ACTIONS, NORMS, SUBACTIONS AND KERNELS OF (FUZZY) NORMS

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Abstract. In this paper, we introduce the notion of an action Y_X as a generalization of the notion of a module, and the notion of a norm $\triangle: Y_X \to F$, where F is a field and $\triangle(xy) \triangle(y') = \triangle(y) \triangle(xy')$ as well as the notion of fuzzy norm, where $\triangle: Y_X \to [0,1] \subseteq \mathbf{R}$, with $\mathbf{R}$ the set of all real numbers. A great many standard mappings on algebraic systems can be modeled on norms as shown in the examples and it is seen that Ker $\triangle = \{y | \triangle(y) = 0\}$ has many useful properties. Some are explored, especially in the discussion of fuzzy norms as they relate to the complements of subactions N_X of Y_X.

1. Introduction

Norms and valuations in a great variety of settings have been investigated for a length of time and the literature on this subject is accordingly enormous. Thus, for example, the "classical" theory of norms and valuations has been discussed in a number of papers and texts, including O. T. O'Meara [5] and the well-known volumes by O. Zariski and P. Samuel [7]. Other than the usual areas, the idea of norms and pseudo-norms has been introduced for highly non-associative systems such as BCK-algebras where J. G. Raftery and T. Sturm [6] introduced the notion of pseudo-norm in BCK-algebras, the idea of a corresponding pseudo-metric and then that of pseudo-normed BCK-algebras. R. A. Borzooei and Y. B. Jun [3] introduced the intuitionisitic fuzzification of (strong, weak, s-weak) hyper BCK-ideals, and they established characterizations of an intuitionisitic fuzzy hyper BCK-ideal. Recently, M. Bakhshi, M. M. Zahedi and R. A. Borzooei [2] defined the notions of fuzzy positive hyper BCK-ideals of several types, and obtained some relationships among fuzzy(strong, weak, reflexive) hyper BCK-ideals. Fuzzy norms on linear spaces have been discussed by A. K. Katsaras [4] with further developments by others [1]. In order to collect all these quite varied notions under one heading we introduce the most general notion one might ever need, i.e., that of an action (defined below) and of a type of mapping satisfying a simple identity involving scalar and vector behavior with respect to these mappings. These "norms" include all known types of norms as well as certain other important mappings, e.g., group characters, not usually recognized as belonging to the same class as the more usual norms. Restricting the range to [0, 1] as a subset of the field F, e.g., when $F = \mathbf{R}$, the real numbers, yields the class of fuzzy norms of most interest in this paper. It

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will be clear from the following that a great deal more can be done along the lines we have established in this paper. We certainly hope to demonstrate this in further development of the theory of norms on actions as initiated here.

2. Preliminaries

Let X and Y be sets and let $\beta : X \times Y \to Y$ be any mapping. We shall denote the image $\beta(x, y)$ by xy and we shall write Y_X instead of $(X, Y, \beta : X \times Y \to Y)$, referring to Y_X as an *action*.

The notion of action is very wide and, in general, somewhat wider than one may want for more specific purposes; however it does allow a considerable body of varied information to be gathered under one heading and so allow for more homogeneous treatment.

The subjects of study in this paper are certain mappings from actions to fields. We shall call these mappings *norms*. Thus a mapping $\triangle: Y_X \to F$, where Y_X is an action and F is a field, is a *norm* provided

$$\triangle\,(xy) \triangle\,(y') = \triangle\,(y) \triangle\,(xy'), \quad \forall\, x \in X,\ \forall\, y, y' \in Y \tag{1}$$

In particular, if $F \supseteq [0, 1]$, then it is called a *fuzzy norm* on the action Y_X.

Example 2.1. Let Y be a Euclidean n-dimensional space E^n and X be the field of real numbers $\mathbf{R}$. By Y_X we understand the vector space E^n over the real field X. For $(\alpha_1, \cdots, \alpha_n) \in E^n$ we have the standard Euclidean norm:

$$\triangle\,(\alpha_1, \cdots, \alpha_n) := \sqrt{\alpha_1^2 + \cdots \alpha_n^2}$$

Then the standard Euclidean norm is certainly a norm in our sense as well since, for vectors $y = (\alpha_1, \cdots, \alpha_n), y' = (\beta_1, \cdots, \beta_n)$ and scalar x, we have $\triangle\,(xy) = |x| \triangle\,(y)$, whence it follows that

$$\triangle\,(xy) \triangle\,(y') = |x| \triangle\,(y) \triangle\,(y') = \triangle\,(y)|x| \triangle\,(y') = \triangle\,(y) \triangle\,(xy').$$

Example 2.2. Let $Y := Z^n - \{(0, \cdots, 0)\}$, where Z is the set of all integers, and $X := N$, the set of all natural numbers. If we define $\triangle$: $Y_X \to [0, 1]$ by $\triangle\,(\alpha_1, \cdots, \alpha_n) := 1/\sqrt{\alpha_1^2 + \cdots \alpha_n^2}$, then it is easy to show that $\triangle$ is a fuzzy norm on Y_X.

Example 2.3. Let $Y = X$ be any group and let C be the field of complex numbers. Y_X denotes the action "group operation". Assume that $\triangle$: $Y_X \to C$ is a norm. Then we have $\triangle\,(xy) \triangle\,(y') = \triangle\,(y) \triangle\,(xy')\ \forall\, x \in X,\ \forall\, y, y' \in Y$. If $y' = e$, then

$$\triangle\,(xy) \triangle\,(e) = \triangle\,(y) \triangle\,(x) = \triangle\,(x) \triangle\,(y)$$

If $\triangle\,(e) = 0$, then $\triangle\,(x^2)0 = \triangle\,(x)^2$ and hence $\triangle\,(x) = 0$ for any $x \in X$, i.e., $\triangle$ is the zero function. Assume that $\triangle\,(e) \neq 0$. Define a map $[\ ] : X \to C$ by $[x] := \frac{\triangle(x)}{\triangle(e)}$. Then $[xy] = \frac{\triangle(xy)}{\triangle(e)} = \frac{\triangle(xy)\triangle(e)}{\triangle(e)^2} = \frac{\triangle(x)\triangle(y)}{\triangle(e)^2} = \frac{\triangle(x)}{\triangle(e)} \cdot \frac{\triangle(y)}{\triangle(e)} = [x][y]$, and $\triangle\,(x) = [x] \triangle\,(e)$, where $[\ ] : X \to C$ is a homomorphism. Such a homomorphism is generally called a *character* and, in this case, the norms are scalar multiples of characters.

Example 2.4. Let Y be any group and Z be the collection of all integers and F be any field. By Y_Z we understand "the exponential action", i.e., the action defined by $ny := y^n$. We have that $(mn)y = y^{mn} = y^{nm} = (y^n)^m = my^n = m(ny)$ and $(m+n)y = y^m y^n = (my)(ny)$. Assume that $\Delta\colon Y_Z \to F$ is a norm. Then $\Delta\,(ny)\,\Delta\,(e) = \Delta\,(y)\,\Delta\,(ne) = \Delta\,(y)\,\Delta\,(e)$ for any $n \in Z$ and $y \in Y$. If $\Delta\,(e) \neq 0$, then $\Delta\,(ny) = \Delta\,(y)$. If we let $n := 0$, then $\Delta\,(y) = \Delta\,(0y) = \Delta\,(y^0) = \Delta\,(e)$ for any $y \in Y$, i.e., Δ is a constant function. If $\Delta\,(e) = 0$, since Δ is not a zero function, there is an element y in Y such that $\Delta\,(y) \neq 0$. If we define a map $\zeta_\Delta(n) := \frac{\Delta(ny)}{\Delta(y)}$ for any $n \in Z$, then we have $\Delta\,(ny) = \zeta_\Delta(n)\,\Delta\,(y)$. Thus, $\zeta_\Delta(m)\zeta_\Delta(n)\,\Delta\,(y) = \zeta_\Delta(m)\,\Delta\,(ny) = \Delta\,(m(ny)) = \Delta\,((mn)y) = \zeta_\Delta(mn)\,\Delta\,(y)$. Since $\Delta\,(y) \neq 0$, we obtain $\zeta_\Delta(m)\zeta_\Delta(n) = \zeta_\Delta(mn)$. Hence $\zeta_\Delta : Z \to F$ is a multiplicative function. In particular, if the field F is the real field and if Δ satisfies the condition $\Delta\,(yy') \leq \Delta\,(y) + \Delta\,(y')$, for any $y \in Y$, then $\Delta\,((m+n)y) = \Delta\,(my \cdot ny) \leq \Delta\,(my) + \Delta\,(ny)$. Thus $\zeta_\Delta(m+n)\,\Delta\,(y) \leq \zeta_\Delta(m)\,\Delta\,(y) + \zeta_\Delta(n)\,\Delta\,(y)$. If $\Delta\,(y) > 0$, then $\zeta_\Delta(m+n) \leq \zeta_\Delta(m) + \zeta_\Delta(n)$. Furthermore, if $\Delta\,(y) > 0$ then $\zeta_\Delta(y) > 0$. Moreover, $\zeta_\Delta(0) = \frac{\Delta(0y)}{\Delta(y)} = \frac{\Delta(y^0)}{\Delta(y)} = \frac{\Delta(e)}{\Delta(y)} = 0$. Hence we obtain a mapping $\zeta_\Delta : Z \to R$ satisfying the following conditions:

(I) $\zeta_\Delta(n) \geq 0$,
(II) $\zeta_\Delta(0) = 0$,
(III) $\zeta_\Delta(mn) = \zeta_\Delta(m)\zeta_\Delta(n)$,
(IV) $\zeta_\Delta(m+n) = \zeta_\Delta(m) + \zeta_\Delta(n)$.

Such a mapping is called a *valuation*.

Example 2.5. Let $X := [0,1]$ and $Y := \{g \mid g \text{ is integrable and } 0 \leq \int_0^1 g(x)dx \leq 1\}$. If we define $\Delta\colon Y \to [0,1]$ by $\Delta\,(g) := \int_0^1 g(x)dx$, then it is easy to show that Δ is a fuzzy norm.

Example 2.6. Let $Y = X = \{\begin{pmatrix} a & c \\ b & d \end{pmatrix} \mid 0 < ad - bc \leq 1, a, b, c, d$ are complex numbers $\}$ and the group operation is matrix multiplication. Y_X denotes the action "group operation". If we define $\Delta\colon Y \to [0,1]$ by $\Delta\,(A)$ is the determinant of A, i.e., $\Delta\,(A) = \det(A)$. Then it is easy to show that Δ is a fuzzy norm on Y_X.

3. Subactions and Kernels of (Fuzzy) Norms

Definition 3.1. Let Y_X be an action. Z_X is called a *subaction* of Y_X, and denoted by $Z_X < Y_X$, if

(i) $Z \subseteq Y$,
(ii) if $x \in X, z \in Z$, then $xz \in Z$, i.e., $XZ \subseteq Z$,

If $\{Z_{iX}\}_{i\in\Lambda}$ is a family of subactions of Y_X and if $Z_X = \cup_{i\in\Lambda} Z_{iX}$, then Z_X is a subaction of Y_X. Similarly, if $Z_X = \cap_{i\in\Lambda} Z_{iX}$, then Z_X is a subaction of Y_X provided we regard $\emptyset$ as a subaction of Y_X. Hence the collection of all subactions of Y_X forms a complete distributive lattice $L(Y_X)$. In fact, the set of all subactions of Y_X is a non-void poset under set inclusion in which any two elements have a meet

$(Z_\alpha \cap Z_\beta)_X$ and a join $(Z_\alpha \cup Z_\beta)_X$. Every subset $\{Z_{iX}\}_{i\in\Lambda}$ has a g.l.b. $\inf\{Z_{iX}\}_{i\in\Lambda}$ given by $Z_X = \cap_{i\in\Lambda} Z_{iX}$, and a l.u.b. $\sup\{Z_{iX}\}_{i\in\Lambda}$ given by $Z_X = \cup_{i\in\Lambda} Z_{iX}$.

Definition 3.2. Let Y_X be an action and $A \subseteq Y$. Then the smallest subaction $[A] = \cap\{Z \,|\, Z \in L(Y_X)$ and $A \subset Z\}$ of Y_X containing A is called the *subaction generated by* A. In particular, we denote $[y] = [\{y\}]$ for any $y \in Y$.

Proposition 3.3. *If A is a subaction of Y_X, then $A = [A] = \cup_{y\in A}[y]$.*

Proof. If $A \subseteq Y$ and $y \in A$, then $[y] \subseteq [A]$ and consequently $A \subseteq \cup_{y\in A}[y] \subseteq [A]$, i.e., $A = [A] = \cup_{y\in A}[y]$. □ □

Definition 3.4. Let $\triangle\colon Y_X \to [0,1]$ be a fuzzy norm. Then the set Ker $\triangle = \{y \in Y \mid \triangle(y) = 0\}$ is called the *kernel* of $\triangle$.

Proposition 3.5. *Let $\triangle\colon Y_X \to [0,1]$ be a fuzzy norm. Then* Ker $\triangle$ *is a subaction of Y_X.*

Proof. Let $y \in$ Ker $\triangle$ and $x \in X$. If $\triangle \neq 0$, then there exists $y' \in Y$ such that $\triangle(y') \neq 0$. Hence $\triangle(xy)\triangle(y') = \triangle(xy')\triangle(y) = 0$. Since $\triangle(y') \neq 0$, we obtain $\triangle(xy) = 0$, i.e., $xy \in$ Ker $\triangle$. If $\triangle = 0$, then $\triangle(xy) = 0$. In either case we have $xy \in$ Ker $\triangle$. □ □

Theorem 3.6. *Let $\triangle\colon Y_X \to F$ be a norm. Define a map $e_F : F \to [0,1]$ by*

$$e_F(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

Then $e_F \circ \triangle\colon Y_X \to [0,1]$ is a fuzzy norm on Y_X.

Proof. We claim that e_F is multiplicative. For any $x \in X$ and $y, y' \in Y$, if $\triangle(y)$ or $\triangle(y')$ is zero or both, then $e_F(\triangle(y)\triangle(y')) = e_F(0) = 0 = e_F(\triangle(y))e_F(\triangle(y'))$. Assume that $\triangle(y)\triangle(y') \neq 0$. Then $e_F(\triangle(y)\triangle(y')$. Moreover, since $\triangle(y) \neq 0 \neq \triangle(y')$, we have $e_F(\triangle(y)\triangle(y')) = 1 = e_F(\triangle(y))e_F(\triangle(y'))$, proving that e_F is multiplicative.

We show that $e_F \circ \triangle$ is a fuzzy norm. Since e_F is multiplicative and $\triangle$ is a norm, for any $x \in X$ and $y, y' \in Y$ we obtain

$$\begin{aligned}(e_F \circ \triangle)(xy)(e_F \circ \triangle)(y') &= e_F(\triangle(xy))e_F(\triangle(y')) \\ &= e_F(\triangle(xy)\triangle(y')) \\ &= e_F(\triangle(y)\triangle(xy')) \\ &= e_F(\triangle(y))e_F(\triangle(xy')) \\ &= (e_F \circ \triangle)(y)(e_F \circ \triangle)(xy'),\end{aligned}$$

proving that $(e_F \circ \triangle)$ is a fuzzy norm on Y_X. □

Proposition 3.7. *Let $\triangle\colon Y_X \to F$ be a norm. Then*

(i) Ker $\triangle =$ Ker$(e_F \circ \triangle)$,
(ii) $1 - e_F \circ \triangle$ *takes value 1 on* Ker $\triangle$ *and value 0 on the complement of* Ker $\triangle$.

Proof. (i). If $y \in \text{Ker}(e_F \circ \triangle)$, then $e_F(\triangle\ (y)) = 0$ and hence $\triangle\ (y) = 0$, i.e., $y \in \text{Ker}\ \triangle$. If $y \in \text{Ker}\ \triangle$, then $e_F(\triangle\ (y)) = 0$, i.e., $y \in \text{Ker}(e_F \circ \triangle)$.

(ii). It is an immediate consequence of the fact that $\text{Ker}\ \triangle = \text{Ker}\ (e_F \circ \triangle)$. □

Definition 3.8. A subaction N_X is called a *fuzzy norm subaction* of Y_X if $\triangle: Y_X \to [0,1]$, the characteristic function of the complement of N_X, is a norm.

Thus, there is a bijection between the fuzzy norm subactions and such norms by the correspondence

$$N_X \longleftrightarrow 1 - \chi_{N_X} = \triangle$$

where χ_S is the characterisitic function of $S \subseteq Y_X$ and 1 is the function which is 1 on all of Y_X. Hence $\chi_{N_X} = 1 - \triangle$ is a linear combination of norms.

Theorem 3.9. *Let $\triangle_i: Y_X \to [0,1]$ be fuzzy norms $(i = 1,2)$ and let $N(Y_X)$ be the set of all fuzzy norms on Y_X. If we define*

$$(\triangle_1 \odot \triangle_2)(y) := \triangle_1\ (y)\ \triangle_2\ (y)$$

for any $y \in Y$, then $(N(Y_X); \odot)$ is a commutative monoid.

Proof. If $\triangle_i$ is a fuzzy norm on Y_X, then

$$\begin{aligned}(\triangle_1 \odot \triangle_2)(xy_1)(\triangle_1 \odot \triangle_2)(y_2) &= \triangle_1\ (xy_1)\ \triangle_2\ (xy_1)\ \triangle_1\ (y_2)\ \triangle_2\ (y_2) \\ &= \triangle_1\ (xy_1)\ \triangle_1\ (y_2)\ \triangle_2\ (xy_1)\ \triangle_2\ (y_2) \\ &= \triangle_1\ (xy_2)\ \triangle_1\ (y_1)\ \triangle_2\ (xy_2)\ \triangle_2\ (y_1) \\ &= (\triangle_1 \odot \triangle_2)(xy_2)(\triangle_1 \odot \triangle_2)(y_1),\end{aligned}$$

proving that $\triangle_1 \odot \triangle_2$ is a fuzzy norm on Y_X. This means that $\triangle_1 \odot \triangle_2 \in N(Y_X)$. Clearly, $(\triangle_1 \odot \triangle_2) \odot \triangle_3 = \triangle_1 \odot (\triangle_2 \odot \triangle_3)$ and $\triangle_1 \odot \triangle_2 = \triangle_2 \odot \triangle_1$. If we define a map $1_{Y_X} : Y_X \to [0,1]$, then it acts as an identity on Y_X. □

Proposition 3.10. *Let $\triangle: Y_X \to [0,1]$ be a fuzzy norm and $\alpha \in [0,1]$. Define $(\alpha\ \triangle)(y) := \alpha\ \triangle\ (y)$ for any $y \in Y$. Then*

(i) *$\alpha\ \triangle$ is a fuzzy norm on Y_X,*
(ii) *$\text{Ker}(\alpha\ \triangle) = \text{Ker}\ \triangle$, where $\alpha \neq 0$,*
(iii) *$\text{Ker}(\triangle_1 \odot \triangle_2) = \text{Ker}\ \triangle_1 \cup \text{Ker}\ \triangle_2$.*

Proof. Straightforward. □

Theorem 3.11. *Let $\{\triangle_i\}$ be the family of characteristic fuzzy norms of complements of subactions $(N_i)_X$ of Y_X. If $\triangle := \inf_i \triangle_i$, then $\triangle$ is also a fuzzy norm on Y_X and $\text{Ker}\ \triangle = \cup_i \text{Ker}\ \triangle_i$.*

Proof. If $\triangle := \inf_i \triangle_i$, then $\triangle\ (y) = \inf_i \{\triangle_i\ (y)\}$. Since $\triangle_i\ (y) \in \{0,1\}$, $\triangle\ (y) \in \{0,1\}$. We show that $\triangle$ is a norm on Y_X. Given $y, y' \in Y, x \in X$, we consider the case $\triangle\ (xy)\ \triangle\ (y') = 1$. It follows that $\inf_i\ \triangle_i\ (xy) = 1 = \inf_i\ \triangle_i\ (y)$ and hence $\triangle_i\ (xy) = 1 = \triangle_i\ (y)\ \forall i$. Now, since $\triangle_i\ (xy) = 1, \forall i$, $xy \notin (N_i)_X$ where $\triangle_i$ is a characteristic fuzzy norm of complement of a subaction $(N_i)_X$, i.e., $\triangle_i = \chi_{Y_X - (N_i)_X}$. By definition of subaction $(N_i)_X$, $y \notin (N_i)_X, \forall i$ and hence $\triangle_i\ (y) = 1, \forall i$, i.e., $\triangle\ (y) = 1$. Now, since $\triangle_i$ is a fuzzy norm, we have $\triangle_i\ (xy)\ \triangle_i$

$(y') = 1 = \triangle_i\ (xy')\ \triangle_i\ (y)$. It follows from $\triangle_i\ (xy') = 1, \forall i$ that $\triangle\ (xy') = 1$. Hence $\triangle\ (xy)\ \triangle\ (y') = 1 = \triangle\ (y)\ \triangle\ (xy')$.

Assume $\triangle\ (xy)\ \triangle\ (y') = 0$. If $\triangle\ (y') = 0$, then $0 = \inf_i\ \triangle_i\ (y)$ and hence there exists i such that $\triangle_i\ (y) = 0$, i.e., $y \in (N_i)_X$. Since $(N_i)_X$ is a subaction of Y_X, we obtain $xy' \in (N_i)_X$. It implies $\triangle_i\ (xy') = 0$, which proves that $\triangle\ (xy') = \inf_i\ \triangle_i\ (xy') = 0$. Hence $\triangle\ (y)\ \triangle\ (xy') = 0$. If $\triangle\ (y') \neq 0$, then $\triangle\ (xy) = 0$. Hence there exists i such that $\triangle_i\ (xy) = 0$. Thus $\triangle$ is a fuzzy norm on Y_X. The proof of Ker $\triangle = \cup_i$Ker $\triangle_i$ is easy, and we omit the proof. □

Theorem 3.12. *Let $\{\triangle_i\}$ be a chain of characteristic fuzzy norms of complements of subactions $(N_i)_X$ of Y_X., i.e., $\max\{\triangle_i, \triangle_j\}$ is $\triangle_i$ or $\triangle_j$. Then $\triangle := \sup_i\ \triangle_i$ is a characteristic fuzzy norm on Y_X and* Ker $\triangle = \cap_i$Ker $\triangle_i$.

Proof. We may assume that $\triangle\ (xy)\ \triangle\ (y') = \triangle_i\ (xy)\ \triangle_j\ (y')$ for some i and j. Let us assume that $\triangle_i \leq \triangle_j$ without loss of generality. Case (I) $\triangle\ (xy)\ \triangle\ (y') = 1$. Then $\triangle_i\ (xy) = 1, \triangle_j\ (y') = 1$ for some i, j. Since $\triangle_i \leq \triangle_j$, we have $\triangle_j\ (xy)\ \triangle_j\ (y') = 1$. Since $\triangle_j$ is a fuzzy norm, $\triangle_j\ (xy)\ \triangle_j\ (y') = 1 = \triangle_j\ (y)\ \triangle_j\ (xy')$ and hence $\triangle\ (y)\ \triangle\ (xy') = \sup_i\ \triangle_i\ (y) \sup_i\ \triangle_i\ (xy') = 1$. Case (II) $\triangle\ (xy)\ \triangle\ (y') = 0$. We consider two cases: (II-i) $\triangle\ (y') = 0$ and (II-ii) $\triangle\ (y') \neq 0$. If $\triangle\ (y') = 0$, then $\triangle_i\ (y') = 0, \forall i$. Since $\triangle_i$ is a norm, we have $\triangle_i\ (y)\ \triangle_i\ (xy') = \triangle_i\ (xy)\ \triangle_i\ (y') = 0$. We claim that $\triangle_i\ (xy') = 0, \forall i$. Assume that $\triangle_j\ (xy') \neq 0$ for some j. Then $xy' \notin (N_j)_X$ for some $(N_j)_X < Y_X$ and hence $y' \notin (N_j)_X$, a contradiction. If $\triangle\ (y') \neq 0$, then $\triangle\ (xy) = 0$. This means that $\triangle_j\ (xy) = 0, \forall j$ and hence $\triangle_j\ (y)\ \triangle_j\ (xy') = \triangle_j\ (xy)\ \triangle_j\ (y') = 0$ for any j. We consider two cases: (II-ii-a) $\triangle_j\ (y) \neq 0$ and (II-ii-b) $\triangle_j\ (y) = 0$. If we consider the case (II-ii-a), then $\triangle_j\ (xy') = 0$. We claim that $\triangle_j\ (xy') = 0$ for any j. Assume that $\triangle_k\ (xy') \neq 0$ for some $k \neq j$. Since $\triangle_j\ (xy') = 0 \leq 1 = \triangle_k\ (xy')$ and $\{\triangle_i\}$ is a chain, we obtain $\triangle_j \leq \triangle_k$. It follows that $0 \neq \triangle_j\ (y) \leq \triangle_k\ (y) = 0$, a contradiction. Hence $\triangle\ (xy') = 0$ and $\triangle\ (y)\ \triangle\ (xy') = 0 = \triangle\ (y')\ \triangle\ (xy)$, proving that $\triangle$ is a fuzzy norm on Y_X. If we consider the case (II-ii-b), then we have two cases $\triangle_j\ (xy') = 0, \forall j$ or $\triangle_k\ (y) \neq 0$ for some k. For the former case, we have $\triangle\ (y)\ \triangle\ (xy') = 0$, proving that $\triangle$ is a fuzzy norm on Y_X. For the latter case, it belongs to the case (II-a), completing that $\triangle$ is a fuzzy norm on Y_X.

Since $\triangle_i$ is a characteristic fuzzy norm, we have

$$\begin{aligned} x \in \text{Ker}\ \triangle &\Longleftrightarrow \triangle\ (x) = 0 \\ &\Longleftrightarrow (\sup_i\ \triangle_i)(x) = 0 \\ &\Longleftrightarrow x \in \text{Ker}\ \triangle_i, \forall i \\ &\Longleftrightarrow x \in \cap_i \text{Ker}\ \triangle_i, \end{aligned}$$

proving that Ker $\triangle = \cap_i$Ker $\triangle_i$. □

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