x_1, x_2, ..., x_k\}$ and respectively $C_2 = \{x_{k+1}, x_{k+2}, ..., x_{k+l}\}$; we respect the convention (7) that $\widetilde{\mu}(x_1) < \widetilde{\mu}(x_{k+1})$.

Then we obtain

$$\forall \;\; i,j \in \{1,2,...,k\}, \;\; \widetilde{\mu}_{\otimes}(x_i,x_j) = \widetilde{\mu}(x_i)\cdot\widetilde{\mu}(x_j) = \widetilde{\mu}(x_1)^2$$
$$\forall \;\; i',j' \in \{k+1,k+2,...,k+l\}, \;\; \widetilde{\mu}_{\otimes}(x_{i'},x_{j'}) = \widetilde{\mu}(x_{i'})\cdot\widetilde{\mu}(x_{j'}) = \widetilde{\mu}(x_{k+1})^2$$
$$\forall \;\; i \in \{1,2,...,k\}, \forall \;\; j' \in \{k+1,...,k+l\}, \;\; \widetilde{\mu}_{\otimes}(x_i,x_{j'}) = \widetilde{\mu}(x_1)\cdot\widetilde{\mu}(x_{k+1}).$$

Since $\widetilde{\mu}(x_1) < \widetilde{\mu}(x_{k+1})$ it follows $\widetilde{\mu}(x_1)^2 < \widetilde{\mu}(x_{k+1})\cdot\widetilde{\mu}(x_1) < \widetilde{\mu}(x_{k+1})^2$, that is $\widetilde{\mu}_{\otimes}(x_i,x_j) < \widetilde{\mu}_{\otimes}(x_i,x_{j'}) < \widetilde{\mu}_{\otimes}(x_{i'},x_{j'})$, for any $i,j \in \{1,2,...,k\}$ and any $i',j' \in \{k+1,...,k+l\}$; therefore with the hypergroupoid $\langle H\times H,\otimes\rangle$ we associate the triple $(k^2, 2kl, l^2)$. □

Generalizing the previous proposition we obtain the following result (its proof is similar to the proof of Proposition 4.1)

Proposition 4.2. *Let us suppose we associate with the hypergroupoid $\langle H,\circ\rangle$ the r-tuple $(k_1,k_2,...,k_r)$. Then we find that the associated tuple of $\langle H\times H,\otimes\rangle$ is the $(2r-1)$-tuple $(k_1^2, 2k_1k_2, k_2^2, 2k_2k_3, k_3^2, ..., k_{r-1}^2, 2k_{r-1}k_r, k_r^2)$.*

Using Theorem 4.5 from [24], we obtain immediately the following result.

Theorem 4.3. *Let $(k_1,k_2,...,k_r)$ be the r-tuple associated with an arbitrary hypergroupoid $\langle H,\circ\rangle$. If, for any j, $1\le j\le\left[\frac{r}{2}\right]$, $k_j = k_{r+1-j}$, then, with the join space ${}^1H$ one associates the l-tuple $(2k_1,2k_2,...,2k_l)$, if $r=2l$, or the $l+1$-tuple $(2k_1,2k_2,...,2k_l,k_{l+1})$, if $r=2l+1$.*

Now, we determine the fuzzy grade of the hypergroupoid $\langle H\times H,\otimes\rangle$ when $|H/R_{\widetilde{\mu}}|=2$.

First we remember which is the fuzzy grade of the hypergroupoid $\langle H,\circ\rangle$, when $|H/R_{\widetilde{\mu}}|\in\{2,3\}$ (see [13]).

Theorem 4.4.

1) *If $|\,H/R_{\widetilde{\mu}}|=2$, that is with H we associate the pair (k,l), then:*
   a) *if $k=l$, we have $s.f.g.(H)=2$;*
   b) *if $k\neq l$, we have $f.g.(H)=1$.*
2) *If $|H/R_{\widetilde{\mu}}|=3$, that is with H we associate the triple (k_1,k_2,k_3), then:*
   a) *if $k_1=k_2=k_3$ we have $f.g.(H)=1$;*
   b) *for $k_1=k_3\neq k_2$ we have $f.g.(H)=2$ if $k_2\neq 2k_3$ and $s.f.g.(H)=3$ if $k_2=2k_3$;*
   c) *if $k_1<k_2=k_3$ we have $f.g.(H)=1$;*
   d) *for $k_1=k_2<k_3$ we have $f.g.(H)=1$ if $P=2k_3^3-8k_1^3-k_1^2k_3+5k_1k_3^2>0$ and $f.g.(H)=3$ if $P<0$;*
   e) *for $k_1\neq k_2\neq k_3$ we have $f.g.(H)\in\{1,2\}$.*

To start with, we determine the fuzzy grade of the direct product $H\times H$ of the same hypergroupoid and secondly, we calculate the fuzzy grade of the direct product of two distinct hypergroupoids.

Proposition 4.5. *Let (k,l) be the pair associated with the hypergroupoid H.*

(i) *If* $k = l$*, then* $s.f.g.(H \times H) = 3$.
(ii) *If* $2k = l$*, then* $f.g.(H \times H) = 1$.
(iii) *If* $k \neq l$ *and* $2k \neq l$*, then* $f.g.(H \times H) \in \{1, 2\}$.

Proof. By Proposition 4.1 it follows that the triple associated with $H \times H$ is $(k^2, 2kl, l^2)$.
If $k = l$ the $s.f.g.(H) = 2$ and $s.f.g.(H \times H) = 3$ (by $(2b)$).
If $k \neq l$ we have two situations (we suppose $k < l$):

(i) if $2k = l$, then $k^2 < 2kl = l^2$ and by $(2c)$ it is clear that $f.g.(H) = f.g.(H \times H) = 1$;
(ii) if $2k \neq l$, then $k^2 \neq 2kl \neq l^2$ and by $(1b)$ and $(2e)$ we obtain $f.g.(H) = 1$ and $f.g.(H \times H) \in \{1, 2\}$.

□

Lemma 4.6. *Let* $\langle H_1, \circ_1 \rangle$ *and* $\langle H_2, \circ_2 \rangle$ *be two distinct hypergroupoids such that* $|H_1/R_{\widetilde{\mu}_1}| = |H_2/R_{\widetilde{\mu}_2}| = 2$*. For more precision we consider:*
$H_1 = C_1 \cup C_2$, $|C_1| = k, |C_2| = l$; $H_2 = C'_1 \cup C'_2$, $|C'_1| = m, |C'_2| = n$.
If, for any $x \in C_1, y \in C_2, x' \in C'_1, y' \in C'_2$ *we have*

a) $\widetilde{\mu}_{\otimes}(x, y') = \widetilde{\mu}_{\otimes}(y, x')$*, then* $|(H_1 \times H_2)/R_{\widetilde{\mu}_{\otimes}}| = 3$;
b) $\widetilde{\mu}_{\otimes}(x, y') \neq \widetilde{\mu}_{\otimes}(y, x')$*, then* $|(H_1 \times H_2)/R_{\widetilde{\mu}_{\otimes}}| = 4$.

Proof. We may write $H_1 \times H_2 = \{(x, x'), (x, y'), (y, x'), (y, y') \mid x \in C_1, y \in C_2, x' \in C'_1, y' \in C'_2\}$ and then $\widetilde{\mu}_{\otimes}(u, v) = \widetilde{\mu}_1(u) \cdot \widetilde{\mu}_2(v)$, for any $u \in H_1$ and any $v \in H_2$.
It is easy to obtain the relations:

$$\widetilde{\mu}_{\otimes}(x, x') < \widetilde{\mu}_{\otimes}(x, y') < \widetilde{\mu}_{\otimes}(y, y')$$
$$\widetilde{\mu}_{\otimes}(x, x') < \widetilde{\mu}_{\otimes}(y, x') < \widetilde{\mu}_{\otimes}(y, y').$$

If, for any $x \in C_1, y \in C_2, x' \in C'_1, y' \in C'_2$ we have

a) $\widetilde{\mu}_{\otimes}(x, y') = \widetilde{\mu}_{\otimes}(y, x')$, then $|(H_1 \times H_2)/R_{\widetilde{\mu}_{\otimes}}| = 3$ and the triple associated with $H_1 \times H_2$ is $(km, kn + lm, ln)$;
b) $\widetilde{\mu}_{\otimes}(x, y') \neq \widetilde{\mu}_{\otimes}(y, x')$, then $|(H_1 \times H_2)/R_{\widetilde{\mu}_{\otimes}}| = 4$ and with $H_1 \times H_2$ is associated the 4-tuple (km, kn, lm, ln).

□

Proposition 4.7. *Let* (k, k) *be the pair associated with the hypergroupoid* $\langle H_1, \circ_1 \rangle$ *and* (l, l) *be the pair associated with the hypergroupoid* $\langle H_2, \circ_2 \rangle$*. Then* $s.f.g.(H_1 \times H_2) = 3$.

Proof. The corresponding join spaces ${}^2H_1$ and ${}^2H_2$ from the sequences associated with H_1 and H_2 respectively are total hypergroups and thus $s.f.g.(H_1) = s.f.g.(H_2) = 2$.

a) If $\widetilde{\mu}_{\otimes}(x, y') = \widetilde{\mu}_{\otimes}(y, x')$, then the triple associated with $H_1 \times H_2$ is $(kl, 2kl, kl)$ and thus the join space ${}^3(H_1 \times H_2)$ is a total hypergroup and $s.f.g.(H_1 \times H_2) = 3$ (by $(2b)$).

b) If $\widetilde{\mu}_{\otimes}(x, y') \neq \widetilde{\mu}_{\otimes}(y, x')$, then the 4-tuple related to $H_1 \times H_2$ has the form (kl, kl, kl, kl); then, by Theorem 4.3, to the join space ${}^1(H_1 \times H_2)$ corresponds the pair $(2kl, 2kl)$ and again the join space ${}^3(H_1 \times H_2)$ is a total hypergroup and $s.f.g.(H_1 \times H_2) = 3$ (by $(1a)$).

$\square$

Example 4.8. Consider the hypergroupoid $H_1 = \{a, b, c, d\}$ with the following hyperoperation:

$\circ_1$	a	b	c	d
a	a	$\{a,b\}$	$\{a,b,c\}$	H
b	$\{a,b\}$	b	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	c	$\{c,d\}$
d	H	$\{b,c,d\}$	$\{c,d\}$	d

and the hypergroupoid $H_2 = \{x, y, z, u, t, v\}$ with the hyperoperation:

$\circ_2$	x	y	z	u	t	v
x	x	$\{x,y,z\}$	$\{x,y,z\}$	H	H	H
y	$\{x,y,z\}$	y	$\{x,y,z\}$	H	H	H
z	$\{x,y,z\}$	$\{x,y,z\}$	z	H	H	H
u	H	H	H	$\{u,t,v\}$	$\{u,t,v\}$	$\{u,t,v\}$
t	H	H	H	$\{u,t,v\}$	$\{u,t,v\}$	$\{u,t,v\}$
v	H	H	H	$\{u,t,v\}$	$\{u,t,v\}$	$\{u,t,v\}$

Then we obtain

$$\widetilde{\mu}_1(a) = \widetilde{\mu}_1(d) = 19/42 > \widetilde{\mu}_1(b) = \widetilde{\mu}_1(c) = 29/66$$

and

$$\widetilde{\mu}_2(x) = \widetilde{\mu}_2(y) = \widetilde{\mu}_2(z) = 6/25 > \widetilde{\mu}_2(u) = \widetilde{\mu}_2(t) = \widetilde{\mu}_2(v) = 6/27,$$

that is the couple associated with H_1 is $(2, 2)$ and the couple associated with H_2 is $(3, 3)$. Using the notations of Lemma 4.6 and the convention (7), we write

$$H_1 = C_1 \cup C_2, C_1 = \{b, c\}, C_2 = \{a, d\},$$

$$H_2 = C_1' \cup C_2', C_1' = \{u, t, v\}, C_2' = \{x, y, z\}$$

and then

$$\widetilde{\mu}_{\otimes}(a, u) = \widetilde{\mu}_1(a) \cdot \widetilde{\mu}_2(u) = 19/189 \neq 29/275 = \widetilde{\mu}_1(b) \cdot \widetilde{\mu}_2(x) = \widetilde{\mu}_{\otimes}(b, x)$$

and then, by Proposition 4.7 b), it follows that 4-tuple associated with $H_1 \times H_2$ has the form $(6, 6, 6, 6)$; then, by Theorem 4.3, with the join space ${}^1(H_1 \times H_2)$ is associated the pair $(12, 12)$; then with the second join space ${}^2(H_1 \times H_2)$ is associated the 1-tuple (24) and thus the join space ${}^3(H_1 \times H_2)$ is a total hypergroup and $s.f.g.(H_1 \times H_2) = 3$.

5. Conclusions

In the last four decades, many mathematicians have defined various relationships between fuzzy sets and algebraic structures. P.Corsini has realized a such connection which leads to a sequence of join spaces and fuzzy sets associated with a hypergroupoid. The principal aim of this paper is to find properties of this sequence for the direct product of hypergroupoids. We saw that in general there is no relation between the fuzzy grade of the hypergroups and the fuzzy grade of their direct product. It would be interesting to find a class of hypergroups for which there is a such relation.

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