REGULAR ORDERED SEMIGROUPS AND INTRA-REGULAR ORDERED SEMIGROUPS IN TERMS OF FUZZY SUBSETS

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Abstract. Let S be an ordered semigroup. A fuzzy subset of S is an arbitrary mapping from S into $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. In this paper, the concept of fuzzy generalized bi-ideals of an ordered semigroup S is introduced. Regular ordered semigroups are characterized by means of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals. Finally, two main theorems which characterize regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy left ideals, fuzzy right ideals, fuzzy bi-ideals or fuzzy quasi-ideals are given. The paper shows that one can pass from results in terms of fuzzy subsets in semigroups to ordered semigroups. The corresponding results of unordered semigroups are also obtained.

1. Introduction

For a set S, a fuzzy subset of S is, by definition, an arbitrary mapping $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. This important concept of a fuzzy set put forth by Zadeh in 1965 [33] has opened up keen insights and applications in a wide range of scientific fields. It offers tools and a new approach to model the imprecision and uncertainty present in phenomena that do not have sharp boundaries. Since then, a series of research on fuzzy sets has come out expounding the importance of the concept and its applications to logic, set theory, algebra theory, real analysis, topology, etc [1, 2]. Rosenfeld explained the notion of a fuzzy subgroup of a group with the notion of fuzzy subsets of a set in 1971 [27].

Following the formulation of fuzzy groups by Rosenfeld [27], many researchers have been engaged in extending the concepts and results of pure algebra to the broader framework of the fuzzy setting, although not all results in algebra can be fuzzified (for example, see [5]). In [22, 23, 24], Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals and fuzzy generalized bi-ideals of semigroups (without order) and characterized regular semigroups, intra-regular semigroups, semigroups that are semilattices of left(right) simple semigroups and so on in terms of fuzzy ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals. In the paper [25] fuzzy analogous of several further important notions, e.g. those of bi-ideals or interior ideals

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in a ring R have been defined and justified in a similar fashion. Furthermore, fuzzy prime ideals on semigroups have also been considered by Kehayopulu, Xie and Tsingelis in [19, 28] and by Kehayopulu and Tsingelis in [17]. Following the works by Kuroki [22, 23, 24], Shih-Chuan Cheng, J. N. Mordeson and Yu Yandong generalized the fuzzy algebras to L-algebras in [4], they accomplished this by taking a complete Brouwerian lattice L as the set of true values and by using a general infinitely union-distributive t-norm T to define TL-subgroups, TL-subrings and so on. As applications of TL-subalgebras, some characterizations of semigroups without orders are described in ([4], Section 4.3). Recently, a theory of fuzzy sets on ordered semigroups has been developed [7, 16, 18, 20] which are relative closely to logic. Following the terminology given by Zadeh, if S is an ordered semigroup, fuzzy sets in ordered semigroups S have been first considered by Kehayopulu and Tsingelis in [16], they then defined fuzzy analogous for several notations that have been proved to be useful in the theory of ordered semigroups. Moreover, each ordered groupoid can be embedded into an ordered groupoid having a greatest element (poe-groupoid) in terms of fuzzy sets [18]. The concept of ordered fuzzy points of an ordered semigroup S has been first introduced by Xie and Tang [31], and studied prime fuzzy ideals of an ordered semigroup S [32]. Authors also introduced the concepts of weakly prime fuzzy ideals, completely prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of an ordered semigroup S, and established the relations among the five classes of the ideals. Furthermore, we characterize weakly prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of S by their level ideals [31].

The concept of regular rings was introduced by von Neumann [26]. As we know, regular rings play an important role in the abstract algebra. In 1956, Kov*á*cs characterized a regular rings as rings satisfying the condition $A \cap B = AB$ for every right ideal A and every left ideal B [21]. Since then, Is*é*ki studied the same properties for semigroups, and proved that a commutative semigroup is regular if and only if every ideal of S is idempotent [6]. In 1961, Calais proved that a semigroup S is regular if and only if the right and the left ideals of S are idempotent and for every right ideal A and every left ideal B of S, the product AB is a quasi-ideal of S [3]. Then Kehayopulu extended those results to ordered semigroups [8, 10]. Recently, Kehayopulu also extended those similar "fuzzy" results to ordered semigroups, and characterized the regular, the left regular and the right regular ordered semigroups by means of fuzzy left ideals, fuzzy right ideals, fuzzy semiprime subsets and fuzzy quasi-ideals [16, 17, 18].

As a continuation of the study undertaken by Kehayopulu [16, 17, 18], we first introduce the concept of the fuzzy generalized bi-ideals of an ordered semigroup S in the present paper. Then regular ordered semigroups are characterized by means of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals. Finally, we give two main theorems which characterize the regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy left ideals, fuzzy right ideals, fuzzy bi-ideals and fuzzy quasi-ideals. As an application of the results of this paper, the corresponding results of unordered semigroups also are obtained.

2. Preliminaries and Some Notations

Throughout this paper, unless stated otherwise, S stands for an ordered semigroup. A function f from S to the real closed interval [0,1] is called a *fuzzy subset* of S. The ordered semigroup S itself is a fuzzy subset of S, its characteristic function, also denoted by S, is defined as follows:

$$S : S \longrightarrow [0,1] | x \rightarrow S(x) := 1.$$

Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ means that

$$f(x) \leq g(x)$$

for all $x \in S$, and $f \cap g$ and $f \cup g$ are defined by

$$(f \cap g)(x) = \min(f(x), g(x)) = f(x) \wedge g(x),$$

$$(f \cup g)(x) = \max(f(x), g(x)) = f(x) \vee g(x),$$

for all $x \in S$. The set of all fuzzy subsets of S is denoted by $F(S)$. One can easily see that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice and the fuzzy subset S of S is the greatest element of the ordered set $(F(S), \subseteq)$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $a \in S$, we define $A_x := \{(y,z) \in S \times S | x \leq yz\}$. The product of $f \circ g$ is defined by

$$(\forall x \in S)\ (f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} \min\{f(y), g(z)\} & , \text{ if } A_x \neq \emptyset , \\ 0 & , \text{ if } A_x = \emptyset. \end{cases}$$

As is well known (cf. [[13], Theorem]), that this operation "$\circ$" is associative.

Lemma 2.1. ([13], Proposition 1) *Let $(S, \cdot, \leq)$ be an ordered semigroup and f_1, f_2, g_1, g_2 fuzzy subsets of S such that $f_1 \subseteq g_1$ and $f_2 \subseteq g_2$, then $f_1 \circ f_2 \subseteq g_1 \circ g_2$.*

We denote by f_A the characteristic function of A, that is, the mapping of S into $[0,1]$ defined by

$$f_A(x) = \begin{cases} 1 & , \text{ if } x \in A, \\ 0 & , \text{ if } x \notin A. \end{cases}$$

Let S be an ordered semigroup. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

A subsemigroup B of an ordered semigroup S is called a *bi-ideal* of S if

(1) $BSB \subseteq B$ and

(2) If $a \in B$ and $S \ni b \leq a$, then $b \in B$.

We denote by $B(a)$ the bi-ideal of S generated by a ($a \in S$). We have $B(a) = (a \cup a^2 \cup aSa]$ [11].

Definition 2.2. Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy subsemigroup* of S if

$f(xy) \geq \min\{f(x), f(y)\}$, $\forall x, y \in S$.

Lemma 2.3. *Let S be an ordered semigroup, f a fuzzy subset f of S. If f is a fuzzy subsemigroup of S such that*

$$x \leq y \Rightarrow f(x) \geq f(y), \ \forall x, y \in S,$$

then $f \circ f \subseteq f$. Conversely, if $f \circ f \subseteq f$, then f is a fuzzy subsemigroup of S.

Proof. Let $x \in S$. If $A_x = \emptyset$, then $f \circ f(x) = 0 \leq f(x)$. If $A_x \neq \emptyset$, then

$$\begin{aligned} f \circ f(x) &= \bigvee_{x \leq bc} \min\{f(b) \wedge f(c)\} \\ &\leq \bigvee_{x \leq bc} f(bc) \\ &\leq \bigvee_{x \leq bc} f(x) \text{ (Since } x \leq ab \text{)} \\ &= f(x). \end{aligned}$$

Then $f \circ f \subseteq f$.

Conversely if $f \circ f \subseteq f$, then for all $x, y \in S$

$$\begin{aligned} f(xy) &\geq (f \circ f)(xy) \\ &= \bigvee_{xy \leq bc} [\min\{f(b) \wedge f(c)\}] \\ &\geq \min\{f(x) \wedge f(y)\}. \end{aligned}$$

□

Definition 2.4. ([13], Definition 1) Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $f(xy) \geq f(y), \forall x, y \in S$.

A fuzzy subset f of S is called a *fuzzy right ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $f(xy) \geq f(x), \forall x, y \in S$.

A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S.

Lemma 2.5. [13] *Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is a left (resp. right) ideal of S if and only if the characteristic function f_A of A is a fuzzy left (resp. right) ideal of S.*

Definition 2.6. ([15], Definition 2) Let S be an ordered semigroup. A fuzzy subsemigroup f of S is called a *fuzzy bi-ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $f(xyz) \geq \min\{f(x), f(z)\}, \forall x, y, z \in S$.

Lemma 2.7 ([21], Theorem 1). *Let S be an ordered semigroup and $\emptyset \neq B \subseteq S$. Then B is a bi-ideal of S if and only if the characteristic function f_B of B is a fuzzy bi-ideal of S.*

Lemma 2.8. *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy left ideal of S if and only if f satisfies that*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.
(2) $S \circ f \subseteq f$.

Proof. $\Rightarrow$. By Proposition 10 in [23], it is easy.
$\Leftarrow$. Assume that (2) holds. Then $f(xy) \geq f(x), \forall x, y \in S$. Indeed: let $x, y \in S$. Put $a = xy$. Then, by $1 \circ f \subseteq f$,

$$\begin{aligned} f(xy) &= f(a) \geq (S \circ f)(a) \\ &= \bigvee_{a \leq bc} [\min\{S(b) \wedge f(c)\}] \\ &\geq \min\{S(x) \wedge f(y)\} \\ &= f(y). \end{aligned}$$

By hypothesis (1) and Definition 2.4, f is a fuzzy left ideal of S. □

Similar to Lemma 2.8, we have the following two lemmas. The proofs are similar to that of Lemma 2.8.

Lemma 2.9. *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy right ideal of S if and only if f satisfies that*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.
(2) $f \circ S \subseteq f$.

Proof. Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy ideal of S if and only if f satisfies that

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, for all $x, y \in S$.
(2) $1 \circ f \subseteq f$, $f \circ S \subseteq f$. □

For the other necessary definitions and notations in this paper, we refer to [29, 30].

3. Fuzzy Bi-ideals and Fuzzy Generalized Bi-ideals of Ordered Semigroups

Definition 3.1. A non-empty subset B of an ordered semigroup S is called a *generalized bi-ideal* of S if
(1) $BSB \subseteq B$.
(2) If $a \in B$ and $S \ni b \leq a$, then $b \in B$.

It is clear that every bi-ideal of an ordered semigroup S is a generalized bi-ideal of S, but not conversely. We denote by $GB(a)$ the generalized bi-ideal generated by a ($a \in S$). One can easily prove that $GB(a) = (a \cup aSa]$.

Furthermore, we now introduce the analogous definition by means of fuzzy sets as follows:

Definition 3.2. Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy generalized bi-ideal* of S if
(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $f(xyz) \geq \min\{f(x), f(z)\}, \forall x, y, z \in S$.

One can easily observe that every fuzzy bi-ideal of an ordered semigroup S is a fuzzy generalized bi-ideal of S, but not conversely. We can illustrate it by the following example:

Example 3.3. We consider the ordered semigroup $S = \{a, b, c, d\}$ defined by the following multiplication " $\cdot$ " and the order " $\leq$ ":

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a), (a,b), (a,c), (a,d), (b,b), (c,c), (d,d)\}.$$

Let f be a fuzzy subset of S such that $f(a) = 0.6, f(b) = 0, f(c) = 0.3, f(d) = 0$. Then we can easily show that f is a fuzzy generalized bi-ideal of S. But f is not a fuzzy bi-ideal of S, since $f(c^2) = f(b) = 0 < 0.3 = \min\{f(c), f(c)\}$, that is, f is not a fuzzy subsemigroup of S.

The following Theorem 3.4 can be proved in a similar way as in the proof of Theorem 1 of [15].

Theorem 3.4. *Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is a generalized bi-ideal of S if and only if the characteristic function f_A of A is a fuzzy generalized bi-ideal of S.*

Proposition 3.5. *If a fuzzy subset f of an ordered semigroup S is a fuzzy generalized bi-ideal of S, then $f \circ S \circ f \subseteq f$.*

Proof. Let f be a fuzzy generalized bi-ideal of an ordered semigroup S. Then $(f \circ S \circ f)(a) \leq f(a)$ for all $a \in S$. Indeed: Since f is a fuzzy subset of S, we have $f(a) \geq 0$ for all $a \in S$. If $(f \circ S \circ f)(a) = 0$, then $(f \circ S \circ f \subseteq f)(a) \leq f(a)$. If $(f \circ S \circ f)(a) \neq 0$, then there exist $x, y, p, q \in S$ such that $(x, y) \in A_a, (p, q) \in A_x$, that is $a \leq xy, x \leq pq$. Since f is a fuzzy generalized bi-ideal of S, we have $f(pqy) \geq \min\{f(p), f(y)\}$. Thus

$$\begin{aligned}(f\circ S\circ f)(a) &= \bigvee_{(x,y)\in A_a} \min\{(f\circ S)(x), f(y)\}\\ &= \bigvee_{(x,y)\in A_a} [\min\{\bigvee_{(p,q)\in A_x} \min\{f(p), S(q)\}], f(y)\}\\ &= \bigvee_{(x,y)\in A_a} \bigvee_{(p,q)\in A_x} \min\{f(p), S(q), f(y)\}\\ &= \bigvee_{(x,y)\in A_a} \bigvee_{(p,q)\in A_x} \min\{f(p), f(y)\}\\ &\leq \bigvee_{(x,y)\in A_a} \bigvee_{(p,q)\in A_x} f(pqy)\\ &\leq \bigvee_{(x,y)\in A_a} f(xy) \leq f(a) \text{ (Since } xy \leq pqy \text{)},\end{aligned}$$

which means that $f\circ S\circ f \subseteq f$. □

In a similar way as in the previous proposition, by Lemma 2.3, we can show the following result:

Proposition 3.6. *If a fuzzy subset f of an ordered semigroup S is a fuzzy bi-ideal of S, then $f\circ f\subseteq f$ and $f\circ S\circ f\subseteq f$.*

Lemma 3.7. *If a fuzzy subset f of an ordered semigroup S and $f\circ S\circ f\subseteq f$, then*

$$(\forall x, y, z\in S)\ f(xyz)\geq min\{f(x), f(z)\}.$$

Proof. For any $x, y, z\in S$, put $a = xyz$. Since $f\circ S\circ f\subseteq f$, we have

$$\begin{aligned}f(xyz) &= f(a)\geq (f\circ S\circ f)(a)\\ &= \bigvee_{(p,q)\in A_a} \min\{(f\circ S)(p), f(q)\}\\ &\geq \min\{(f\circ S)(xy), f(z)\}\\ &= \min\{\bigvee_{(u,v)\in A_{xy}} \min\{f(u), S(v)\}, f(z)\}\\ &\geq \min\{\min\{f(x), S(y)\}, f(z)\}\\ &= \min\{f(x), S(y), f(z)\}\\ &= \min\{f(x), f(z)\}.\end{aligned}$$

□

Proposition 3.8. *The product of two fuzzy generalized bi-ideals of an ordered semigroup S is a fuzzy bi-ideal of S.*

Proof. Let f, g be two fuzzy generalized bi-ideals of S. Then, by Proposition 3.5, $f \circ S \circ f \subseteq f$ and $g \circ S \circ g \subseteq g$. Thus

$$(f \circ g) \circ (f \circ g) \subseteq f \circ (g \circ S \circ g) \subseteq f \circ g.$$

Therefore, by Lemma 2.3, $f \circ g$ is a fuzzy subsemigroup of S. Furthermore, by Lemma 2.1 we have

$$(f \circ g) \circ S \circ (f \circ g) = f \circ g \circ (S \circ f) \circ g \subseteq f \circ (g \circ S \circ g) \subseteq f \circ g.$$

By Lemma 3.7,

$$(\forall x, y, z \in S)\ (f \circ g)(xyz) \geq \min\{(f \circ g)(x), (f \circ g)(z)\}.$$

Furthermore, if $x \leq y$, then $(f \circ g)(x) \geq (f \circ g)(y)$. Indeed: If $A_y = \emptyset$, then $(f \circ g)(y) = 0$. Since $f \circ g$ is a fuzzy subset of S, we have $(f \circ g)(x) \geq 0 = (f \circ g)(y)$. If $A_y \neq \emptyset$, then, since $x \leq y$, we have $A_y \subseteq A_x$. Thus we have

$$(f \circ g)(y) = \bigvee_{(u,v) \in A_y} \min\{f(u), g(v)\} \leq \bigvee_{(u,v) \in A_x} \min\{f(u), g(v)\} = (f \circ g)(x).$$

□

An ordered semigroup $(S, \cdot, \leq)$ is called *regular* if, for each element a of S, there exists an element x in S. such that $a \leq axa$ (cf. [9, 10]). Equivalent definition:

(1) $A \subseteq (ASA], \forall A \in S$.

(2) $a \in (aSa], \forall a \in S$.

Theorem 3.9. *Every fuzzy generalized bi-ideal of a regular ordered semigroup S is a fuzzy bi-ideal of S.*

Proof. Let f be any fuzzy generalized bi-ideals of S and $a, b \in S$. Then, since S is regular, there exists $x \in S$ such that $b \leq bxb$. Then

$$f(ab) \geq f(a(bxb)) = f(a(bx)b) \geq \min\{f(a), f(b)\}.$$

Thus f is a fuzzy subsemigroup of S, and so it is a fuzzy bi-ideal of S. □

4. Characterizations of Regular Ordered Semigroups

The characterization of regular semigroups or ordered semigroups in terms of left ideals, right ideals and (generalized) bi-ideals is well known (for example, see [8, 9, 10, 11]). In this section we give the similar results by means of fuzzy sets of ordered semigroups.

Lemma 4.1. *An ordered semigroup S is regular if and only if $B = (BSB]$ for any bi-ideal of S.*

Proof. Let S be regular. Then $x \in (xSx]$ for all $x \in S$. Thus $B \subseteq (BSB]$. Since B is a bi-ideal of S, we have $(BSB] \subseteq B$. Then $B = (BSB]$.

Conversely, if $B = (BSB]$ for any bi-ideal of S, then, by Lemma 9 in [14], $(B(x)SB(x)] = B(x) \subseteq (xSx]$. Thus $x \in B(x) \subseteq (xSx]$ implies that S is regular. □

Now we give characterizations of regular ordered semigroups by fuzzy (generalized) bi-ideals.

Theorem 4.2. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is regular.*
(2) *$f = f \circ S \circ f$ for any fuzzy bi-ideal f of S.*

Proof. (1) $\Rightarrow$ (2). Let f be any fuzzy bi-ideal of S and $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa$, and $(ax, a) \in A_a$. Thus

$$\begin{aligned} (f \circ S \circ f)(a) &= \bigvee_{(y,z)\in A_a} \min\{(f \circ S)(y), f(z)\} \\ &\geq \min\{(f \circ S)(ax), f(a)\} \\ &= \min\{\bigvee_{(p,q)\in A_{ax}} \min\{f(p), S(q)\}, f(a)\} \\ &\geq \min\{\min\{f(a), S(x)\}, f(a)\} \\ &= \min\{\min\{f(a), 1\}, f(a)\} = f(a), \end{aligned}$$

and so we have $f \circ S \circ f \supseteq f$. Since f is a fuzzy bi-ideal of S, by Proposition 3.6, we have $f \circ S \circ f \subseteq f$. Thus $f \circ S \circ f = f$.

(2) $\Rightarrow$ (1). Let B be any bi-ideal of S. Then , by Lemma 2.7, f_B is a fuzzy bi-ideal of S, and for each $a \in B$

$$\bigvee_{(y,z)\in A_a} [\min\{(f_B \circ S)(y), f_B(z)\}] = [(f_B \circ S) \circ f_B](a) = f_B(a) = 1,$$

which implies that $A_a \neq \emptyset$, and there exist $b, c \in S$ such that $a \leq bc$, $(f_B \circ S)(b) = 1$ and $f_B(c) = 1$. Then $c \in B$, and

$$\bigvee_{(p,q)\in A_b} \min\{f_B(p), S(q)\} = (f_B \circ S)(b) = 1,$$

which implies that $A_b \neq \emptyset$, and there exist $u, v \in S$ such that $b \leq uv$, $f_B(u) = 1$, and $S(v) = 1$. Then $u \in B$, and

$$a \leq bc \leq uvc \in BSB,$$

and so $B \subseteq (BSB]$. On the other hand, since B is a bi-ideal of S, we have $(BSB] \subseteq (B] = B$. Therefore, $B = (BSB]$. By Lemma 4.1, S is regular. ☐

In a similar way, the following theorem can be proved.

Theorem 4.3. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is regular.*
(2) *$f = f \circ S \circ f$ for any fuzzy generalized bi-ideal f of S.*

Theorem 4.4. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$f \circ g \circ f = f \cap g$ for any fuzzy bi-ideal f and any fuzzy ideal g of S.*

Proof. (1) $\Rightarrow$ (2). Let f, g be a fuzzy bi-ideal and a fuzzy ideal of S, respectively. Then by Lemma 2.1 and Proposition 3.6 we have $f \circ g \circ f \subseteq f \circ S \circ f \subseteq f$. Since g is a fuzzy ideal of S, by Lemma 2.10 we have

$$f \circ g \circ f \subseteq S \circ g \circ S \subseteq 1 \circ g \subseteq g.$$

Thus $f \circ g \circ f \subseteq f \cap g$. On the other hand, let $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa (\leq axaxa)$, so $(a, xaxa) \in A_a$. Since g is a fuzzy ideal of S, we have $g(xax) \geq g(ax) \geq g(a)$. Thus

$$\begin{aligned}(f \circ g \circ f)(a) &= \bigvee_{(y,z) \in A_a} \min\{f(y), (g \circ f)(z)\} \\ &\geq \min\{f(a), (g \circ f)(xaxa)\} \\ &= \min\{f(a), \bigvee_{(p,q) \in A_{xaxa}} \min\{g(p), f(q)\}\} \\ &\geq \min\{f(a), \min\{g(xax), f(a)\}\} \\ &\geq \min\{f(a), \min\{g(a), f(a)\}\} \\ &= \min\{f(a), g(a)\} = (f \cap g)(a),\end{aligned}$$

which means that $f \circ g \circ f \supseteq f \cap g$. Therefore $f \circ g \circ f = f \cap g$.

(2) $\Rightarrow$ (1). Since S itself is a fuzzy ideal of S, by hypothesis we have

$$f = f \cap S = f \circ S \circ f.$$

By Theorem 4.2, S is regular. □

In a similar way we can show the following theorem:

Theorem 4.5. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$f \circ g \circ f = f \cap g$ for every fuzzy generalized bi-ideal f and every fuzzy ideal g of S.*

Lemma 4.6. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$B \cap L \subseteq (BL]$ for every bi-ideal B and every left ideal L of S.*

(3) *$R \cap B \cap L \subseteq (RBL]$ for every bi-ideal B, every right ideal R and every left ideal L of S.*

Proof. (1) $\Rightarrow$ (2). If S is regular, then for any $a \in B \cap L$ there exists $x \in S$ such that

$$a \leq axa = a(xa) \in B(SL) \subseteq BL.$$

Thus, $a \in (BL]$. Therefore $B \cap L \subseteq (BL]$.

(2) $\Rightarrow$ (1). Let $a \in S$. By hypothesis, we have

$$\begin{aligned} a &\in B(a) \cap L(a) \subseteq (B(a)L(a)] = ((a \cup a^2 \cup aSa](a \cup Sa]] \\ &= ((a \cup a^2 \cup aSa)(a \cup Sa)] \subseteq (a^2 \cup aSa]. \end{aligned}$$

Then $a \leq t$ for some $t \in a^2 \cup aSa$. If $t = a^2$, then $a \leq a^2 = aa \leq a^2 a = aaa \in aSa$. If $t \in aSa$, then $a \in (aSa]$. Thus S is regular.

(1) $\Rightarrow$ (3). If S is regular, then for any $a \in R \cap B \cap L$ there exists $x \in S$ such that

$$a \leq axa = (axa)xa = (ax)a(xa) \in (RS)B(SL) \subseteq RBL.$$

Thus, $a \in (RBL]$. Therefore $R \cap B \cap L \subseteq (RBL]$.

(3) $\Rightarrow$ (1). Similar to (2) $\Rightarrow$ (1). □

Now we give characterizations of regular ordered semigroups by fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals.

Theorem 4.7. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$f \cap g \subseteq f \circ g$ for every fuzzy generalized bi-ideal f and every fuzzy left ideal g of S.*

(3) *$f \cap g \subseteq f \circ g$ for every fuzzy bi-ideal f and every fuzzy left ideal g of S.*

(4) *$h \cap f \cap g \subseteq h \circ f \circ g$ for every fuzzy generalized bi-ideal f, every fuzzy left ideal g and every fuzzy right ideal h of S.*

(5) *$h \cap f \cap g \subseteq h \circ f \circ g$ for every fuzzy bi-ideal f, every fuzzy left ideal g and every fuzzy right ideal h of S.*

Proof. (1) $\Rightarrow$ (2). Let f and g be a fuzzy generalized bi-ideal and a fuzzy left ideal of S, respectively, let $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa$. Since $(a, xa) \in A_a$, we have

$$\begin{aligned} (f \circ g)(a) &= \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} \\ &\geq \min\{f(a), g(xa)\} \\ &\geq \min\{f(a), g(a)\} (\text{since } g \text{ is a fuzzy left ideal of } S) \\ &= (f \cap g)(a), \end{aligned}$$

which means that $f \cap g \subseteq f \circ g$.

(2) $\Rightarrow$ (3). Clearly.

(3) $\Rightarrow$ (1). Let B and L be a bi-ideal and a left ideal of S, respectively. Let $a \in B \cap L$. Then, by Lemma 2.5 and Lemma 2.7, f_B is a fuzzy bi-ideal of S and f_L is a fuzzy left ideal of S. Thus, by hypothesis $f_B \cap f_L \subseteq f_B \circ f_L$, and

$$(f_B \circ f_L)(a) \geq (f_B \cap f_L)(a) = \min\{f_B(a), f_L(a)\} = 1$$

for all $a \in B \cap L$, and $A_a \neq \emptyset$. Since $f_B \circ f_L$ is a fuzzy subset of S, we have $(f_B \circ f_L)(a) \leq 1$ for all $a \in S$. Then

$$\bigvee_{(y,z)\in A_a} \min\{f_B(y), f_L(z)\} = (f_B \circ f_L)(a) = 1,$$

which implies that there exist $b, c \in S$ such that $a \leq bc$, $f_B(b) = 1$ and $f_L(c) = 1$. Then $a \leq bc \in BL$, and so $B \cap L \subseteq (BL]$. By Lemma 4.6, S is regular.

(1) $\Rightarrow$ (4). Let f, g and h be a fuzzy generalized bi-ideal, fuzzy left ideal and a fuzzy right ideal of S, respectively. Let $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa$. Thus $(a, xa) \in A_a$, and

$$\begin{aligned}
(h \circ f \circ g)(a) &= \bigvee_{(y,z)\in A_a} \min\{h(y), (f \circ g)(z)\} \\
&\geq \min\{h(ax), (f \circ g)(a)\} \\
&\geq \min\{h(a), \bigvee_{(p,q)\in A_a} \min\{f(p), g(q)\}\} \\
&\geq \min\{h(a), \min\{f(a), g(xa)\}\} \\
&\geq \min\{h(a), \min\{f(a), g(a)\}\} \\
&= \min\{h(a), f(a), g(a)\} = (h \cap f \cap g)(a).
\end{aligned}$$

Therefore $h \cap f \cap g \subseteq h \circ f \circ g$.

(4) $\Rightarrow$ (5). Clearly.

(5) $\Rightarrow$ (1). Let B, L and R be a bi-ideal, a left ideal and a right ideal of S, respectively. Let $a \in R \cap B \cap L$. Then, by Lemma 2.5 and Lemma 2.7, f_B, f_L and f_R be a fuzzy bi-ideal, a fuzzy left ideal and a fuzzy right ideal of S, respectively. Thus, by hypothesis, $f_R \cap f_B \cap f_L \subseteq f_R \circ f_B \circ f_L$. If $a \in R \cap B \cap L$, then

$$(f_R \circ f_B \circ f_L)(a) \geq (f_R \cap f_B \cap f_L)(a) = \min\{f_R(a), f_B(a), f_L(a)\} = 1.$$

Thus $A_a \neq \emptyset$. Furthermore, since $f_R \circ f_B \circ f_L$ is a fuzzy subset of S, we have $(f_R \circ f_B \circ f_L)(a) \leq 1$ for all $a \in S$. Thus

$$\bigvee_{(y,z)\in A_a} \min\{f_R \circ f_B(y), f_L(z)\} = (f_R \circ f_B \circ f_L)(a) = 1,$$

which implies that there exist $b, c \in S$ such that $a \leq bc$, and $(f_R \circ f_B)(b) = 1$ and $f_L(c) = 1$. Then $A_b \neq \emptyset$,

$$\bigvee_{(p,q)\in A_b} \min\{f_R(p), f_B(q)\} = (f_R \circ f_B)(b) = 1,$$

which implies that there exist $d, e \in S$ such that $b \leq de$, and $f_R(d) = f_B(e) = 1$. Then $a \leq bc \leq dec \in RBL$, that is $a \in (RBL]$. Thua $R \cap B \cap L \subseteq (RBL]$. By Lemma 4.6, S is regular. $\square$

5. Characterizations of Regular Ordered Semigroups and Intra-regular Ordered Semigroups

An ordered semigroup $(S, \cdot, \leq)$ is called *intra-regular* if, for each element a of S, there exists $x, y \in S$ such that $a \leq xa^2y$.

Equivalent Definition: $a \in (Sa^2S], \forall a \in S$ [13].
As we know, the characterizations of regular semigroups and intra-regular semigroups in terms of left ideals, right ideals, bi-ideals or quasi-ideals was investigated [8, 10]. In this section, we shall characterize them in terms of the corresponding "fuzzy" analogous notions.

Lemma 5.1. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is intra-regular.*
(2) *$R \cap L \subseteq (LR]$ for every left ideal L and every right ideal R of S.*

Proof. (1) $\Rightarrow$ (2). If S is a intra-regular ordered semigroup, then for any $a \in R \cap L$ there exist $x, y \in S$ such that

$$a \leq xa^2y = (xa)(ay) \in (SL)(RS) \subseteq LR.$$

Thus, $a \in (LR]$.

(2) $\Rightarrow$ (1). Assume that (2) holds and $a \in S$. Then, by hypothesis and Lemma 2.2,

$$\begin{aligned} a &\in R(a) \cap L(a) \subseteq (L(a)R(a)] \\ &= ((a \cup Sa](a \cup aS]] = ((a \cup Sa)(a \cup aS)] \\ &= (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]. \end{aligned}$$

Thus $a \leq t$ for some $t \in a^2 \cup Sa^2 \cup a^2S \cup Sa^2S$. If $t = a^2$, then $a \leq a^2 \leq a^4 \in Sa^2S$, that is $a \in (Sa^2S]$. If $t = xa^2$ for some $x \in S$, then $a \leq xa^2 \leq x(xa^2)a = x^2a^2a \in (Sa^2S]$. If $t = a^2y$ for some $y \in S$, then $a \leq a^2y \leq a(a^2y)y = aa^2y^2 \in Sa^2S$. If $t \in (Sa^2S]$, then $a \in (Sa^2S]$. Thus S is intra-regular. □

Now we give a characterization of an intra-regular ordered semigroup by fuzzy left ideals and fuzzy right ideals.

Theorem 5.2. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is intra-regular.*
(2) *$f \cap g \subseteq g \circ f$ for every fuzzy left ideal g and every fuzzy right ideal f of S.*

Proof. (1) $\Rightarrow$ (2). Suppose that S is an intra-regular ordered semigroup, f and g are a fuzzy right ideal and a fuzzy left ideal of S, respectively. Let $a \in S$. Then there exists $x, y \in S$ such that $a \leq xa^2y$, that is $(xa, ay) \in A_a$. Thus

$$\begin{aligned}(g \circ f)(a) &= \bigvee_{(y,z)\in A_a} \min\{g(y), f(z)\} \\ &\geq \min\{g(xa), f(ay)\} \\ &\geq \min\{g(a), f(a)\} \\ &= (f \cap g)(a),\end{aligned}$$

which means that $f \cap g \subseteq g \circ f$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let R and L be any right ideal and left ideal of S, respectively, and $a \in R \cap L$. Then , by Lemma 2.5, f_R and f_L be a fuzzy right ideal and a fuzzy left ideal of S, respectively. Thus , by hypothesis,

$$(f_L \circ f_R)(a) \geq (f_R \cap f_L)(a) = \min\{f_R(a), f_L(a)\} = 1,$$

and so $A_a \neq \emptyset$. Since $f_L \circ f_R$ is a fuzzy subset of S, we have $(f_L \circ f_R)(a) \leq 1$ for any $a \in S$. Thus we have

$$\bigvee_{(y,z)\in A_a} \min\{f_L(y), f_R(z)\} = (f_L \circ f_R)(a) = 1,$$

which implies that there exist $b, c \in S$ such that $a \leq bc$, $f_L(b) = 1$ and $f_{LR}(c) = 1$. Then $a \leq bc \in LR$, and so $R \cap L \subseteq (LR]$. It follows, by Lemma 5.1, that S is intra-regular. □

Theorem 5.3. *An ordered semigroup S is intra-regular if and only if*

$$(\forall a \in S)\ f(a) = f(a^2)$$

for every fuzzy ideal f of S.

Proof. $\Longrightarrow$. Let f be a fuzzy ideal of S and $a \in S$. Then, by hypothesis, there exist $x, y \in S$ such that $a \leq xa^2y$, and

$$f(a) \geq f(xa^2y) \geq f(a^2y) \geq f(a^2) \geq f(a),$$

which implies that $f(a) = f(a^2)$.
$\Longleftarrow$. By Lemma 2.5, $f_{I(a^2)}$ is a fuzzy ideal of S. Since $f(a) = f(a^2)$, we have $f_{I(a^2)}(a) = f_{I(a^2)}(a^2) = 1$, so $a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$. Then, as is stated in the proof of Lemma 5.1, S is intra-regular. □

Theorem 5.4. *Let S be an intra-regular ordered semigroup. Then, for any fuzzy ideal f of S, we have*

$$(\forall a, b \in S)\ f(ab) = f(ba).$$

Proof. Let f be any fuzzy ideal of S and $a, b \in S$. Then, by Theorem 5.3 and hypothesis, we have

$$f(ab) = f((ab)^2) = f(a(ba)b) \geq f(ba) = f((ba)^2) = f(b(ab)a) \geq f(ab).$$

Thus we obtain that $f(ab) = f(ba)$. □

Lemma 5.5. *An ordered semigroup S is regular and intra-regular if and only if $B = (B^2]$ for every bi-ideal B of S.*

Proof. Let S be both regular and intra-regular, and B a bi-ideal of S. Then $BSB \subseteq B, B \subseteq (BSB]$ and $B \subseteq (SB^2S]$. Thus

$$\begin{aligned} B &\subseteq (BSB] \subseteq (BS(BSB]] \\ &\subseteq ((BS](BSB]] = (BSBSB] \\ &\subseteq ((BS](SB^2S](SB]] \subseteq ((BS)(SB^2S)(SB)] \\ &\subseteq ((BSB)(BSB)] \subseteq (B^2]. \end{aligned}$$

On the other hand, since B is a bi-ideal of S, we have $(B^2] \subseteq (B] = B$. Hence we obtain that $B = (B^2]$.

Conversely, let $a \in S$. Since $B = (B^2]$ for every bi-ideal B of S, we have

$$\begin{aligned} a &\in B(a) \subseteq (B^2(a)] \\ &= ((a \cup a^2 \cup aSa](a \cup a^2 \cup aSa]] \\ &= ((a \cup a^2 \cup aSa)(a \cup a^2 \cup aSa)] \\ &\subseteq (a^2 \cup aSa]. \end{aligned}$$

Then, as is stated in the proof of Lemma 4.6, S is regular. Furthermore, we also have

$$\begin{aligned} a &\in B(a) \subseteq (B^2(a)] \subseteq (B^2(a)B(a)] \\ &\subseteq ((a^2 \cup aSa](a \cup a^2 \cup aSa]] \\ &= ((a^2 \cup aSa)(a \cup a^2 \cup aSa)] \\ &= (a^3 \cup a^4 \cup a^3Sa \cup aSa^2 \cup aSa^3 \cup aSa^2Sa] \\ &\subseteq (a^3 \cup aSa^2 \cup Sa^2S]. \end{aligned}$$

Then $a \leq t$ for some $t \in a^3 \cup aSa^2 \cup Sa^2S$. If $t = a^3$, then $a \leq a^3 = aa^2 \leq a^5 \in Sa^2S$, that is $a \in (Sa^2S]$. If $t = axa^2$ for some $x \in S$, then $a \leq axa^2 \leq (axa^2)xa^2 = (ax)a^2(xa^2) \in Sa^2S$. If $t \in Sa^2S$, then $a \in (Sa^2S]$. Thus, S is intra-regular. □

Lemma 5.6. *(cf.* [17]*, Theorem 1) An ordered semigroup S is regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of S, we have $f \circ g = f \cap g$.*

Now we shall give some characterizations of an ordered semigroup that is both regular and intra-regular by fuzzy left ideals, fuzzy right ideals and fuzzy bi-ideals.

Theorem 5.7. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(1) *S is regular and intra-regular.*

(2) *$f \circ f = f$ for every fuzzy bi-ideal of S.*

(3) *$f \cap g \subseteq (f \circ g) \cap (g \circ f)$ for any fuzzy bi-ideals f and g of S.*

(4) *$f \cap g \subseteq (f \circ g) \cap (g \circ f)$ for every fuzzy bi-ideal f and every fuzzy left ideal g of S.*

(5) *$f \cap g \subseteq (f \circ g) \cap (g \circ f)$ for every fuzzy right ideal f and every fuzzy bi-ideal g of S.*

(6) *$f \cap g \subseteq (f \circ g) \cap (g \circ f)$ for every fuzzy right ideal f and every fuzzy left ideal g of S.*

Proof. (1) $\Rightarrow$ (3). Let f and g be fuzzy bi-ideals of S and $a \in S$. Then, since S is both regular and intra-regular, there exists $x \in S$ such that $a \le axa (\le axaxa)$, and there exist $y, z \in S$ such that $a \le ya^2z$. Thus

$$a \le axa \le axaxa \le ax(ya^2z)xa = (axya)(azxa),$$

that is $(axya, azxa) \in A_a$. Since f and g are both fuzzy bi-ideals of S, we have

$$f(axya) \ge \min\{f(a), f(a)\} = f(a) \text{ and } g(azxa) \ge \min\{g(a), g(a)\} = g(a).$$

Then

$$\begin{aligned}(f \circ g)(a) &= \bigvee_{(p,q)\in A_a} \min\{f(p), g(q)\} \\ &\ge \min\{f(axya), g(azxa)\} \\ &\ge \min\{f(a), g(a)\} = (f \cap g)(a),\end{aligned}$$

which means that $f \cap g \subseteq f \circ g$. In a same way, we can show that $f \cap g \subseteq g \circ f$. Hence $f \cap g \subseteq (f \circ g) \cap (g \circ f)$.

Since every fuzzy left (right) ideal of S is a fuzzy bi-ideal of S, and so (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (6), (3) $\Rightarrow$ (2), (3) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (1). Let f and g be a fuzzy right ideal and a fuzzy left ideal of S, respectively. By hypothesis, $f \cap g \subseteq (f \circ g) \cap (g \circ f) \subseteq g \circ f$. By Theorem 5.2, S is intra-regular. On the other hand,

$$\begin{aligned} f \cap g &\subseteq (f \circ g) \cap (g \circ f) \subseteq f \circ g. \\ f \circ g &\subseteq f \circ S \subseteq f. \text{ (By Lemma 2.9)} \\ f \circ g &\subseteq S \circ g \subseteq g. \text{ (By Lemma 2.8)} \end{aligned}$$

Thus $f \circ g \subseteq f \cap g$, and$f \circ g = f \cap g$. By Lemma 5.6, S is regular.

(2) $\Rightarrow$ (1). Let B be a bi-ideal of S and $a \in B$. By Lemma 2.7, f_B is a fuzzy bi-ideal of S. Then , by hypothesis, $(f_B \circ f_B)(a) = f_B(a) = 1$, which implies that $A_a \neq \emptyset$, and

$$\bigvee_{(y,z)\in A_a} \min\{f_B(y), f_B(z)\} = (f_B \circ f_B)(a) = 1.$$

Then there exist $b, c \in S$ such that $a \le bc$, and $f_B(b) = f_B(c) = 1$. Then $a \le bc \in BB = B^2$, that is $a \in (B^2]$, and so $B \subseteq (B^2]$. By Lemma 5.5, S is regular and intra-regular. □

A non-empty subset Q of an ordered semigroup S is called a *quasi-ideal* of S if

(1) $(QS] \cap (SQ] \subseteq Q$.

(2) If $a \in Q$ and $S \ni b \le a$, then $b \in Q$ [9].

Let $(S, \cdot, \le)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if

(1) $x \le y \Rightarrow f(x) \ge f(y)$.

(2) $(f \circ S) \cap (S \circ f) \subseteq f$ [17].

By Lemma 2.8 and Lemma 2.9, every fuzzy left (right) ideal of an ordered semigroup S is a fuzzy quasi-ideal of S.

Lemma 5.8. *(cf. [17], Proposition 9) Let S be an ordered semigroup and $\emptyset \neq Q \subseteq S$. Then Q is a quasi-ideal of S if and only if the characteristic function f_Q of Q is a fuzzy quasi-ideal of S.*

Lemma 5.9. *(cf. [22], Theorem 3) Let S be an ordered semigroup. Then the following conditions are equivalent:*

(1) *S is regular and intra-regular.*
(2) *$B \cap Q \subseteq (BQB]$ for every bi-ideal B and every quasi-ideal Q of S.*
(3) *$B \cap L \subseteq (BLB]$ for every bi-ideal B and every left ideal L of S.*
(4) *$B \cap R \subseteq (BRB]$ for every bi-ideal B and every right ideal R of S.*
(5) *$B \cap Q \subseteq (QBQ]$ for every bi-ideal B and every quasi-ideal Q of S.*
(6) *$L \cap Q \subseteq (QLQ]$ for every left ideal L and every quasi-ideal Q of S.*
(7) *$R \cap Q \subseteq (QRQ]$ for every right ideal R and every quasi-ideal Q of S.*

By Lemma 5.8 and Lemma 5.9, we give characterizations of an ordered semigroup that is both regular and intra-regular by fuzzy left ideals, fuzzy right ideals, fuzzy bi-ideals and fuzzy quasi-ideals.

Theorem 5.10. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is regular and intra-regular.*
(2) *$f \cap g \subseteq f \circ g \circ f$ for every fuzzy bi-ideal f and every fuzzy quasi-ideal g of S.*
(3) *$f \cap g \subseteq f \circ g \circ f$ for every fuzzy bi-ideal f and every fuzzy left ideal g of S.*
(4) *$f \cap g \subseteq f \circ g \circ f$ for every fuzzy bi-ideal f and every fuzzy right ideal g of S.*
(5) *$f \cap g \subseteq g \circ f \circ g$ for every fuzzy bi-ideal f and every fuzzy quasi-ideal g of S.*
(6) *$f \cap g \subseteq g \circ f \circ g$ for every fuzzy left ideal f and every fuzzy quasi-ideal g of S.*
(7) *$f \cap g \subseteq g \circ f \circ g$ for every fuzzy right ideal f and every fuzzy quasi-ideal g of S.*

Proof. (1) $\Rightarrow$ (2). Let f and g be a fuzzy bi-ideal and a fuzzy quasi-ideal of S, respectively, let $a \in S$. Since S is both regular and intra-regular, there exist $x, y, z \in S$ such that $a \leq axa$ and $a \leq ya^2z$. Then

$$a \leq (axa)x(axa) \leq ax(ya^2z)x(ya^2z)xa = (axya)(azxyz)(azxa).$$

Since f is a fuzzy bi-ideal of S and g is a fuzzy quasi-ideal of S, we have

$$\begin{aligned} g(azxya) &\geq (g \circ S) \cap (S \circ g)(azxya) \\ &= \min\{(g \circ S)(azxya), (S \circ g)(azxya)\} \\ &= \min\{\bigvee_{(p,q)\in A_{azxya}} \min\{g(p), S(q)\}, \bigvee_{(u,v)\in A_{azxya}} \min\{S(u), g(v)\}\} \\ &\geq \min\{\min\{g(a), S(zxya)\}, \min\{S(azxy), g(a)\}\} \\ &= \min\{\min\{g(a), 1\}, \min\{1, g(a)\}\} \\ &= \min\{g(a), g(a)\} = g(a). \\ f(axya) &\geq \min\{f(a), f(a)\} = f(a). \\ f(azxa) &\geq \min\{f(a), f(a)\} = f(a). \end{aligned}$$

Thus

$$\begin{aligned}
(f\circ g\circ f)(a) &= \bigvee_{(m,n)\in A_a} \min\{(f\circ g)(m), f(n)\}\\
&\geq \min\{(f\circ g)(axyaazxya), f(azxa)\}\\
&\geq \min\{\bigvee_{(s,t)\in A_{axyaazxya}} \min\{f(s), g(t)\}, f(azxa)\}\\
&\geq \min\{\min\{f(axya), g(azxya)\}, f(azxa)\}\\
&\geq \min\{\min\{f(a), g(a)\}, f(a)\}\\
&= \min\{f(a), g(a)\} = (f\cap g)(a),
\end{aligned}$$

which implies that $f\cap g\subseteq f\circ g\circ f$.

Since every fuzzy left (right) ideal of S is a fuzzy quasi-ideal of S, and so (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are clear.

(3) $\Rightarrow$ (1). Let B and L be a bi-ideal and a left ideal of S, respectively. By Lemma 2.6 and Lemma 2.8, f_B and f_L are a fuzzy bi-ideal and a fuzzy left ideal of S, respectively. By hypothesis, we have

$$(f_B\circ f_L\circ f_B)(a)\geq (f_B\cap f_L(a) = \min\{f_B(a), f_L(a)\} = 1,$$

and so $A_a\neq\emptyset$. Since $f_B\circ f_L\circ f_B$ is a fuzzy subset of S, we have $(f_B\circ f_L\circ f_B)(a)\leq 1$ for all $a\in S$. Thus we have

$$\bigvee_{(y,z)\in A_a} \min\{f_B\circ f_L(y), f_B(z)\} = (f_B\circ f_L\circ f_B)(a) = 1,$$

which implies that there exist $b, c\in S$ such that $a\leq bc$, $(f_B\circ f_L)(b) = 1$ and $f_B(c) = 1$. Then $A_b\neq\emptyset$, $c\in B$ and

$$\bigvee_{(p,q)\in A_b} \min\{f_B(p), f_L(q)\} = (f_B\circ f_L)(b) = 1,$$

which implies that there exist $d, e\in S$ such that $b\leq de$, $f_B(d) = 1$ and $f_L(e) = 1$. Then $d\in B$, $e\in L$ and $a\leq bc\leq dec\in BLB$, that is $a\in (BLB]$, and so $B\cap L\subseteq (BLB]$. By Lemma 5.9, S is both regular and intra-regular.

In a same way, we can show that (4) $\Rightarrow$ (1), (6) $\Rightarrow$ (1) and (7) $\Rightarrow$ (1) hold.

(1) $\Rightarrow$ (5). Similar to (1) $\Rightarrow$ (2), we omit.

Since every fuzzy left (right) ideal of S is a fuzzy bi-ideal of S, and so (5) $\Rightarrow$ (6) and (5) $\Rightarrow$ (7) are clear. This completes the proof. □

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