FUZZY SUBGROUPS OF RANK TWO ABELIAN P-GROUP

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ABSTRACT. In this paper we enumerate fuzzy subgroups, up to a natural equivalence, of some finite abelian p-groups of rank two where p is any prime number. After obtaining the number of maximal chains of subgroups, we count fuzzy subgroups using inductive arguments. The number of such fuzzy subgroups forms a polynomial in p with pleasing combinatorial coefficients. By exploiting the order, we label the subgroups of maximal chains in a special way which enables us to count the number of fuzzy subgroups.

1. **Introduction**

1°. Finite abelian group.
A finite abelian group G of rank two is of the form $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ for two primes p and q, and two natural numbers n and m. If G is cyclic, then p and q are distinct. If not, it is of the form $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$ which is a p-group for a prime p. The enumeration of fuzzy subgroups in the above two cases are vastly different. In [3] we dealt with the cyclic case and here we deal with the non-cyclic case. In [2] we introduced a technique for counting the fuzzy subgroups of $\mathbb{Z}_p + \mathbb{Z}_p$, using keychains. A further analysis of that technique enables us to count the fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ for any positive integer n and represent it by means of a combinatorial formula. Proceeding inductively it is possible to represent the number of fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$ as a polynomial in p with coefficients as functions of n and $m = 1, 2, 3$. For general m we briefly indicate how to carry forward the inductive steps.

2°. Fuzzy subgroup.
We use $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$, the real unit interval as a chain with the usual ordering in which $\wedge$ stands for infimum and $\vee$ stands for supremum. A *fuzzy subset* of a set G is a mapping $\mu : G \to \mathbf{I}$, see [6]. Throughout this paper we take G to be a finite abelian p-group of rank two. We denote the trivial subgroup $\{0\}$ by G_0.
A fuzzy set μ is said to be a *fuzzy subgroup* if $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$ and $\mu(x) = \mu(-x)$ [5]. In this paper we assume that $\mu(0) = 1$ without loss of generality. By *core* and *support* of μ we mean the crisp subsets of G given by $core(\mu) = \{x \in G : \mu(x) = 1\}$ and $supp(\mu) = \{x \in G : \mu(x) \neq 0\}$ respectively. These are indeed crisp subgroups.

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3°. Equivalence, keychain and flag of fuzzy subgroups.
We now recall from [2] the definition of an *equivalence relation* on the set $\mathbf{I}^G$ of fuzzy subgroups of G. $\mu \sim \nu$ if and only if the following two conditions are satisfied:

$$\begin{array}{ll} \text{(i)} & \forall x, y \in G,\ \mu(x) > \mu(y) \quad \text{if and only if} \quad \nu(x) > \nu(y) \\ \text{(ii)} & \mu(x) = 0 \quad \text{if and only if} \quad \nu(x) = 0. \end{array} \tag{1.1}$$

We denote this equivalence relation by $\mu \sim \nu$, and the equivalence class containing μ is denoted by $[\mu]$. A. Jain in [1] had discussed the above equivalence relation together with some of its variants thoroughly. The equivalence relation reduces to equality of subgroups when two truth-values $\{0, 1\}$ are used. Therefore counting fuzzy subgroups up to the above equivalence is like counting the number of subgroups in the crisp case. This way of counting is dependent on the notion of a keychain. We refer the reader to [3] for results on keychain but state only its definition here. The way keychains were used to count fuzzy subgroups can be found in [4].

By a *keychain* $\ell = (\lambda_i)_0^n$ we mean an (n+1)-tuple $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ of real numbers in $\mathbf{I}$ of the form

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \tag{1.2}$$

where the λ_i's are not all necessarily distinct. The λ_i's are called *pins* . Note that there are $2^{n+1} - 1$ distinct keychains of length $n + 1$.
By a *flag* $\mathcal{C}$ we mean a maximal chain of subgroups of G and by a *pinned-flag* we mean a pair ($\mathcal{C}, \ell$), of a flag $\mathcal{C}$ on G and a keychain ℓ from $\mathbf{I}$, written as follows:

$$G_0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \subset \cdots \subset G_n^{\lambda_n} \tag{1.3}$$

We call $G_i^{\lambda_i}$ for $i = 0, 1, \ldots, n$, the $(i + 1)$-th component of the pinned-flag.
By *length* of a keychain ℓ or a flag $\mathcal{C}$ or a pinned-flag ($\mathcal{C}, \ell$) we mean the number of components, which is $n+1$. Further we want to talk about extensions of keychains, flags, and pinned-flags in the following sense. A keychain $\ell_1 = (\lambda_i)_0^n$ is an extension of $\ell_2 = (\beta_j)_0^m$ if $n \geq m$ and $\lambda_i = \beta_i$ for $i = 0, 1, \ldots, m$. Similar definitions apply for extensions of flags and pinned-flags. Thus a fuzzy subgroup μ of G is an extension of a fuzzy subgroup ν of H if $H \subseteq G$ and the keychain of μ is an extension of the keychain of ν. There is a one-to-one correspondence between fuzzy subgroups and pinned-flags up to equivalence, for details see [4]. We refer to fuzzy subgroups and pinned-flags interchangebly as refering to the same object.

Remark 1.1. Using the order of subgroups with carefully identified generators we arrive at the notion of levels. Since the orders of subgroups in a maximal chain of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$ are of the form p^i, $i = 0, 1, 2, \ldots, n + m$, the number of subgroups in any maximal chain is $n + m + 1$. Any subgroup of order p^i is referred to as being in the $(i + 1)$-th level of that maximal chain. For instance the $(i + 1)$-th level of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ has $p + 1$ subgroups for every $i = 1, 2, 3, \ldots, n$, given by p cyclic subgroups generated by (p^i, j) for $j = 0, 1, 2, \ldots, p-1$, and one non-cyclic subgroup generated by $(p^i, 0)$ and $(0, 1)$.

2. Fuzzy Subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_p$

1°. The case $\mathbb{Z}_{p^2} + \mathbb{Z}_p$.

The case $\mathbb{Z}_p + \mathbb{Z}_p$ has already been dealt with in our earlier paper [2], page 263. In that case the number of fuzzy subgroups turned out to be $4p+7$. We use Mathematical Induction to prove a counting formula for the number of fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_p$. For convenience we define $\binom{n}{k} = 0$ whenever $k > n$ and $\binom{0}{0} = 0$.

Theorem 2.1. *The number of fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ is*

$$2^{n+1}\binom{n}{1}p + 2^{n+2} - 1 \tag{2.4}$$

Proof. Induction on n. For $n = 1$ the formula gives the value $4p+7$ which agrees with that of Proposition 3.6, [2], page 263. Assume that the theorem is valid for a positive integer k. There are $k+2$ levels in $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ and the $(k+2)$-th level consists of a single subgroup, namely the whole group, whereas there are $p+1$ subgroups of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$ in the $(k+2)$-th level and the whole group $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$ is in the $(k+3)$-th level by Remark 1.1. Firstly we identify $\mathbb{Z}_{p^k} + \{0\}$ with $\mathbb{Z}_{p^k}$. Secondly every pinned-flag of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$ can be considered as an extension of a pinned-flag of $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ or $\mathbb{Z}_{p^k}$. Thirdly we determine the number of fuzzy subgroups of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$ which are extensions of fuzzy subgroups of $\mathbb{Z}_{p^k} + \mathbb{Z}_p$. Our assumption implies that there are $2^k\binom{k}{1}p + 2^{k+1}$ fuzzy subgroups whose support is precisely $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ and there are $2^k\binom{k}{1}p + 2^{k+1} - 1$ fuzzy subgroups whose support is strictly contained in $\mathbb{Z}_{p^k} + \mathbb{Z}_p$. We denote the former set of fuzzy subgroups by S_1 and the latter by S_2,(see Proposition 3.3. and Theorem 3.4. in [2]). Each fuzzy subgroup of S_1 gives rise to three fuzzy subgroups of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$. Their keychains are extended as follows: $(\lambda_0, \cdots, \lambda_{k+1}, \lambda_{k+1})$, $(\lambda_0, \ldots, \lambda_{k+1}, \lambda_{k+2})$ where $0 < \lambda_{k+2} < \lambda_{k+1}$ and $(\lambda_0, \ldots, \lambda_{k+1}, 0)$. Clearly each fuzzy subgroup of S_2 gives rise to only one fuzzy subgroup of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$. Thus the number of extensions is $4\left(2^k\binom{k}{1}p + 2^{k+1}\right) - 1$. Fourthly we consider extensions of $\mathbb{Z}_{p^k}$. There are $2^{k+1} - 1$ fuzzy subgroups of $\mathbb{Z}_{p^k}$. The case of those fuzzy subgroups whose supports are strictly contained in $\mathbb{Z}_{p^k}$ has already been included in the previous case. So we need only consider those fuzzy subgroups, whose set is denoted by S_3, having the support exactly $\mathbb{Z}_{p^k}$. Now $|S_3| = 2^k$. Each fuzzy subgroup in S_3 yields 4 fuzzy subgroups of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$; their keychains are extended as follows: $(\lambda_0, \ldots, \lambda_k, \lambda_{k+1}, \lambda_{k+2})$ where $0 \leq \lambda_{k+2} < \lambda_{k+1} \leq \lambda_k$. Thus the number of extensions resulting from S_3 is $4\left(2^k p\right)$. Hence the total number of fuzzy subgroups of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$ is

$$4\left(2^k\binom{k}{1}p + 2^{k+1}\right) - 1 + 4\left(2^k p\right) = 2^{k+3} + p(k+1)2^{k+2} - 1 \tag{2.5}$$

$$= 2^{k+2}\binom{k+1}{1}p + 2^{k+3} - 1$$

as required. This completes the induction. For $m = 2$, we have the following □

Theorem 2.2. *The number of fuzzy subgroups of* $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{p^2}$ *is*

$$2^{n+1}\left(\binom{n}{2} + 2\binom{n-1}{1}\right)p^2 + 2^{n+2}\binom{n+1}{1}p + 2^{n+3} - 1 \tag{2.6}$$

Proof. By induction on n. For $n = 1$, the above theorem is seen to be true by taking $n = 2$ in Theorem 2.1. Assume the theorem is true for $n = k$. To show that it is true for $n = k+1$, we count all possible extensions of fuzzy subgroups in three disjoint lots, S_1, S_2 and S_3 which will exhaust all fuzzy subgroups of G with $n = k+1$. The number of S_1-extensions from $\mathbb{Z}_{p^k} + \mathbb{Z}_{p^2}$ to $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_{p^2}$ is

$$4\left(2^k\left(\binom{k}{2} + 2\binom{k-1}{1}\right)p^2 + 2^{k+1}\binom{k+1}{1}p + 2^{k+2}\right) - 1 \tag{2.7}$$

For S_2-extensions from $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ to $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_{p^2}$ we use the previous theorem for the number of fuzzy subgroups of $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ together with 4 keychain extensions ending with pins of the form $\lambda\lambda\beta$, $\lambda\lambda 0$, $\lambda\beta\gamma$ and $\lambda\beta 0$ where $0 < \gamma < \beta < \lambda$, giving

$$4\left(2^k\binom{k}{1}p + 2^{k+1}\right)p \tag{2.8}$$

(Here we multiply by p since there are p flags extending from $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ to $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_{p^2}$ excluding the maximal chain through $\mathbb{Z}_{p^k} + \mathbb{Z}_{p^2}$ which has already been counted in S_1).

There are p cases in S_3 corresponding to level $k+1$. Each one of them gives rise to $8\,(2^k)\,p$ fuzzy subgroups. We prove this fact for only one of them for illustration, namely the number of possible extensions of fuzzy subgroups from $\mathbb{Z}_{p^k}$ to $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_{p^2}$. The 8 in the product arises from the 8 keychain extensions ending with $\lambda\lambda\beta\beta$, $\lambda\lambda 00$, $\lambda\lambda\beta\gamma$, $\lambda\lambda\beta 0$, $\lambda\beta\beta\gamma$, $\lambda\beta\beta 0$, $\lambda\beta\gamma\delta$ and $\lambda\beta\gamma 0$ where $0 < \delta < \gamma < \beta < \lambda$. There is only one maximal chain connecting $\{0\}$ to $\mathbb{Z}_{p^k}$ and it gives $2^{k+1} - 1$ fuzzy subgroups on that flag, but, only 2^k keychains ending with non-zero pins will contribute to fuzzy subgroups (not already counted) of $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_{p^2}$. Now the factor p accounts for the number of flags from $\mathbb{Z}_{p^k}$ to $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_{p^2}$ which were not counted before. Therefore there are $p(8\,(2^k)\,p)$ which is $8\,(2^k)\,p^2$ fuzzy subgroups in S_3. The total number of fuzzy subgroups in all (exhaustive) three cases S_1, S_2 and S_3 is ,

$$2^{k+2}\left(\binom{k+1}{2} + 2\binom{k}{1}\right)p^2 + 2^{k+3}\binom{k+2}{1}p + 2^{k+4} - 1 \tag{2.9}$$

This completes the induction. □

For $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$, combinatorial formulae can be developed along similar lines by considering extensions of fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$ to fuzzy subgroups of $\mathbb{Z}_{p^{n+1}} + \mathbb{Z}_{p^m}$ using induction on s-levels for any positive integer s satisfying

$m+1 \leq s \leq n+m+1$ to $(n+1)+m+1$. For instance we state a formula for $m=3$ which can be verified using such an induction.

$$2^{n+1}\left(\binom{n}{3}+2\left(2\binom{n-1}{2}+3\binom{n-2}{1}\right)\right)p^3+2^{n+2}\left(\binom{n+1}{2}+2\binom{n}{1}\right)p^2$$
$$+2^{n+3}\binom{n+2}{1}p+2^{n+4}-1$$

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