SOLVING BEST PATH PROBLEM ON MULTIMODAL TRANSPORTATION NETWORKS WITH FUZZY COSTS

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Abstract. Numerous algorithms have been proposed to solve the shortest-path problem; many of them consider a single-mode network and crisp costs. Other attempts have addressed the problem of fuzzy costs in a single-mode network, the so-called fuzzy shortest-path problem (FSPP). The main contribution of the present work is to solve the optimum path problem in a multimodal transportation network, in which the costs of the arcs are fuzzy values. Metropolitan transportation systems are multimodal in that they usually contain multiple modes, such as bus, metro, and monorail. The proposed algorithm is based on the path algebra and dioid of k-shortest fuzzy paths. The approach considers the number of mode changes, the correct order of the modes used, and the modeling of two-way paths. An advantage of the method is that there is no restriction on the number and variety of the services to be considered. To track the algorithm step by step, it is applied to a pseudo-multimodal network.

1. Introduction

In graph theory, the shortest-path problem is the problem of finding a path between two nodes such that the summation of the costs of its constituent arcs is minimal. An example is identifying the shortest way from one location to another on a street map; in this case, the nodes represent locations, the arcs represent segments of streets, and the length is the considered cost. As metropolitan transportation services expand, passengers are confronted with an increasing number of considerations in addition to the shortest distance of an urban trip, such as aesthetic quality of the scenery, travel time, and preference for using a particular mode of transportation. Introducing these parameters, especially into a multimodal network, makes the current shortest-path algorithms, such as those of Dijkstra, Bellman-Ford, and A^* [9, 10, 25], insufficient to guide travelers properly.

Various extensions have been proposed to overcome such deficiencies [2, 4, 16, 19, 21]. Generally, multimodal transportation is defined as the transportation of a particular entity by different transportation services. For example, in an urban area, passengers might combine various transportation modes. Beilli *et al.* described a multimodal travel system (MTS) that focuses on network object modeling and the multimodal shortest-path algorithm [2]. They proposed a tool for detecting the facility of using the different travel modes offered by a given transportation

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network. Keshtiarast *et al.* studied the viable paths, providing passengers with a guidance system along a multimodal journey [16]. Their algorithm was based on a generalization of the label-correction method. Miller *et al.* developed an innovative multimodal network design in which the likelihood of GIS database consistency is maximized [19].

Transportation networks often include fuzzy information along their arcs, such as transportation time and aesthetic quality of the scenery. Since these values are inexact, they are well-presented by fuzzy sets [11, 23]. A fuzzy shortest path is, by definition, one that has the minimum overall fuzzy cost. Since the word "minimum" has different meanings in fuzzy theory, the problem can be solved from different points of view [1, 7, 8, 14, 15, 17, 20, 22, 24]. Okada and Soper developed an algorithm based on the multiple-labeling method and applied it to a multicriteria shortest-path problem [24]. They introduced an order relation between fuzzy numbers based on a fuzzy min operator and then defined a non-dominated path or Pareto optimal path from the specified node to every other node. Okada solved the FSPP by means of possibility theory, introducing the concept of the degree of possibility that an arc is located along the shortest path [23]. A recently published paper by Hernandes *et al.* addressed the drawbacks of previous algorithms, such as finding the fuzzy shortest path without the corresponding path(s) and the need to model the negative weights [14]. Their proposed algorithm is similar to the Bellman-Ford algorithm in the case of crisp numbers. Boulmakoul proposed a new algebra that can be regarded as the environment in which FSPP can be modeled [3]. He generalized Gauss-Seidel's shortest-path algorithm to discrete fuzzy numbers.

The contribution of this study is to generalize FSPP to multimodal networks. The solution is investigated with the aid of dioids and path algebra. A network containing various modes, in which costs are fuzzy numbers, is examined and best paths with respect to the desired cost are presented. The proposed approach divides the multimodal transportation network into sub-graphs. Best paths are then obtained on the basis of the k-shortest fuzzy paths algorithm. Multimodal challenges are also considered by the algorithm.

2. Dioid and Fuzzy Shortest Path

The term "dioid" is most often used in the context of an algebra in which path analysis can be developed. Dioid structures are applied to model optimization problems [6, 12]. The FSPP is an analysis that can be developed by a fuzzy dioid. A dioid is denoted by $\langle \Omega, \oplus, \otimes, \varepsilon, e \rangle$ where Ω is the base set, $\oplus$ and $\otimes$ are the dioid's sum and multiplication operators and ε and e are the neutral elements of $\oplus$ and $\otimes$, respectively. A dioid is a special kind of semi-ring structure [13]. The predefined dioid $\langle \mathbb{R} \cup \{+\infty\}, min, +, +\infty, 0 \rangle$ represents the shortest-path problem in a single-mode graph with crisp costs. The FSPP can also be seen as an optimization problem. In the following, the elements of the dioid are defined and then the extended k-shortest fuzzy paths algorithm is presented.

Definition 2.1. *Fuzzy-valued Graph:* The structure $G(V, E, \Phi)$ is a fuzzy-valued graph where:

V, as the nodes set, is a finite countable set, e.g. $\{v_1, v_2, ..., v_n\}$
Φ, as the weighting function, images the elements of $V \times V$ to fuzzy sets
E, as the set of arcs, is a subset of $V \times V$ such that function Φ maps its members to the fuzzy sets with non-empty support set (Θ) [3, 5].

Figure 1 shows a fuzzy-valued graph.

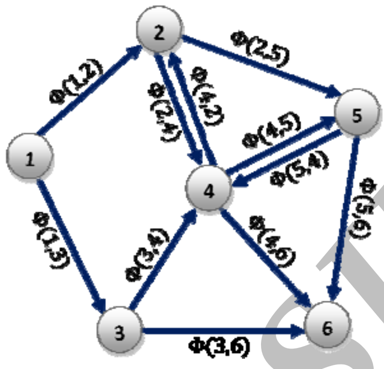

FIGURE 1. An Example of a Fuzzy-valued Graph

The parameters of the fuzzy valued graph are:
$V = \{1, 2, 3, 4, 5, 6\}, \Theta = \{1, 2, 3\}$
$E = \{(1, 2)(1, 3)(2, 4)(2, 5)(3, 4)(3, 6)(4, 2)(4, 5)(4, 6)(5, 4)(5, 6)\}$
The set of arcs are mapped to the fuzzy weights like $\Phi(1, 2) = \{.1/1, .2/2, .3/3\}$.
Let V_k be the set of all fuzzy sets of natural numbers whose support cardinality is less than or equal to k:

$$V_k = \{\tilde{A} \in [0, 1]^{\mathbb{N}} | |\Theta(\tilde{A})| \leqslant k\} \tag{1}$$

The operator $\Pi_k(A)$ is defined as a sorting operator, where A is a crisp subset of natural numbers. The operator returns only k-smallest elements of A. Also, let us define the operator $[\tilde{A}]_k$, where $\tilde{A}$ is a fuzzy set, as follows:

$$[\tilde{A}]_k = \Pi_k(\Theta(\tilde{A}))\tilde{\cap}\tilde{A} \tag{2}$$

where $\tilde{\cap}$ is the fuzzy intersection operator according to any arbitrary t-norm. The membership function for each $v \in V_k$ is:

$$\mu_{[\tilde{A}]_k} = 1_{\prod_k(\Theta(\tilde{A}))}(v) \times \mu_{\tilde{A}}(v) \tag{3}$$

where:

$$1_A(v) = \{ \begin{matrix} 1 & if v \in A \\ 0 & else. \end{matrix} \tag{4}$$

According to these definitions, the binary operators of the dioid of k-shortest fuzzy paths can be given for $\tilde{A} \in V_k$ and $\tilde{B} \in V_k$, as follows:

$$\begin{aligned} \tilde{A} \oplus \tilde{B} &= [\tilde{A} \tilde{\cup} \tilde{B}]_k, \\ \forall v \in V_k \to \mu_{\tilde{A} \oplus \tilde{B}}(v) = \mu_{[\tilde{A} \tilde{\cup} \tilde{B}]_k}(v) &= 1_{\prod_k \Theta(\tilde{A} \tilde{\cup} \tilde{B})}(v) \times \mu_{\tilde{A} \tilde{\cup} \tilde{B}}(v) \\ &= 1_{\prod_k \Theta(\tilde{A} \tilde{\cup} \tilde{B})}(v) \times max(\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v)) \end{aligned} \tag{5}$$

$$\begin{aligned} \tilde{A} \otimes \tilde{B} &= [\tilde{A} \mp \tilde{B}]_k, \\ \forall v \in V_k \to \mu_{\tilde{A} \otimes \tilde{B}}(v) = \mu_{[\tilde{A} \mp \tilde{B}]_k}(v) &= 1_{\prod_k \Theta(\tilde{A} \mp \tilde{B})}(v) \times \mu_{\tilde{A} \mp \tilde{B}}(v) \\ &= 1_{\prod_k \Theta(\tilde{A} \mp \tilde{B})}(v) \times max_{v=x+y}(min(\mu_{\tilde{A}}(v), \mu_{\tilde{B}}(v))) \end{aligned} \tag{6}$$

where $\tilde{\cup}$ and $\mp$ are the fuzzy union and the fuzzy sum operators, respectively. To complete the dioid, its neutral elements are defined as:

$$\varepsilon = \emptyset \tag{7}$$

$$e = \{1/0\} \tag{8}$$

Algorithm 2.1. *Solution of the FSPP:* Let $G(V, E, \Phi)$ be the intended fuzzy-valued graph. In the following algorithm, $\pi(i)$ denotes the fuzzy cost from the origin to node i. $\Gamma(i)$ returns all the nodes connected to the outgoing arcs from i. The inverse of $\Gamma(i)$ brings the nodes connected to the incoming arcs to i. Other parameters were already defined.

α) $\pi(1) = e$

$$\pi(i) = \left\{ \begin{array}{ll} \Phi(1,i) & if (1,i) \in E, \\ \varepsilon & else. \end{array} \right. \quad for\ i \geqslant 2$$

β) *at step t*

for $i = 1...n$ do

$$\pi(1) = \textstyle\sum^{\oplus}_{j \in \Gamma^{-1}(1)} [\pi(j) \otimes \Phi(j,1) \oplus e]$$
$$\pi(i) = \textstyle\sum^{\oplus}_{j \in \Gamma^{-1}(i)} [\pi(j) \otimes \Phi(j,i)]$$

χ) repeat β until *stabilization of* $\Pi(i)$

The algorithm initializes the values in step α. In step β, the costs of all predecessors of i ($\pi(j)$) are multiplied by the cost from j to i ($\Phi(j,i)$). The least cost is then selected by the operator $\oplus$ above the sigma. $\oplus$ functions as a selector operator. Step β is repeated for all nodes until the convergence of results is achieved. The js selected by the operator $\oplus$ determine the shortest paths to i [3].

3. Solution of the Fuzzy Multimodal Transportation Network (FMTN)

Let $G(V, E, \Phi)$ be the intended FMTN, where V is a countable finite set of all nodes of every mode, Φ is the fuzzy weighting function, and E is a subset of V^2 according to Definition 2.1. Here, G is composed of various networks ($M^i, i = 1, 2, ..., m$) each of which represents one of the metropolitan transportation services.

$$G = M^1 \cup M^2 ... \cup M^m \tag{9}$$

$$V = V^1 \cup V^2 ... \cup V^m = \{v_1, v_2, ..., v_n\} \tag{10}$$

$$E = E^1 \cup E^2 ... \cup E^m \cup E^T \tag{11}$$

where m and n are the number of modes and nodes, respectively. In the notation, symbols such as χ^i imply that χ is an element related to mode M^i.

The origin and destination nodes (v_O & v_D), the stations and parking lots, and the boundary nodes of each mode through which a mode is connected to the others form the set of the model's nodes.

$$V^i = \{v_p^i | p = 1, 2, ..., n_{V^i}\} \tag{12}$$

There are two kinds of arcs in Equation 11; Traverse arcs (E^i) are those connecting the nodes of the same mode. Transfer arcs (E^T) are those connecting adjacent heterogeneous modes. Walking from a bus to a metro station, for example, is represented by a transfer arc in the model. The ordered pairs representing the arcs make the definition of two-way paths possible.

$$\begin{array}{l} E^i = \{(v_p^i, v_q^i) \in E | p \neq q\}, \\ E^T = \{(v_p^i, v_q^j) \in E | i \neq j\}. \end{array} \forall p, q \in \{1, 2, ..., n\} \tag{13}$$

Definition 3.1. *Connected Fuzzy-valued Graph:* Let $G(V, E, \Phi)$ be a fuzzy-valued graph. G is said to be connected if, for any arbitrary pair of nodes such as v^a and v^b, there is at least one string of nodes (v^i) such that:

$$\Theta(C) \neq \emptyset \tag{14}$$

where:

$$C = \Phi(v^a, v^1) \otimes \left(\sum_{(i=2,...,b-1)}^{\otimes} \Phi(v^{i-1}, v^i) \right) \otimes \Phi(v^{b-1}, v^b) \tag{15}$$

Definition 3.2. *Sub-graph (SG):* The connected fuzzy-valued graphs that form a mode (M^i) are the sub-graphs of that mode. The number of sub-graphs and the x^{th} sub-graph of mode M^i are, respectively, denoted by nsg^i and $SG_x^i(V_{SG_x^i}, E_{SG_x^i}, \Phi)$.

Definition 3.3. *Incoming Boundary Nodes (IN.BNs):* The set of nodes connected to the incoming arcs of a sub-graph and the origin node from which at least one arc enters the sub-graph are incoming boundary nodes.

Definition 3.4. *Outgoing Boundary Nodes (OUT.BNs):* The set of nodes connected to the outgoing arcs of a sub-graph and the destination node to which at least one arc leaves the sub-graph are outgoing boundary nodes.

According to Definition 3.2, G can be uniquely divided into sub-graphs. The least number of sub-graphs occurs when each mode is completely a connected fuzzy-valued graph, i.e.:

$$nsg^1 + nsg^2 + ... + nsg^m \geqslant m \tag{16}$$

Each sub-graph is a single-mode connected fuzzy-valued graph. Thus, for each sub-graph, k-shortest fuzzy paths from all members of IN.BNs to all OUT.BNs are computed by Algorithm 2.1. As a result, there is one arc for each member of the Cartesian product $IN.BNs \times OUT.BNs$ in a sub-graph schema (SGS), which is the reduced form of the sub-graph whose set of nodes ($V^{SGS_x^i}$) comprises the boundary nodes of the sub-graph. The results of the k-shortest fuzzy paths algorithm give the SGS's arcs ($E^{SGS_x^i}$) and their related costs.

$$E^{SGS_x^i} = \{(v_p, v_q) | v_p \in (IN.BNs)_{SG_x^i}, v_q \in (OUT.BNs)_{SG_x^i}, p \neq q\} \qquad (17)$$

Figure 2 shows a SGS and its related sub-graph, for which the reduced arcs, boundary nodes, and transfer arcs are schematically presented.

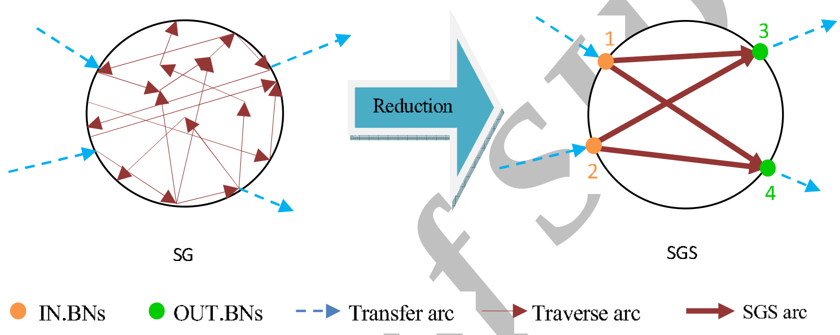

FIGURE 2. Reduction of Sub-Graphs

One of the particular concerns regarding multimodal networks is the number of mode changes. The problem is to develop the algorithm such that users are not subjected to an excess number of mode changes for a slight decrease in the cost. Generally, more than three or four changes frustrate passengers. Sub-graph level is defined as the minimum number of used transfer arcs from the origin to the sub-graph. All sub-graphs that contain the origin node are at zero level, the connected sub-graphs to the outgoing arcs of zero levels are at the first level, etc. One of the possible solutions to prevent an excess number of changes is to associate heavy costs with the transfer arcs. The problem can be treated dynamically by increasing the costs of all transfer arcs once a transfer arc is used, since passengers become increasingly frustrated with each mode change. Also, due to the algorithm's multiple results, passengers can select a path with fewer, or the fewest, changes.

All SGSs connected by the transfer arcs is referred to as the reduced G, which can be solved through Algorithm 2.1. In this step, the viability of paths or the correct order of modes used is also considered [18]. For example, a passenger who begins the trip by his/her private car and then changes to the metro will not be able to use his/her private car again. This problem is solved by temporary disablement of the corresponding arcs. In this step, the fuzzy k-best fuzzy paths and their corresponding fuzzy costs are obtained as the final product of the algorithm. The user can select a unique path among these paths according to his/her preferred modes, and the number of mode changes in each path.

3.1. **Computational Complexity Analysis.** Let r and s denote the number of nodes and arcs of a sub-graph, respectively. Algorithm 2.1 uses $O(r)$ operations for the first loop (α). There are $r-1$ iterations in loop β with $s(\oplus, \otimes)$ operations. Thus, there are $O(r \times s)$ operations at this step. Operation $\oplus$ needs k comparisons, and operation $\otimes$ uses $k \times \log(k)$ operations for a sorting algorithm [3]. Algorithm 2.1 is a one-to-many type i.e. each time it computes shortest paths from one origin node to all nodes. So we apply Algorithm 2.1 once for each member of IN.BNs of all sub-graphs and once for the reduced G. Hence, the total complexity of our approach is estimated as $1 + \sum_{i=1}^{m} \sum_{x=1}^{nsg^i} |(IN.BNs)_{SG_x^i}|$ times that of Algorithm 2.1.

4. Implementation

In order to clearly track each step of the algorithm, a pseudo-network is considered. The network contains four modes; bus, metro, taxi, and private car. Walking arcs serve as the transfer mode. The model is configured as:

$$V = \{v_1, v_2, v_3, ..., v_{23}\}$$

$$V^1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}\} \mapsto Bus$$

$$V^2 = \{v_1, v_2, v_3, v_6, v_7, v_8, v_9, v_{11}, v_{12}\} \mapsto Private\ car$$

$$V^3 = \{v_8, v_9, v_{10}, v_{11}, v_{12}\} \mapsto Metro$$

$$V^4 = \{v_{13}, v_{14}, v_{15}, v_{16}, v_{17}\} \mapsto Taxi$$

To summarize, the arcs are specified in their related sub-graphs. Here, it is assumed that the sub-graph recognition process has been accomplished.

$$E_{SG_1^1} = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_2, v_5), (v_3, v_4), (v_3, v_6), (v_4, v_2), (v_4, v_5) \\ , (v_4, v_6), (v_5, v_4), (v_5, v_6)\}$$

$$E_{SG_2^1} = \{(v_{18}, v_{19}), (v_{18}, v_{22}), (v_{19}, v_{20}), (v_{20}, v_{21}), (v_{21}, v_{23}), (v_{22}, v_{23})\}$$

$$E_{SG_1^2} = \{(v_1, v_2), (v_1, v_3), (v_1, v_6), (v_2, v_6), (v_3, v_7), (v_7, v_6)\}$$

$$E_{SG_2^2} = \{(v_8, v_9), (v_8, v_{12}), (v_9, v_{11}), (v_{11}, v_{12}), (v_{12}, v_8)\}$$

$$E_{SG_1^3} = \{(v_8, v_9), (v_8, v_{10}), (v_9, v_8), (v_9, v_{10}), (v_9, v_{11}), (v_{10}, v_{11}), (v_{10}, v_{12}) \\ , (v_{11}, v_9), (v_{11}, v_{12})\}$$

$$E_{SG_1^4} = \{(v_{13}, v_{14}), (v_{13}, v_{17}), (v_{14}, v_{13}), (v_{14}, v_{15}), (v_{15}, v_{16}), (v_{15}, v_{17}), (v_{16}, v_{17}), (v_{17}, v_{13})\}$$

$$E^T = \{(v_2^1, v_2^2), (v_2^2, v_2^1), (v_6^1, v_{14}^4), (v_{14}^4, v_6^1), (v_6^1, v_{11}^3), (v_6^2, v_{14}^4), (v_{14}^4, v_6^2), (v_6^2, v_{11}^3), (v_9^2, v_9^3), (v_9^3, v_9^2), (v_8^3, v_3^1), (v_8^3, v_3^2), (v_{12}^3, v_{18}^1), (v_{12}^2, v_{18}^1), (v_{16}^4, v_{18}^1), (v_{19}^1, v_{17}^4)\}$$

A closer look at the elements of E^T indicates the concordance of the model with the real world. For example, arcs such as (v_2^1, v_2^2) and their corresponding costs represent the cost spent on the mode change for the case in which the stations are located in the same place, such as walking from a metro to a bus station that is located in the same place. One-way and two-way arcs with different costs in each direction are other samples of concordances. The function Φ images the arcs to the fuzzy costs. To summarize, only the weights of the sub-graph SG_1^1 and the transfer arcs are listed below:

SG_1^1

$\Phi(v_1, v_2) = \{.1/1, .2/2, .3/3\}$ $\quad$ $\Phi(v_1, v_3) = \{.2/1, .2/2, .4/3\}$
$\Phi(v_2, v_4) = \{.3/1, .2/2, .2/3\}$ $\quad$ $\Phi(v_2, v_5) = \{.3/1, .6/2, .4/3\}$
$\Phi(v_3, v_4) = \{.3/1, .1/2, .1/3\}$ $\quad$ $\Phi(v_3, v_6) = \{.4/1, .3/2, .3/3\}$
$\Phi(v_4, v_2) = \{.4/1, .4/2, .2/3\}$ $\quad$ $\Phi(v_4, v_5) = \{.7/1, .1/2, .3/3\}$
$\Phi(v_4, v_6) = \{.1/1, .1/2, .2/3\}$ $\quad$ $\Phi(v_5, v_4) = \{.4/1, .4/2, .4/3\}$
$\Phi(v_5, v_6) = \{.5/1, .5/2, .4/3\}$

E^T

$\Phi(v_2^1, v_2^2) = \{.3/3, .1/4, .5/5\}$ $\quad$ $\Phi(v_2^2, v_2^1) = \{.4/3, .2/4, .6/5\}$
$\Phi(v_6^1, v_{14}^4) = \{.1/5, .3/6, .4/7\}$ $\quad$ $\Phi(v_{14}^4, v_6^1) = \{.1/5, .3/6, .4/7\}$
$\Phi(v_6^1, v_{11}^3) = \{.5/5, .3/6, .6/7\}$ $\quad$ $\Phi(v_6^2, v_{14}^4) = \{.1/5, .3/6, .4/7\}$
$\Phi(v_{14}^4, v_6^2) = \{.1/5, .3/6, .4/7\}$ $\quad$ $\Phi(v_6^2, v_{11}^3) = \{.5/5, .3/6, .6/7\}$
$\Phi(v_9^2, v_9^3) = \{.5/3, .2/4, .6/5\}$ $\quad$ $\Phi(v_9^3, v_9^2) = \{.5/3, .2/4, .6/5\}$
$\Phi(v_8^3, v_3^1) = \{.4/6, .5/7, .6/8\}$ $\quad$ $\Phi(v_8^3, v_3^2) = \{.4/6, .5/7, .6/8\}$
$\Phi(v_{12}^3, v_{18}^1) = \{.4/8, .5/9, .5/10\}$ $\quad$ $\Phi(v_{12}^2, v_{18}^1) = \{.5/5, .4/6, .7/7\}$
$\Phi(v_{16}^4, v_{18}^1) = \{.1/6, .2/7, .3/8\}$ $\quad$ $\Phi(v_{19}^1, v_{17}^4) = \{.3/6, .5/7, .5/8\}$

Figure 3 shows the network. Each mode is determined by its color, with some modes composed of two sub-graphs. It should be noted that the lengths of the arcs are independent of the associated costs.

Based on the definitions, the sets of incoming and outgoing boundary nodes are listed in Table 1.

Algorithm 2.1 is then applied to provide the reduced sub-graphs. The results of the algorithm are the shortest paths from IN.BNs to OUT.BNs of each sub-graph. Table 2 shows the derived shortest paths from IN.BNs to OUT.BNs, after the convergence. Note that the paths in each cell are arranged according to the number of arcs. To summarize, only the results in sub-graph SG_1^1 (except $v_3 \to v_6$) are listed.

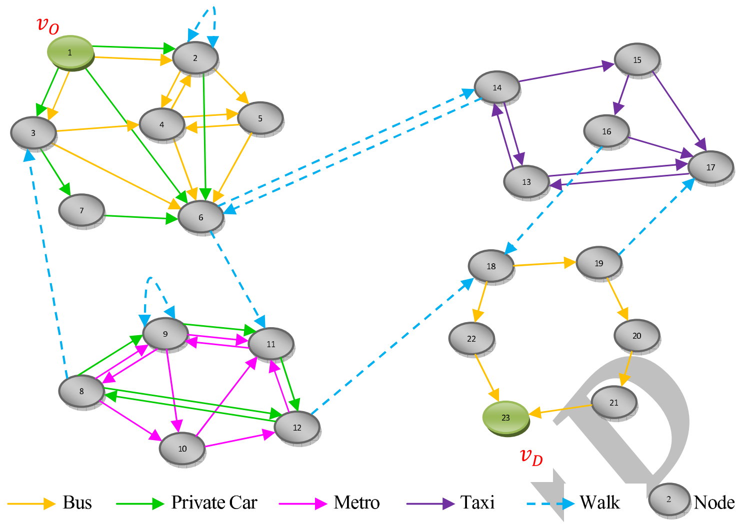

FIGURE 3. A Pseudo Multimodal Network

Sub-Graph	Incoming Boundary Nodes	Outgoing Boundary Nodes
SG_1^1	$\{v_O, v_2, v_3\}$	$\{v_2, v_6\}$
SG_2^1	$\{v_{18}\}$	$\{v_D, v_{19}\}$
SG_1^2	$\{v_O, v_2, v_3\}$	$\{v_2, v_6\}$
SG_2^2	$\{v_9\}$	$\{v_{12}\}$
SG_1^3	$\{v_9, v_{11}\}$	$\{v_8, v_9, v_{12}\}$
SG_1^4	$\{v_{14}, v_{17}\}$	$\{v_{14}, v_{16}\}$

TABLE 1. Boundary Nodes

The output of the algorithm covers all possible paths that are listed under the title of "All Corresponding Paths" in Table 2. The algorithm may result in several paths having similar costs from an origin to a destination. Some of them like $v_2 \to v_4 \to v_2$ repeat their destinations. From a fuzzy point of view such paths are valid; however, in practice the paths are ignored due to the repetition. Hence "Filter Paths" column of Table 2 indicates all logical choices that a passenger can consider.

There are several noteworthy points about paths such as $v_2 \to v_2$ and their related costs $\{1/0, .3/2, .3/3\}$; the term $1/0$ represents the ordinary cost of the path from v_2 to v_2. Other terms of the fuzzy number $(.3/2, .3/3)$ originate from the path $v_2 \to v_4 \to v_2$. In addition, there is no $*/1$-like term in the fuzzy cost, since a cost of 1 is impossible along the two above-mentioned paths. Although this fuzzy number is not absolutely zero ($\{1/0\}$), it plays exactly the same role as zero in the aggregation of costs, due to the definition of the operator $\otimes$. The reduced

Sub-Graph	IN. BNs	OUT. BNs	Cost	All Corresponding Paths	Filtered Paths
SG_1^1	v_O	v_2	$\{.1/1, .2/2, .3/3\}$	$1 \to 2$	$1 \to 2$
		v_6	$\{.2/2, .2/3, .4/4\}$	$1 \to 3 \to 6$	$1 \to 3 \to 6$
	v_2	v_2	$\{1/0, .3/2, .3/3\}$	$\begin{pmatrix} 2 \\ 2 \to 4 \end{pmatrix} \to 2$	$2 \to 2$
		v_6	$\{.3/2, .5/3, .5/4\}$	$\begin{pmatrix} 2 \\ 2 \to 4 \to 2 \end{pmatrix} \to 5 \to 6$	$2 \to 5 \to 6$
	v_3	v_2	$\{.3/2, .3/3, .3/4\}$	$\begin{pmatrix} 3 \\ 3 \to 4 \to 2 \\ \begin{pmatrix} 3 \to 4 \\ 3 \to 4 \to 2 \end{pmatrix} \to 5 \end{pmatrix} \to 4 \to 2$	$3 \to 4 \to 2$

Table 2. Shortest Fuzzy Paths in Sub-graphs

G is shown in Figure 4. Each color presents a mode as in Figure 3 and each circle belongs to a sub-graph.

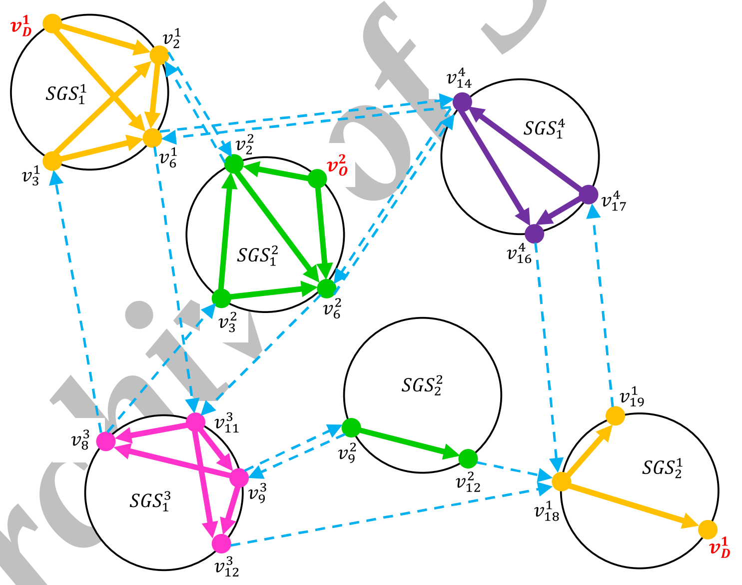

Figure 4. The Reduced G

Algorithm 2.1 is applied to the reduced G. The results are shown in Table 3.

The full paths are obtained by insertion of the internal shortest paths from Table 2 into the highlighted paths in Table 3:

$$\begin{pmatrix} v_O^2 \\ v_O^2 \to v_3^2 \to v_7^2 \end{pmatrix} \to v_6^2 \to \begin{pmatrix} v_{11}^3 \to v_{12}^3 \\ v_{14}^4 \to v_{15}^4 \to v_{16}^4 \end{pmatrix} \to v_{18}^1 \to v_{22}^1 \to v_D^1$$

Origin	$\mathbf{v_p^i}$	**Fuzzy Cost**	**Final Paths**
	v_O^1	e	$v_O^1 \to v_O^1$
	v_2^1	$\{.1/1, .2/2, .3/3\}$	$v_O^1 \to v_2^1$
	v_3^1	$\{.4/14, .4/15, .4/16\}$	$\begin{pmatrix} v_O^2 \to v_6^2 \to v_{11}^3 \\ v_O^2 \to v_6^2 \to v_{11}^3 \to v_9^3 \end{pmatrix} \to v_8^3 \to v_3^1$
	v_6^1	$\{.2/2, .2/3, .2/4\}$	$v_O^1 \to v_6^1$
	v_{18}^1	$\{.1/14, .4/15, .5/16\}$	$\begin{pmatrix} v_O^2 \to v_6^2 \to v_{14}^4 \to v_{16}^4 \\ v_O^2 \to v_6^2 \to v_{11}^3 \to v_{12}^3 \end{pmatrix} \to v_{18}^1$
	v_{19}^1	$\{.1/15, .2/16, .3/17\}$	$\begin{pmatrix} v_O^2 \to v_6^2 \to v_{14}^4 \to v_{16}^4 \\ v_O^2 \to v_6^2 \to v_{11}^3 \to v_{12}^3 \end{pmatrix} \to v_{18}^1 \to v_{19}^1$
	v_D^1	$\{.1/16, .2/17, .3/18\}$	$\begin{pmatrix} v_O^2 \to v_6^2 \to v_{14}^4 \to v_{16}^4 \\ v_O^2 \to v_6^2 \to v_{11}^3 \to v_{12}^3 \end{pmatrix} \to v_{18}^1 \to v_D^1$
v_O	v_O^2	e	$v_O^2 \to v_O^2$
	v_2^2	$\{.3/1, .2/2, .1/3\}$	$v_O^2 \to v_2^2$
	v_3^2	$\{.4/14, .4/15, .4/16\}$	$\begin{pmatrix} v_O^2 \to v_6^2 \to v_{11}^3 \\ v_O^2 \to v_6^2 \to v_{11}^3 \to v_9^3 \end{pmatrix} \to v_8^3 \to v_3^2$
	v_6^2	$\{.5/1, .7/2, .4/3\}$	$v_O^2 \to v_6^2$
	v_9^2	$\{.4/10, .5/11, .5/12\}$	$v_O^2 \to v_6^2 \to v_{11}^3 \to v_9^3 \to v_9^2$
	v_{12}^2	$\{.3/12, .4/13, .4/14\}$	$v_O^2 \to v_6^2 \to v_{11}^3 \to v_9^3 \to v_9^2 \to v_{12}^2 \quad *$
	v_8^3	$\{.4/8, .4/9, .5/10\}$	$\begin{pmatrix} v_O^2 \to v_6^2 \to v_{11}^3 \\ v_O^2 \to v_6^2 \to v_{11}^3 \to v_9^3 \end{pmatrix} \to v_8^3$
	v_9^3	$\{.4/7, .5/8, .5/9\}$	$v_O^2 \to v_6^2 \to v_{11}^3 \to v_9^3$
	v_{11}^3	$\{.5/6, .5/7, .5/8\}$	$v_O^2 \to v_6^2 \to v_{11}^3$
	v_{12}^3	$\{.5/7, .5/8, .5/9\}$	$v_O^2 \to v_6^2 \to v_{11}^3 \to v_{12}^3$
	v_{14}^4	$\{.1/6, .3/7, .4/8\}$	$v_O^2 \to v_6^2 \to v_{14}^4$
	v_{16}^4	$\{.1/8, .2/9, .2/10\}$	$v_O^2 \to v_6^2 \to v_{14}^4 \to v_{16}^4$
	v_{17}^4	$\{.1/21, .2/22, .3/23\}$	$\begin{pmatrix} v_O^2 \to v_6^2 \to v_{14}^4 \to v_{16}^4 \\ v_O^2 \to v_6^2 \to v_{11}^3 \to v_{12}^3 \end{pmatrix} \to v_{18}^1 \to v_{19}^1 \to v_{17}^4$

Table 3. Best Paths of the Entire Network

It can be seen that there are at least two mode changes in each obtained path. The path marked by $*$ is not a viable path due to the incorrect order of modes used.

5. Conclusions and Future Work

Modern transportation services, such as those that include monorail, metro, and high-speed trains, imply very high running costs and are economically justified only under certain circumstances, such as efficiency of utility. The present study was aimed at supporting decision-making with respect to inexact quantities like "heavy" and "light" traffic, which are usually available and free. The fuzzy theory and related methods were applied in order to model these quantities. The proposed algorithm is based on fuzzy path algebra and the dioid of k-shortest fuzzy paths. Any type of fuzzy value can serve as algorithm input. The algorithm determines the shortest paths according to both the defined cost and multimodal concerns, such as the number of mode changes and the correct order of the modes used. There is no restriction on the number of modes that can be modeled. Two-way paths, even those with different costs in each direction, can be considered as well. Future work will focus on the problem under multicriteria conditions. Use of higher-level fuzzy numbers will provide users with more flexible guidance. Negative fuzzy costs can be modeled through intuitionistic fuzzy sets. Also, the proposed approach can be deployed in the context of client/server architecture, servicing users based on their location (location-based services, LBS).

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