OPTIMIZATION OF LINEAR OBJECTIVE FUNCTION SUBJECT TO FUZZY RELATION INEQUALITIES CONSTRAINTS WITH MAX-PRODUCT COMPOSITION

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ABSTRACT. In this paper, we study the finitely many constraints of the fuzzy relation inequality problem and optimize the linear objective function on the region defined by the fuzzy max-product operator. Simplification operations have been given to accelerate the resolution of the problem by removing the components having no effect on the solution process. Also, an algorithm and some numerical and applied examples are presented to abbreviate and illustrate the steps of the problem resolution.

1. **Introduction**

Fuzzy relation equations (FRE), fuzzy relation inequalities (FRI) and their connected problems have been investigated by many researchers in both theoretical and applied areas [1, 5, 6, 7, 9, 12, 14, 19, 29, 32, 33, 34, 36, 37, 38, 40, 43, 45]. Sanchez [35] started a development of the theory and applications of FRE treated as a formalized model for non-precise concepts. Generally, FRE and FRI have a number of properties that make them suitable for formulating the uncertain information upon which many applied concepts are usually based. The application of (FRE) and (FRI) can be seen in many areas, for instance, fuzzy control, fuzzy decision making, systems analysis, fuzzy modeling, fuzzy arithmetic, fuzzy symptom diagnosis, and especially fuzzy medical diagnosis and so on (see [2, 3, 4, 8, 9, 10, 11, 25, 27, 28, 29, 30, 31, 39, 42]).

An interesting extensively investigated kind of such problems is the optimization of the objective functions on the region whose sets of feasible solutions have been defined as FRE or FRI constraints [4, 11, 15, 17, 19, 23, 24]. Fang and Li solved the linear optimization problem with respect to the FRE constraints by considering the max-min composition [11]. The max-min composition is commonly used when a system requires conservative solutions in the sense that the goodness of one value cannot compensate for the badness of another value [23]. Recent results in the literature, however, show that the min operator is not always the best choice for the intersection operation. Instead, the max-product composition has provided results better than or equivalent to the max-min composition in some applications [2].

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The fundamental result for fuzzy relation equations with max-product composition goes back to Pedrycz [28]. A recent study in this regard can be found in Bourk and Fisher [4]. They extended the study of an inverse solution of a system of fuzzy relation equations with max-product composition. They provided theoretical results for determining the complete sets of solutions, as well as the conditions for the existence of resolutions. Their results showed that such complete sets of solutions can be characterized by one maximum solution and a number of minimal solutions. Furthermore, the monograph by Di Nola, Sessa, Pedrycz and Sanchez [9] contains a thorough discussion of this class of equations. A problem of optimization was studied by Loetamonphong and Fang with max-product composition [23], which was improved by Guu and Wu by shrinking the search region [17]. The linear objective optimization problem with FRI was investigated by Zhang et al. [44], where the fuzzy operator is considered as the max-min composition. Also, Guo and Xia presented an algorithm to accelerate the resolution of this problem [15].

A system of fuzzy relation equations, namlely $A \bullet x = b$, has considered by Perfilieva [32]. In this system "$\bullet$" is max-$*$ composition, and "$*$" denotes a continues t-norm, which in special case it can be considered as max -product composition, and also in this paper solvability of this system is discussed by introducing fixed point of the shrivel operator. A system of fuzzy relation equations $A \rhd x = b$ has studied by Perfilieva [33], where " $\rhd$ " denotes inf (min) composition. However, inf (min)$\rightarrow$composition is completely different from max-product composition. Also, a kind of optimization problem as

$$\begin{aligned} & minimum && c^t * x \\ & subject\ to && \\ & && A * x \geqslant b \end{aligned}$$

has considered by Hosseinyazi [20], where " $*$ " denotes max-min composition. The objective function is $c^t * x$ which is mix-min composition, where x is a feasible solution point of $A * x \geqslant b$. The feasible solution itself is obtained by max-min composition.

Here, the constraints of our problem are more general than one in [32]; moreover, a necessary and sufficient condition is presented for solvability of the constraints (see corollarly 5). In this paper, we generalize the linear optimization problem of the FRE with the max-product operator [23] by considering fuzzy relation inequalities instead of the equations in the constraints. This problem can be formulated as follows:

$$\begin{aligned} & minimum && c^t x \\ & subject\ to && \\ & && A \bullet x \geqslant d^1 \\ & && B \bullet x \leqslant d^2 \\ & && x \in [0,1]^n \end{aligned} \tag{1}$$

where $A = (a_{ij})_{m\times n}$, $a_{ij} \in [0,1]$, $B = (b_{ij})_{l\times n}$, $b_{ij} \in [0,1]$ are fuzzy matrices,$d^1 = (d_i^1)_{m\times 1} \in [0,1]^m$, $d^2 = (d_i^2)_{l\times 1} \in [0,1]^l$ are fuzzy vectors, $c = (c_j)_{n\times 1} \in \mathbb{R}^n$ is the vector of cost coefficients, and $x = (x_j)_{n\times 1} \in [0,1]^n$ is an unknown vector and "$\bullet$" denotes the fuzzy max-product operator as defined below and also the objective function is $c^t x = c_1.x_1 + c_2.x_2 + ... + c_n.x_n$, where the operations "." and "+" denote the ordinary multiplication and addition, respectively. Problem (1) can be rewritten as the following problem in detail;

$$\begin{aligned} &min \quad c^t x \\ &s.\ t \\ &\qquad a_i \bullet x \geqslant d_i^1 \quad i \in I^1 = \{1,2,3,...,m\} \\ &\qquad b_i \bullet x \leqslant d_i^2 \quad i \in I^2 = \{1,2,3,...,l\} \\ &\qquad 0 \leqslant x_j \leqslant 1 \quad j \in J = \{1,2,3,...,n\} \end{aligned} \tag{2}$$

where a_i and b_i are the ith row of matrices A and B, respectively and the constraints are expressed by the max-product operator definition as:

$$\begin{aligned} a_i \bullet x = max_{j\in J}\{a_{ij} \cdot x_j\} \geqslant d_i^1 \quad \forall i \in I^1 \\ b_i \bullet x = max_{j\in J}\{b_{ij} \cdot x_j\} \leqslant d_i^2 \quad \forall i \in I^2 \end{aligned} \tag{3}$$

However, it is tried to explain motivation behind using max- product composition in the introduction and Example 3.

In section 2, the set of the feasible solutions of problem (2) and its properties are studied. A necessary and a sufficient conditions are given to realize the feasibility of problem (2). In section 3, some simplification operations are presented to accelerate the resolution process. Also, in section 4 an algorithm is introduced to solve the problem by using the results of the previous sections, and three numerical and applied examples are given to illustrate the algorithm in this section. Finally, the conclusion is stated in section 5.

2. The Characteristics of the Set of Feasible Solutions

We shall use the following notations during this paper

$$\forall i \in I^1 : S(A,d^1)_i = \{x \in [0,1]^n : a_i \bullet x \geqslant d_i^1\}$$
$$\forall i \in I^2 : S(B,d^2)_i = \{x \in [0,1]^n : b_i \bullet x \leqslant d_i^2\}$$
$$S(A,d^1) = \bigcap_{i\in I^1} S(A,d^1)_i = \{x \in [0,1]^n : A \bullet x \geqslant d^1\}$$
$$S(B,d^2) = \bigcap_{i\in I^2} S(B,d^2)_i = \{x \in [0,1]^n : B \bullet x \leqslant d^2\}$$
$$S(A,B,d^1,d^2) = S(A,d^1) \cap S(B,d^2) = \{x \in [0,1]^n : A \bullet x \geqslant d^1, B \bullet x \leqslant d^2\}$$

Corollary 2.1. *$x \in S(A,d^1)_i$ for every $i \in I^1$ if and only if there exists some $j_i \in J$ such that $x_{j_i} \geqslant \frac{d_i^1}{a_{ij_i}}$, similarly, $x \in S(B,d^2)_i$ for every $i \in I^2$ if and only if $x_j \leqslant \frac{d_i^2}{b_{ij}}$, $\forall j \in J$.*

Proof. This clearly results from relations (3). □

Lemma 2.2. *(a) $S(A,d^1) \neq \phi$ if and only if for every $i \in I^1$ there exists $j_i \in J$ such that $a_{ij_i} \geqslant d_i^1$. (b) If $S(A,d^1) \neq \phi$ then $\bar{1} = [1,1,...,1]_{1\times n}^t$ is the greatest element in the set $S(A,d^1)$.*

Proof. (a) Suppose $S(A,d^1) \neq \phi$ and $x \in S(A,d^1)$. Thus,$x \in S(A,d^1)_i$, $\forall i \in I^1$ and then by Corollary 2.1 for every $i \in I^1$ we have $x_{j_i} \geqslant \frac{d_i^1}{a_{ij_i}}$ for some $j_i \in J$. Therefore, since $x \in S(A,d^1)$, $x \in [0,1]^n$ and thus $\frac{d_i^1}{a_{ij_i}} \leqslant 1$, $\forall i \in I^1$, which implies that there exists $j_i \in J$ such that $a_{ij_i} \geqslant d_i^1$. Conversely, suppose that there exists some $j_i \in J$ such that $a_{ij_i} \geqslant d_i^1$, $\forall i \in I^1$. Set $\bar{1} = [1,1,...,1]_{1\times n}^t$, since $x \in [0,1]^n$ and $x_{j_i} = 1 \geqslant \frac{d_i^1}{a_{ij_i}}$,$\forall i \in I^1$, by Corollary 2.1 $x \in S(A,d^1)_i$, $\forall i \in I^1$, and therefore $x \in S(A,d^1)$. (b) Proof is attained from part (a) and Corollary 2.1. □

Lemma 2.3. *(a) $S(B,d^2) \neq \phi$. (b) The smallest element in set $S(B,d^2)$ is $\bar{0} = [0,0,...,0]_{1\times n}^t$.*

Proof. Set $x = \bar{0} = [0,0,...,0]_{1\times n}^t$. Since $d_i^2 \geqslant 0$ and $b_{ij} \geqslant 0$(if $b_i = 0$, then it is well defined and is clear), $\frac{d_i^2}{b_{ij_i}} \geqslant 0$. Therefore $x_j \leqslant \frac{d_i^2}{b_{ij_i}}$, $\forall i \in I^2$, $\forall j \in J$, thus Corollary 1 implies that $x \in S(B,d^2)$ and so parts (a) and (b) are proved. □

Theorem 2.4. *(Necessary condition) If $S(A,B,d^1,d^2) \neq \phi$, then for every $i \in I^1$ there exists $j \in J$ such that $a_{ij} \geqslant d_i^1$.*

Proof. Suppose that $S(A,B,d^1,d^2) \neq \phi$. Since $S(A,B,d^1,d^2) = S(A,d^1)\cap S(B,d^2)$, $S(A,d^1) \neq \phi$, and so it is proved by using part (a) of Lemma 2.2. □

Definition 2.5. Set $\bar{x} = (\bar{x}_j)_{n\times 1}$ where

$$\bar{x}_j = \begin{cases} 1, & \forall i : b_{ij} \leqslant d_i^2, \\ min_{i=1,2,...,l}\{\frac{d_i^2}{b_{ij}} : b_{ij} \geqslant d_i^2\}, & otherwise. \end{cases}$$

Lemma 2.6. *$\bar{x}$, is the greatest element in the set $S(B,d^2)$.*

Proof. See [23] page 348. □

Corollary 2.7. *$S(B,d^2) = \{x \in [0,1]^n : B \bullet x \leqslant d^2\} = [\bar{0},\bar{x}]$, where, $\bar{x}$ and $\bar{0}$ are defined as in Definition 2.5 and Lemma 2.3, respectively.*

Proof. Since $S(B,d^2) \neq \phi$ by Lemmas 2.3 and 2.6. $\bar{0}$ and $\bar{x}$ are the smallest and the greatest elements, respectively, Let $x \in [\bar{0},\bar{x}]$, then $x \in [0,1]^n$ and $x \leqslant \bar{x}$, thus, $b_i \bullet x \leqslant b_i \bullet \bar{x} \leqslant d_i^2$, $\forall i \in I^2$, which implies $x \in S(B,d^2)$. Conversely, let $x \in S(B,d^2)$ by part (b) of Lemma 2.3, $\bar{0} \leqslant \bar{x}$ and also $x \in S(B,d^2)_i$, $\forall i \in I^2$. Then, Corollary 2.1 requires $x_j \leqslant \frac{d_i^2}{b_{ij}}$, $\forall i \in I^2$ and $\forall j \in J$. Hence, $x_j \leqslant \bar{x}_j$, $\forall j \in J$, which means that $x \leqslant \bar{x}$. Thus, $x \in [\bar{0},\bar{x}]$. □

Definition 2.8. Let $J_i = \{j \in J : a_{ij} \geqslant d_i^1\}$, $\forall i \in I^1$. For every $j \in J_i$, we define $i_{x(j)} = (i_{x(j)_k})_{n\times 1}$ such that

$$i_{x(j)_k} = \begin{cases} \frac{d_i^1}{a_{ij}}, & k = j, \\ 0, & k \neq j \end{cases}$$

Lemma 2.9. *Consider a fixed* $i \in I^1$. *(a) If* $d_i^1 \neq 0$ *then the vectors* $i_{x(j)}$ *are the only minimal elements of* $S(A,d^1)_i$ *for every* $j \in J_i$. *(b) If* $d_i^1 = 0$ *then* $\bar{0}$ *is the smallest element in* $S(A,d^1)_i$.

Proof. (a) Suppose $j \in J_i$ and $i \in I^1$. Since $i_{x(j)_j} = \frac{d_i^1}{a_{ij}}$, by Corollary 2.2. $i_{x(j)} \in S(A,d^1)_i$, suppose by contradiction that $x \in S(A,d^1)_i$ and $x \leqslant i_{x(j)}$. Hence we must have $x_j \leqslant \frac{d_i^1}{a_{ij}}$ and $x_k = 0$ for $k \in J$ and $k \neq j$. Thus $x_j \leqslant \frac{d_i^1}{a_{ij}}$, $\forall j \in J$, and then by Corollary 2.1, $x \notin S(A,d^1)_i$ which is a contradiction. (b) It is clear from Corollary 2.1 and the fact that $x_j \geqslant 0$, $\forall j \in J$. □

Corollary 2.10. *If* $S(A,d^1)_i \neq \phi$, *then* $S(A,d^1)_i = \{x \in [0,1]^n : a_i \bullet x \geqslant d_i^1\} = \bigcup_{j \in J_i}[i_{x(j)}, \bar{1}]$, *where* $i \in I^1$ *and* $i_{x(j)}$ *are as defined in Definition 2.8.*

Proof. If $S(A,d^1)_i \neq \phi$, then from Lemmas 2.2 and 2.9, vector $\bar{1}$ is the maximum solution and the vectors $i_{x(j)}$, $\forall j \in J_i$ are the minimal solutions in $S(A,d^1)_i$. Let $x \in \bigcup_{j \in J_i}[i_{x(j)}, \bar{1}]$. Then $x \in [i_{x(j)}, \bar{1}]$ for some $j \in J_i$ and therefore $x \in [0,1]^n$ and $x_j \geqslant i_{x(j)_j} = \frac{d_i^1}{a_{ij}}$ from Definition 2.8, hence $x \in S(A,d^1)_i$ from Corollary 2.1. Conversely, let $x \in S(A,d^1)_i$. Then there exits $j\prime \in J$ such that $x_{j\prime} \geqslant \frac{d_i^1}{a_{ij\prime}}$ from Corollary 2.1. Since $x \in [0,1]^n$, $\frac{d_i^1}{a_{ij\prime}} \leqslant 1$, and so $j\prime \in J_i$. Thus, $i_{x(j)} \leqslant x \leqslant \bar{1}$, which implies $x \in \bigcup_{j \in J_i}[i_{x(j)}, \bar{1}]$. □

Definition 2.11. Let $e = (e(1), e(2), ..., e(m)) \in J_1 \times J_2 \times ... \times J_m$ such that $e(i) = j \in J_i$. We define $x(e) = (x(e)_j)_{n \times 1}$, where $x(e)_j = max_{i \in I_j^e}\{i_{x(e(i))_j}\} = max_{i \in I_j^e}\{\frac{d_i^1}{a_{ij}}\}$ if $I_j^e \neq \phi$ and $x(e)_j = 0$ if $I_j^e = \phi$, where $I_j^e = \{i \in I^1 : e(i) = j\}$.

Corollary 2.12. *(a) If* $d_i^1 = 0$ *for some* $i \in I^1$, *then we can remove the* i*th row of the matrix* A *with no effect on the calculation of the vectors* $x(e)$ *for each* $e \in J_I = J_1 \times J_2 \times ... \times J_m$. *(b) If* $j \notin J_i$, $\forall i \in I^1$, *then we can remove the* j*th column of the matrix* A *before calculating the vectors* $x(e)$, $\forall e \in J_I$, *and set* $x(e)_j = 0$ *for each* $e \in J_I$.

Proof. (a) It is proved by Definition 2.11 and part (b) of Lemma 2.9, because we will get the minimal elements of $S(A,d^1)$. (b) It is proved by using Definition 2.11. □

Lemma 2.13. *Suppose* $S(A,d^1) \neq \phi$, *then* $S(A,d^1) = \bigcup_{X(e)}[x(e), \bar{1}]$, *where* $X(e) = \{x(e) : e \in J_I\}$.

Proof. If $S(A,d^1) \neq \phi$, then $S(A,d^1)_i \neq \phi$, $\forall i \in I^1$. Thus by Corollary 2.10 and Definition 2.11, we have

$$S(A,d^1) = \bigcap_{i \in I^1} S(A,d^1)_i = \bigcap_{i \in I^1}[\bigcup_{j \in J_i}[i_{x(j)}, \bar{1}]] = \bigcap_{i \in I^1}[\bigcup_{e(i) \in J_i}[i_{x(e(i))}, \bar{1}]] =$$

$$\bigcup_{e \in J_I}[\bigcap_{i \in I^1}[i_{x(e(i))}, \bar{1}]] = \bigcup_{e \in J_I}[max_{i \in I^1}\{i_{x(e(i))}\}, \bar{1}] = \bigcup_{e \in J_I}[x(e), \bar{1}] = \bigcup_{X(e)}[x(e), \bar{1}].$$

□

By Lemma 2.13, it is obvious that $S(A,d^1)=\bigcup_{X_0(e)}[x(e),\bar{1}]$ and $X_0(e)=\{x: x\in S_0(A,d^1)\ \textit{and}\ \ x\leqslant\bar{x}\}$, where $S_0(A,d^1)$ are the sets of minimal solutions in $X(e)$ and $S(A,d^1)$, respectively.

Theorem 2.14. *If $S(A,B,d^1,d^2)\neq\phi$, then $S(A,B,d^1,d^2)=\bigcup_{X_0(e)}[x(e),\bar{x}]$.*

Proof. By using Corollary 2.7 and the result of Lemma 2.13, we have

$$S(A,B,d^1,d^2)=S(A,d^1)\bigcap S(B,d^2)=\{\bigcup_{X_0(e)}[x(e),\bar{1}]\}\bigcap[\bar{0},\bar{x}]=\bigcup_{X_0(e)}[x(e),\bar{x}]$$

and the proof is complete. □

Corollary 2.15. *(Necessary and sufficient condition) $S(A,B,d^1,d^2)\neq\phi$ if and only if $\bar{x}\in S(A,d^1)$. Equivalently, $S(A,B,d^1,d^2)\neq\phi$ if and only if there exists $e\in J_I$ such that $x(e)\leqslant\bar{x}$.*

Proof. Suppose that $S(A,B,d^1,d^2)\neq\phi$, then $S(A,B,d^1,d^2)=\bigcup_{X_0(e)}[x(e),\bar{x}]$ by Theorem 2.14, thus $\bar{x}\in S(A,B,d^1,d^2)$, and hence $\bar{x}\in S(A,d^1)$. Conversely, let $\bar{x}\in S(A,d^1)$. We know that $\bar{x}\in S(B,d^2)$, thus $x\in S(A,d^1)\bigcap S(B,d^2)=S(A,B,d^1,d^2)$. □

3. Simplification Operations and the Resolution Algorithm

In order to prove the problem (2), we first convert it into the two sub-problems below:

$$\begin{array}{ll} \textit{minimum} & {c^+}^t x \\ \textit{subject to} & \\ & A\bullet x\geqslant d^1 \\ & B\bullet x\leqslant d^2 \\ & x\in[0,1]^n \end{array} \tag{4a}$$

$$\begin{array}{ll} \textit{minimum} & {c^-}^t x \\ \textit{subject to} & \\ & A\bullet x\geqslant d^1 \\ & B\bullet x\leqslant d^2 \\ & x\in[0,1]^n \end{array} \tag{4b}$$

where

$$c_j^+=\begin{cases} c_j, & c_j>0, \\ 1, & c_j\leqslant 0. \end{cases} \qquad \textit{and} \qquad c_j^-=\begin{cases} c_j, & c_j<0, \\ 1, & c_j\geqslant 0. \end{cases}$$

By Theorem 2.14, it is obvious that $\bar{x}$ is the optimal solution of problem (4b). Also, problem (4a) achieves its optimal points at some $x(e)\in X_0(e)$. If $x(e_0)$ optimizes problem (4a), then we set $x^*=(x_j^*)_{n\times 1}$ such that

$$x_j^*=\begin{cases} \bar{x}_j, & c_j<0, \\ x(e_0), & c_j\geqslant 0. \end{cases}$$

Now the following lemma in below, gives an optimal point of problem (2).

Lemma 3.1. *If $x(e_0) \in X_0(e)$ optimizes problem (4a), then x^* defined as above, is an optimal solution of the problem (2). (It is possible to find many optimal solutions for the problem (4a) and so for the problem (2), too.)*

Proof. Assume that $S(A, B, d^1, d^2) \neq \phi$. Also, suppose that $x(e_0)$ and $\bar{x}$ are the optimal solutions of (4a) and (4b), respectively. Then, for any x satisfying inequalities $A \bullet x \geqslant d^1$ and $B \bullet x \leqslant d^2$ ($x \in S(A, B, d^1, d^2)$) we have

$$\begin{aligned}\sum_{j=1}^{n} c_j x_j^* &= \sum_{j=1}^{n} c_j^- \bar{x}_j + \sum_{j=1}^{n} c_j^+ x(e_0)_j \\ &\leqslant \sum_{j=1}^{n} c_j^- x_j + \sum_{j=1}^{n} c_j^+ x_j \\ &= \sum_{j=1}^{n} c_j x_j\end{aligned}$$

Therefore, x^* is an optimal solution of problem (2), and so the proof is completed. □

For calculating x^* it is sufficient to find $\bar{x}$ and $x(e_0)$ from Lemma 3.1. While $\bar{x}$ is easily attained by Definition 2.5, $x(e_0)$ is usually hard to find X. Since $X_0(e)$ is attained by pairwise comparison between the members of set $X_(e)$, the finding process of set $X_0(e)$ is time-consuming if $X_(e)$ has many members. Therefore, a simplification operation can accelerate the resolution of problem (4a) by removing the vectors $e \in J_I$ such that $x(e)$ is not optimal in (4a). One of such operations is given by Corollary 2.12. Other operations are attained by the following theorems.

Theorem 3.2. *The set of feasible solutions for problem (1), namely $S(A, B, d^1, d^2)$, is nonempty if and only if for every $i \in I^1$ the set $\bar{J}_i = \{j \in J_i : \frac{d_i^1}{a_{ij}} \leqslant \bar{x}_j\}$ is nonempty, where $\bar{x}$ is defined by Definition 2.5.*

Proof. Suppose $S(A, B, d^1, d^2) \neq \phi$. By Corollary 2.15, $\bar{x} \in S(A, B, d^1, d^2)$ and so we have $\bar{x} \in S(A, d^1)_i$, $\forall i \in I^1$. Thus, by Corollary 2.1 for every $i \in I^1$ there exists $j \in J$ such that $\bar{x}_j \geqslant \frac{d_i^1}{a_{ij}}$, which means that $\bar{J}_i \neq \phi$, $\forall i \in I^1$. Conversely suppose $\bar{J}_i \neq \phi$, $\forall i \in I^1$. Then there exists $j \in J$ such that $\bar{x}_j \geqslant \frac{d_i^1}{a_{ij}}$, $\forall i \in I^1$. Hence, by Corollary 2.1 $\bar{x} \in S(A, d^1)_i$, $\forall i \in I^1$, which implies $\bar{x} \in S(A, d^1)$. These facts together with Lemma 2.6 imply $\bar{x} \in S(A, B, d^1, d^2)$, and therefore $S(A, B, d^1, d^2) \neq \phi$. □

Theorem 3.3. *If $S(A, B, d^1, d^2) \neq \phi$, then $S(A, B, d^1, d^2) = \bigcup_{\bar{X}(e)} [x(e), \bar{x}]$ where $\bar{X}(e) = \{x(e) : e \in \bar{J}_I = \bar{J}_1 \times \bar{J}_2 \times ... \times \bar{J}_m\}$.*

Proof. By Theorem 2.14, it is sufficient to show $x(e) \notin S(A, B, d^1, d^2)$ if $e \notin \bar{J}_I$. Suppose $e \notin \bar{J}_I$. Thus, there exist $i' \in I^1$ and $j' \in J_{i'}$ such that $e(i') = j'$ and $\bar{x}_{j'} < \frac{d_{i'}^1}{a_{i'j'}}$. Then $i' \in I_{j'}^e$ and by Definition 2.11 we have $x(e)_{j'} = max_{i \in I_{j'}^e} \{\frac{d_i^1}{a_{ij'}}\} \geqslant$

$\frac{d^1_{i'}}{a_{i'j'}} > \bar{x}_{j'}$. Therefore, $x(e) \leqslant \bar{x}$ is not correct, which by Theorem 2.14 implies that $x(e) \notin S(A, B, d^1, d^2)$. □

From the notation in Theorem 3.2, $\bar{J}_i \subseteq J_i, \forall i \in I^1$, which requires $\bar{X}(e) \subseteq X(e)$. Also, by Theorem 3.3, $S_0(A, B, d^1, d^2) \subseteq \bar{X}(e)$ where $S_0(A, B, d^1, d^2)$ is the set of minimal elements of $S(A, B, d^1, d^2)$, thus Theorem 3.3 reduces the search region to find set $S_0(A, B, d^1, d^2)$.

Definition 3.4. Let $J^*_i = \{j \in \bar{J}_i : c^-_j \neq 0\}, \forall i \in I^1$.

Theorem 3.5. *Suppose $x(e_0)$ is the optimal solution of (4a) and $J^*_{i'} \neq \phi$ for some $i' \in I^1$, then there exist $x(e')$ such that $e'(i') \in J^*_{i'}$, and also $x(e')$is the optimal solution of (4a).*

Proof. Suppose $J^*_{i'} \neq \phi$ for some $i' \in I^1$ and $e_0(i') = j'$. Define $e' \in \bar{J}_I$ such that $e'(i') = k \in J^*_{i'}$ and $e'(i) = e_0(i)$ for every $i \in I^1$ and $i \neq i'$. From Definition 2.11 we have:

$$x(e_0)_{j'} = max_{i \in I^{e_0}_{j'}} \{\frac{d^1_i}{a_{ij'}}\} \geqslant max_{i \in I^{e_0}_{j'},\ i \neq i'} \{\frac{d^1_i}{a_{ij'}}\} = x(e')_{j'}$$

Also, $x(e_0)_j = x(e')_j$ for every $j \in J$ and $j \neq j', k$. Thus, by noting that $c^+_k = 0$ we have

$$c^{+^t} x(e_0) = c^+_{j'} x(e_0)_{j'} + \sum_{j \in J,\ j \neq j'} c^+_j x(e_0)_j \geqslant c^+_{j'} x(e')_{j'} + \sum_{j \in J,\ j \neq j'} c^+_j x(e')_j = c^{+^t} x(e')$$

Therefore, $x(e')$ is the optimal solution of (4a), and so the proof is completed. □

Corollary 3.6. *If $J^*_i \neq \phi$ for some $i \in I^1$, then we can remove the ith row of matrix A without any effect on finding the optimal solution of problem (4a).*

Proof. The proof results from Theorem 3.5 and noting that $c^+_j = 0$ for every $j \in J^*_i$. □

Definition 3.7. Let $j_1, j_2 \in J$, $c_{j_1} \geqslant 0$ and $c_{j_2} \geqslant 0$. j_2 is said to be dominate j_1 if and only if
(a) $j_1 \in \bar{J}_i$ implies $j_2 \in \bar{J}_i, \forall i \in I^1$.
(b) for each $i \in I^1$, such that $j_1 \in \bar{J}_i$, we have $c_{j_1}(\frac{d^1_i}{a_{ij_1}}) \geqslant c_{j_2}(\frac{d^1_i}{a_{ij_2}})$.

Theorem 3.8. *Suppose $x(e_0)$ is the optimal solution of (4a) and j_2 dominates j_1 for $j_1, j_2 \in J$, then there exists $x(e')$ such that $I^{e'}_{j_1} = \phi$, and also $x(e')$ is the optimal solution of (4a).*

Proof. Define $e' = (e'(i))_{m \times 1}$ such that

$$e'(i) = \begin{cases} e_0(i), & i \notin I^{e_0}_{j_1}, \\ j_2, & i \in I^{e_0}_{j_1}. \end{cases}$$

It is obvious that $I_{j_1}^{e'} = \phi$ and then $x(e')_{j_1} = 0$. Also, $x(e_0)_j = x(e')_j$ for every $j \in J$ and $j \neq j_1, j_2$. By Definition 3.2, $x(e')_{j_2} = \frac{d_{i_0}^1}{a_{i_0 j_2}}$. Now, if $i_0 \notin I_{j_1}^{e_0}$, then

$$x(e_0)_{j_2} = x(e')_{j_2} = \frac{d_{i_0}^1}{a_{i_0 j_2}}$$

So we have

$$c^{+^t} x(e_0) = c_{j_1}^+ x(e_0)_{j_1} + \sum_{j \in J,\ j \neq j_1} c_j^+ x(e_0)_j \geqslant c_{j'}^+ x(e')_{j'} + \sum_{j \in J,\ j \neq j_1} c_j^+ x(e')_j = c^{+^t} x(e')$$

The proof is complete in this case. Otherwise, suppose $i_0 \in I_{j_1}^{e_0}$. We show $c^{+^t} x(e_0) \geqslant c^{+^t} x(e')$. By Definition 2.11, let $x(e_0)_{j_2} = \frac{d_i^1}{a_{ij_2}}$. Then we have $c_{j_2}^+ x(e_0)_{j_2} \geqslant 0$ from part (a) of Corollary 2.12 and Definition 3.7. Therefore, since

$$c^{+^t} x(e_0) = c_{j_1}^+ x(e_0)_{j_1} + c_{j_2}^+ x(e_0)_{j_2} + \sum_{j \neq j_1, j_2} c_j^+ x(e_0)_j$$

and

$$c^{+^t} x(e') = c_{j_2}^+ x(e')_{j_2} + \sum_{j \neq j_1, j_2} c_j^+ x(e')_j$$

it is sufficient to show $c_{j_1}^+ x(e_0)_{j_1} \geqslant c_{j_2}^+ x(e')_{j_2}$. Let $x(e_0)_{j_1} = \frac{d_{i'}^1}{a_{i' j_1}}$ from Definition 2.11. Since j_2 dominates j_1 we have

$$c_{j_1}^+ (\frac{d_{i'}^1}{a_{i' j_1}}) \geqslant c_{j_2}^+ (\frac{d_{i_0}^1}{a_{i_0 j_2}})$$

which means $c_{j_1}^+ x(e_0)_{j_1} \geqslant c_{j_2}^+ x(e')_{j_2}$ if $i_0 = i'$. Otherwise, suppose $i_0 \neq i'$. Since $i_0 \in I_{j_1}^{e_0}$ and j_2 dominates j_1,

$$c_{j_1}^+ (\frac{d_{i_0}^1}{a_{i_0 j_1}}) \geqslant c_{j_2}^+ (\frac{d_{i_0}^1}{a_{i_0 j_2}})$$

Also, by Definition 2.11, we have

$$x(e_0)_{j_1} = max_{i \in I_{j_1}^{e_0}} \frac{d_i^1}{a_{ij_1}} = \frac{d_{i'}^1}{a_{i' j_1}}$$

This implies

$$\frac{d_{i'}^1}{a_{i' j_1}} \geqslant \frac{d_i^1}{a_{ij_1}}, \forall i \in I_{j_1}^{e_0}$$

Therefore

$$c_{j_1}^+ (\frac{d_{i'}^1}{a_{i' j_1}}) \geqslant c_{j_1}^+ (\frac{d_{i_0}^1}{a_{i_0 j_1}}) \geqslant c_{j_2}^+ (\frac{d_{i_0}^1}{a_{i_0 j_2}})$$

which requires $c_{j_1}^+ x(e_0)_{j_1} \geqslant c_{j_2}^+ x(e')_{j_2}$. Hence, $c^{+^t} x(e_0) \geqslant c^{+^t} x(e')$ and the proof is complete. □

Corollary 3.9. *If j_2 dominates j_1 for some $j_1, j_2 \in J$, then we can remove the j_1th column of matrix A without any effect on finding the optimal solution $x(e_0)$ in (4a).*

Proof. It is the result of Theorem 3.8. □

4. An Algorithm for Finding an Optimal Solution and Some Examples

Definition 4.1. Consider the problem (1). We call $\bar{A} = (\bar{a}_{ij})_{m\times n}$ and $\bar{B} = (\bar{b}_{ij})_{l\times n}$ the characteristic matrices of matrix A and matrix B, respectively, where $\bar{a}_{ij} = \frac{d_i^1}{a_{ij}}$ for each $i \in I^1$ and $j \in J$, also $\bar{b}_{ij} = \frac{d_i^2}{b_{ij}}$ for each $i \in I^2$ and $j \in J$ (set $\frac{0}{0} = 0$ and, $\frac{k}{0} = \infty$ for $0 < k \leqslant 1$).

Algorithm:
Given problem (1),
1- Find matrices $\bar{A}$ and $\bar{B}$ by Definition 4.1.
2- If there exists $i \in I^1$ such that $\bar{a}_{ij} > 1$, $\forall j \in J$, then stop. Problem 2 is infeasible (see Theorem 2.4).
3- Calculate $\bar{x}$ from $\bar{B}$ by Definition 2.5, if the constraints in problem (1) are just in the form $A \bullet x \geqslant d^1$, set $\bar{x} = \bar{1}$.
4- If the constraints in problem (1) are just in the form $B \bullet x \leq d^2$, set $x(e_0) = \bar{0}$ and go to step 13.
5- If there exists $i \in I^1$ such that $d_i^1 = 0$, then remove the ith row of matrix $\bar{A}$ (see part (a) of Corollary 2.12).
6- If $\bar{a}_{ij} \geqslant \bar{x}_j$, then set $\bar{a}_{ij} = 0$, $\forall i \in I^1$ and $\forall j \in J$.
7- If there exists $i \in I^1$ such that $\bar{a}_{ij} = 0$, $\forall i \in I^1$, then stop. Problem (2) is infeasible (see Theorems 3.2 and 3.3)
8- If there exists $j' \in J$ such that $\bar{a}_{ij'} = 0$, $\forall i \in I^1$, then remove the j'th column of matrix $\bar{A}$ (see part (b) of Corollary 2.12) and set $x(e_0)_{j'} = 0$.
9- For each $i \in I^1$, if $j_i^* \neq \phi$ then remove the ith row of matrix A (see Corollary 3.6).
10- Remove each column $j \in J$ from $\bar{A}$ such that $c_j < 0$ and set $x(e_0)_j = 0$.
11- If j_2 dominates j_1, then remove column j_1 from $\bar{A}$, $\forall j_1, j_2 \in J$ (see Corollary 3.9) and set $x(e_0)_{j_1} = 0$.
12- Let $J_i^{new} = \{j \in \bar{J}_i : \bar{a}_{ij} \neq 0\}$ and $J_I^{new} = J_1^{new} \times J_2^{new} \times ... \times J_m^{new}$. Find the vectors $x(e)$, $\forall e \in J_I^{new}$, by Definition 2.11 from $\bar{A}$, and $x(e_0)$ by pairwise comparison between the vectors $x(e)$.
13- Find x^* from Lemma 3.1.

Guu and Wu [17] considered the optimization problem as follows:

$$\begin{aligned} &\textit{minimum} \quad yc \\ &\textit{subject to} \\ &\qquad y \bullet B = f \end{aligned}$$

If $x = y^t$, $A = B^t$ and $b = f^t$, then the problem is converted to the special case of our problem as follows:

$$\begin{aligned} &minimum \quad c^t x \\ &subject\ to \\ &\qquad A \bullet x = b \end{aligned} \tag{4}$$

Firstly, presented algorithm in the present paper is provided for solving problem (1) and then in the special case it can be used to solve (4), too, because $A \bullet x = b$ is equivalent to $A \bullet x \leqslant b$ and $A \bullet x \geqslant b$. Therefore we can compare the presented algorithm with the algorithm in [17]. The algorithm of Guu and Wu is based on Rule (1) in their paper. By Rule (1), they removed some columns of matrix A in (4), then they reduced problem (4) and converted it to a simpler one. In section 3, we modified Rule (1) in order to solve the problem (1) and stated Theorem 3.1, which is the same as Rule (1) that is used to solve the problem (4). Now our algorithm1 is used to solve Example 4.2 that was presented by Guu and Wu ([17], p.353). We will observe that results obtained by the two methods are the same.

Example 4.2.

$$minimum Z = 0x_1 + 3x_2 + 2x_3 + 3x_4 + 5x_5 + 2x_6 + x_7 + 2x_8 + 5x_9 + 6x_{10}$$

$$subjectto$$

$$\begin{pmatrix} 0.6 & 0.5 & 0.1 & 0.1 & 0.3 & 0.8 & 0.4 & 0.6 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.9 & 0.6 & 0.8 & 0.4 & 0.5 & 0.3 & 0.5 & 0.3 \\ 0.5 & 0.9 & 0.4 & 0.2 & 0.8 & 0.1 & 0.4 & 0.4 & 0.7 & 0.6 \\ 0.3 & 0.5 & 0.7 & 0.5 & 0.8 & 0.1 & 0.8 & 0.3 & 0.4 & 0.6 \\ 0.7 & 0.8 & 0.5 & 0.4 & 0.8 & 0.2 & 0.4 & 0.1 & 0.9 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.1 & 0.5 & 0.8 & 0.7 & 0.2 & 0.9 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.7 & 0.5 & 0.8 & 0.3 & 0.5 & 0.7 & 0.4 \\ 0.8 & 0.8 & 0.7 & 0.5 & 0.8 & 0.3 & 0.4 & 0.7 & 0.2 & 0.8 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} = \begin{pmatrix} 0.48 \\ 0.56 \\ 0.72 \\ 0.56 \\ 0.64 \\ 0.72 \\ 0.42 \\ 0.64 \end{pmatrix}$$

$$0 \leqslant x_j \leqslant 1, j = 1,2,3,4,5,6,7,8,9,10$$

Matrices A and B are equal in this problem, which means the constraints are $A \bullet x \geqslant b$ and $A \bullet x \leqslant b$. Matrix $\bar{A}$ and vector $\bar{x}$ are as follows:

$$\bar{A} = \begin{pmatrix} 0.8 & 0.96 & 4.8 & 4.8 & 1.6 & 0.6 & 1.2 & 0.8 & 2.4 & 4.8 \\ 2.8 & 0.933 & 0.622 & 0.933 & 0.7 & 1.4 & 1.12 & 1.866 & 1.12 & 1.866 \\ 1.44 & 0.8 & 1.8 & 3.6 & 0.9 & 7.2 & 1.8 & 1.8 & 1.028 & 1.2 \\ 1.866 & 1.12 & 0.8 & 1.12 & 0.7 & 5.6 & 0.7 & 1.866 & 1.4 & 0.933 \\ 0.914 & 0.8 & 1.28 & 1.6 & 0.8 & 3.2 & 1.6 & 6.4 & 0.711 & 1.066 \\ 1.44 & 0.8 & 1.028 & 7.2 & 1.44 & 0.9 & 1.028 & 3.6 & 0.8 & 1.8 \\ 2.1 & 1.4 & 1.05 & 0.6 & 0.84 & 0.525 & 1.4 & 0.84 & 0.6 & 1.05 \\ 0.8 & 0.8 & 0.914 & 1.28 & 0.8 & 2.133 & 1.6 & 0.914 & 3.2 & 0.8 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 0.8 & 0.8 & 0.622 & 0.6 & 0.7 & 0.525 & 0.7 & 0.8 & 0.6 & 0.8 \end{pmatrix}$$

Also, this example does not satisfy in steps 2 and 4. By step 5, $\bar{A}$ is converted to the following:

$$\bar{A} = \begin{pmatrix} 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.622 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.525 & 0 & 0 & 0.6 & 0 \\ 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 \end{pmatrix}$$

The matrix above does not satisfy steps 6-9, but in the obtained matrix, according to the step10, the first column dominates the eighth and tenth columns, and the sixth column dominates the fourth and ninth columns. Therefore, by removing columns 4, 8, 9 and 10, the matrix is converted into the following matrix and we set $x(e_0)_4 = x(e_0)_8 = x(e_0)_9 = x(e_0)_{10} = 0$.

$$\bar{A} = \begin{pmatrix} 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.622 & 0.7 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 & 0.7 \\ 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.525 & 0 \\ 0.8 & 0.8 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the new matrix, we have $J_1^{new} = \{1\}$, $J_2^{new} = \{3,5\}$, $J_3^{new} = \{2\}$, $J_4^{new} = \{5,7\}$, $J_5^{new} = \{2\}$, $J_6^{new} = \{2\}$, $J_7^{new} = \{6\}$ and $J_8^{new} = \{1,2\}$. Now, by considering the step11, the minimal solutions are $x(e_1) = (0.8, 0.8, 0.622, 0, 0, 0.525, 0.7, 0, 0, 0)$ and $x(e_2) = (0.8, 0.8, 0, 0, 0.7, 0.525, 0, 0, 0, 0)$. By comparison pairwise, since $x(e_1)$ optimizes the problem with the objective function ${c^+}^t x = c^t x$, $x(e_1) = x(e_0)$, and then $x^* = (0.8, 0.8, 0.622, 0, 0, 0.525, 0.7, 0, 0, 0)$ by step12.

The obtained result is the same as that of Guu and Wu [17]. In special case of FREs with max- product composition.

Example 4.3. Consider the problem below:

$$minimum Z = 2x_1 - x_2 + x_3 - 3x_4$$
$$subject to$$
$$\begin{pmatrix} 0.5 & 0.8 & 0.35 & 0.25 \\ 0.9 & 0.92 & 1 & 0.86 \\ 0.2 & 1 & 0.45 & 0.8 \\ 0.55 & 0.6 & 0.8 & 0.64 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geqslant \begin{pmatrix} 0.4 \\ 0.9 \\ 0.8 \\ 0.65 \end{pmatrix}$$
$$\begin{pmatrix} 0.6 & 0.5 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.6 & 0.5 \\ 0.5 & 0.9 & 0.8 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leqslant \begin{pmatrix} 0.48 \\ 0.56 \\ 0.72 \end{pmatrix}$$
$$0 \leqslant x_j \leqslant 1, j = 1, 2, 3, 4$$

Matrices $\bar{A}$, $\bar{B}$ and vector $\bar{x}$ are as follows:

$$\bar{A} = \begin{pmatrix} 0.8 & 0.5 & 1.14 & 1.6 \\ 1 & 0.97 & 0.9 & 1.04 \\ 4 & 0.8 & 1.77 & 1 \\ 1.18 & 1.08 & 0.81 & 1.01 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0.8 & 0.96 & 4.8 & 4.8 \\ 2.8 & 0.93 & 0.93 & 1.12 \\ 1.44 & 0.8 & 0.9 & 1.8 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 0.8 & 0.8 & 0.9 & 1 \end{pmatrix}$$

By considering step 5, matrix $\bar{A}$ is converted to the following:

$$\bar{A} = \begin{pmatrix} 0.8 & 0.5 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0.8 & 0 & 1 \\ 0 & 0 & 0.81 & 0 \end{pmatrix}$$

According to the step 8, since $J_1^* = \{2\}$ and $J_3^* = \{2, 4\}$, we can remove the first and the third rows, then we have:

$$\bar{A} = \begin{pmatrix} 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.81 & 0 \end{pmatrix}$$

By step 9, we remove the second and the fourth columns, and we set $x(e_0)_2 = x(e_0)_4 = 0$ by the step 10 then:

$$\bar{A} = \begin{pmatrix} 0 & 0.9 \\ 0 & 0.81 \end{pmatrix}$$

By steps 11, 12 and 13, $x(e_0)$ and x^* are calculated as follows:

$$x(e_0) = (0, 0, 0.9, 0)$$

$$x^* = (0, 0.8, 0.9, 1)$$

Example 4.4. (*Application*) Consider a city with eight educational zones (1-8) as shown in Figure 1. A schoolmaster decides to cover the six zones 1-6 by enhancing the educational quality and diminishing educational shortcoming of his school (A). He considers the four criteria below to convince the parents to select school A (we call them positive criteria).
(1) The quality of cultural activities.
(2) The quality of the athletic-recreational facilities (such as playground, pool, etc.).
(3) The educational quality of school A.
(4) The quality of cleanliness of school A.
Also he considers two shortcomings that cause the parents not to select school A (we call them negative criteria).
(5) Shortage of enough laboratories in school A.
(6) Shortage of enough space in school A.
We evaluate the quality of the athletic-recreational facilities of school A versus the schools in zones 1-6, separately. The example has been illustrated in Figure 2, where school A has been compared with all schools in zones 1-6, concerning the athletic-recreational facilities. Based on the four positive criteria, the methods of the evaluation of schools evaluation are shown in Figures 2, 3, 4, 5. Also, based

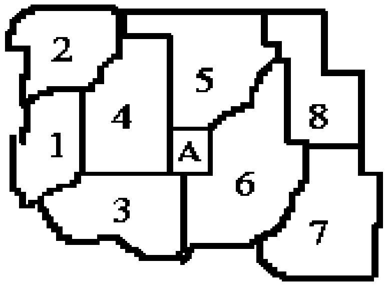

FIGURE 1. Educational Zones and the Situation of School A

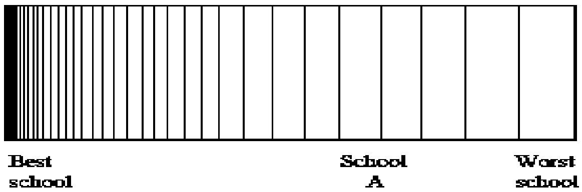

FIGURE 2. Evaluation of School A from the Viewpoint of the Quality of Cultural Activities

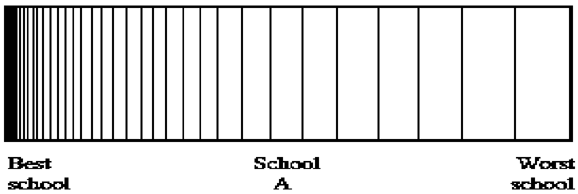

FIGURE 3. Evaluation of School A from the Viewpoint of the Quality of Athletic Facilities

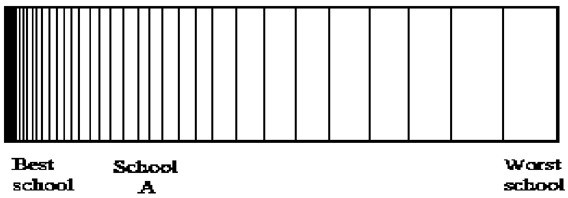

FIGURE 4. Evaluation of School A from the Viewpoint of Educational Quality

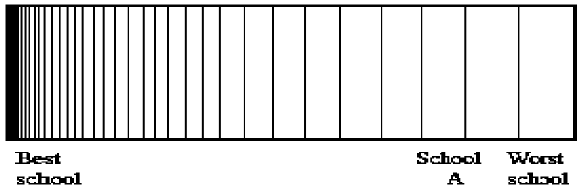

FIGURE 5. Evaluation of School A from the Viewpoint of the Quality of Cleanliness

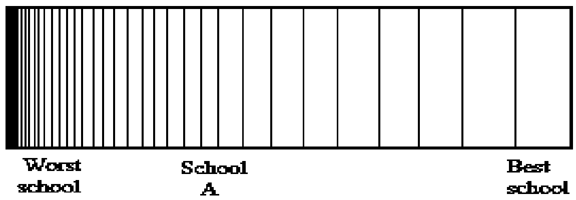

FIGURE 6. Evaluation of School A from the Viewpoint of Having Enough Laboratories

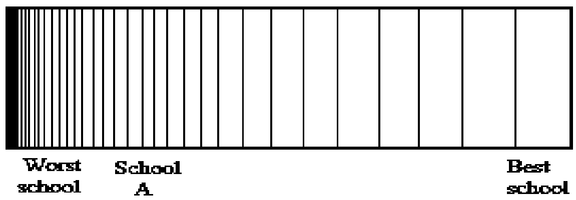

FIGURE 7. Evaluation of School A from the Viewpoint of Having Enough Space

on the two negative criteria, the methods of the evaluation of schools are shown in Figures 6 and 7.

Also, the schoolmaster has some plans for each potentially poor positive criterion:
$(\acute{1})$ If the cultural activities are poor, then he increases the cultural programs.
$(\acute{2})$ If the quality of the athletic-recreational facilities is poor, he contracts athletic-recreational places to enhance this quality. Considering that school A has no appropriate spaces for building the athletic-recreational places.
$(\acute{3})$ If the level of the educational quality of school A is poor, he employs experienced teachers.
$(\acute{4})$ If the quality level of cleanliness is poor, he employs more workmen to clean school A.
By considering positive criteria 1-4, we can categorize the parents' expectations in

four classes: (1) the problem of the cultural activities of students, (2) the athletic-recreational activities of the students, (3) the educational problems of the students, and (4) the cleanliness problem of school A.
Now, suppose that a_{ij} denotes the required quality level of the positive criteria ($i = 1, 2, 3, 4$) from the viewpoint of the students' parents in zone j. The matrix for the six zones in Figure 1 and the four positive criteria is as follows:

$$A = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.8 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.2 & 0.8 & 0.5 & 0.5 \\ 0.5 & 0.8 & 0.4 & 0.6 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.5 & 0.8 & 0.8 & 0.6 \end{pmatrix}$$

In addition, the schoolmaster has some plans for diminishing each negative criterion:
($\acute{5}$) he decides to equip the laboratories in order to compensate for the shortage of enough laboratories in school A.
($\acute{6}$) he decides to change the school building (A) in order to generating some spaces.
Also, by considering the negative criteria 5-6, we can categorize the parents' other expectation in two classes: (1) the problem of equipping laboratories, (2) the problem lack of enough space in school A.
Now, suppose that b_{ij} denotes the deficiency level of the negative criteria ($i = 5, 6$) from the viewpoint of the students' parents in zone j. The matrix for the six zones in Figure 1 and the two negative criteria is as follows:

$$B = \begin{pmatrix} -0.3 & -0.2 & -0.4 & -0.5 & -0.4 & -0.3 \\ -0.5 & -0.6 & -0.3 & -0.3 & -0.5 & -0.4 \end{pmatrix}$$

The schoolmaster estimates that if he expends cost x_j (x_j has been normalized in $[0, 1]$) to overcome the expectation of kind (i=1, 2, 3, 4, 5, 6) by doing activity (í=$\acute{1}$, $\acute{2}$, $\acute{3}$, $\acute{4}$, $\acute{5}$, $\acute{6}$), then he will obtain the quality level $a_{ij} \cdot x_j$ from the viewpoint of the parents in zone (j) for criterion (i=1, 2, 3, 4) and level $b_{ij} \cdot x_j$ from the viewpoint of the parents in zone (j) for criterion (i=5, 6). Also, the schoolmaster estimates levels b_i, for i=1, 2, 3, 4, such that if he make quality levels b_i, i=1, 2, 3, 4, for criterion (i) to meet the expectations of at least the parents of the students' of one of the zones, then he will overcome the difficulties in the expectations of kind (i) by doing activity (í). Vector b is as follows:

$$b = \begin{pmatrix} 0.4 & 0.5 & 0.3 & 0.6 \end{pmatrix}^t$$

Also, he estimates d_i, for i=5, 6, such that for compensating for the shortages of (i=5, 6) by doing activities (í=$\acute{5}$, $\acute{6}$) he must fulfill level d_i, i=5, 6, for criterion (i) at least for the parent of the students' of one of the zones. Vector d is as follows:

$$d = \begin{pmatrix} -0.3 & -0.6 \end{pmatrix}^t$$

The schoolmaster wants to spend a minimum cost, too. Therefore, we can formulate the problem as follows:

$$minimum Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$
$$subject to$$

$$A \bullet x = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.8 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.2 & 0.8 & 0.5 & 0.5 \\ 0.5 & 0.8 & 0.4 & 0.6 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.5 & 0.8 & 0.8 & 0.6 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geqslant \begin{pmatrix} 0.4 \\ 0.5 \\ 0.3 \\ 0.6 \end{pmatrix}$$

$$B \circ x = \begin{pmatrix} -0.3 & -0.2 & -0.4 & -0.5 & -0.4 & -0.3 \\ -0.5 & -0.6 & -0.3 & -0.3 & -0.5 & -0.4 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -0.3 \\ -0.6 \end{pmatrix} = d$$

The above notation, "$\bullet$" is the max-product composition and, "$\circ$" is defined by $b_i \circ x = min_{j \in J}\{b_{ij} \cdot x_j\}$, but we can change the second part of the constraints as follows:

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix}$$

Also, the above constraint is equivalent to two systems

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \leqslant \begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix}$$

and

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geqslant \begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix}$$

Finally, by combining the constraints we come to

$$minimumZ = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$
$$subjectto$$
$$\begin{pmatrix} 1 & 0.2 & 0.5 & 0.8 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.2 & 0.8 & 0.5 & 0.5 \\ 0.5 & 0.8 & 0.4 & 0.6 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.5 & 0.8 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geqslant \begin{pmatrix} 0.4 \\ 0.5 \\ 0.3 \\ 0.6 \\ 0.3 \\ 0.6 \end{pmatrix}$$
$$\begin{pmatrix} 0.3 & 0.2 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.5 & 0.6 & 0.3 & 0.3 & 0.5 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \leqslant \begin{pmatrix} 0.3 \\ 0.6 \end{pmatrix}$$

According to the presented algorithm, the solution of the above problem is $x = \begin{pmatrix} 0.4 & 0.375 & 0 & 0 & 0 & 1 \end{pmatrix}$ and the value of the objective function is 1.775.

Example 4.5. Consider the problem

$$minimumZ = x_1 - x_2 + x_3$$
$$subjectto$$
$$\begin{pmatrix} 0.4 & 0.6 & 0.3 \\ 0.8 & 0.27 & 0.9 \\ 0.3 & 0.9 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geqslant \begin{pmatrix} 0.6 \\ 0.36 \\ 0.18 \end{pmatrix}$$
$$0 \leqslant x_j \leqslant 1, j = 1, 2, 3$$

Matrix $\bar{A}$ and vector $\bar{x}$ from the steps 1 and 3 respectively, are as follows:

$$\bar{A} = \begin{pmatrix} 0.67 & 1 & 2 \\ 0.45 & 1.33 & 0.4 \\ 0.6 & 0.2 & 0.45 \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

By considering step 6, matrix $\bar{A}$ is converted to the following:

$$\bar{A} = \begin{pmatrix} 0.67 & 1 & 0 \\ 0.45 & 0 & 0.4 \\ 0.6 & 0.2 & 0.45 \end{pmatrix}$$

According to step 9, since $J_1^* = \{2\}$ and $J_3^* = \{2\}$, we can remove the first and the third rows, then we have:

$$\bar{A} = \begin{pmatrix} 0.45 & 0 & 0.4 \end{pmatrix}$$

By step 10, we remove the second column, and set $x(e_0)_2 = 0$, then:

$$\bar{A} = \begin{pmatrix} 0.45 & 0.4 \end{pmatrix}$$

The first column of the above matrix is removed by step 11, and only the second column (in original matrix third column) remains.

$$\bar{A} = \begin{pmatrix} 0.4 \end{pmatrix}$$

By steps 12 and 13, $x(e_0)$ and x^* are calculated as follows:

$$x(e_0) = (0, 0, 0.4), \quad x^* = (0, 1, 0.4)$$

and finally, the minimum value of the objective function is given by $Z^* = -0.6$.

Example 4.6. Consider the problem

$$\begin{aligned} & minimum Z = x_1 - x_2 + x_3 \\ & subject to \\ & \begin{pmatrix} 0.4 & 0.6 & 0.3 \\ 0.8 & 0.27 & 0.9 \\ 0.3 & 0.9 & 0.4 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqslant \begin{pmatrix} 0.6 \\ 0.36 \\ 0.18 \end{pmatrix} \\ & 0 \leqslant x_j \leqslant 1, j = 1, 2, 3 \end{aligned}$$

Matrix $\bar{A}$ and vector $\bar{x}$ from steps 1 and 3 respectively, are as follows:

$$\bar{B} = \begin{pmatrix} 0.67 & 1 & 2 \\ 0.45 & 1.33 & 0.4 \\ 0.6 & 0.2 & 0.45 \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} 0.45 & 0.2 & 0.4 \end{pmatrix}$$

By considering step 4, $x(e_0) = \bar{0} = (0, 0, 0)$, and by step 13 the optimal solution is as follows:

$$x^* = (0, 0.2, 0)$$

and finally, the minimum value of the objective function is given by $Z^* = -0.2$.

5. **Conclusion**

In this paper, we studied the linear optimization problem with fuzzy relational inequalities constraints defined by the max-product operator. Since the difficulty of this problem is finding the minimal solutions optimizing the same problem with the objective function c^{+^t}, we presented an algorithm together with some simplification operations to accelerate the problem resolution. At last, we gave some examples to illustrate the proposed algorithm.

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