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A RELATED FIXED POINT THEOREM IN n FUZZY METRIC SPACES

F. MERGHADI AND A. ALIOUCHE

Abstract. We prove a related fixed point theorem for n mappings which are not necessarily continuous in n fuzzy metric spaces using an implicit relation one of them is a sequentially compact fuzzy metric space which generalize results of Aliouche, et al. [2], Rao et al. [14] and [15].

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [20] in 1965. George and Veeramani [5] modified the concept of fuzzy metric space introduced by [7] and defined the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and $E-$infinity theory which were given and studied by El- Naschie [8, 9, 10, 11, 12] and [18]. They showed also that every metric induces a fuzzy metric.

Recently, Fisher [4], Aliouche and Fisher [1], Aliouche et.al [2], Rao et.al [14], [15], and [19] proved some related fixed point theorems in compact metric spaces and sequentially compact fuzzy metric spaces. Motivated by a work due to Popa [13], we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition.

Definition 1.1. [17] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of a continuous $t-$norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 1.2. [5] The triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous $t-$norm, and M is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,

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(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 1.3. Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Definition 1.4. [5] Let $(X, M, *)$ be a fuzzy metric space.

1) A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(x_n, x, t) > 1 - \epsilon$; i.e., $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $M(x_n, x_m, t) > 1 - \epsilon$; i.e., $M(x_n, x_m, t) \to 1$ as $n, m \to \infty$ for all $t > 0$.

3) A fuzzy metric space (X, M, t) in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.5. [6] *For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7. [6] *M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.8. $(X, M, *)$ is said to be sequentially compact fuzzy metric space if every sequence in X has a convergent sub- sequence in it.

Let Φ be the set of all functions $\phi : [0, 1]^6 \longrightarrow [0, 1]$ such that if either $\phi(u, 1, u, v, v, 1) > 0$ or $\phi(u, u, 1, v, 1, v) > 0$ for all $u, v \in [0, 1)$, then $u > v$.

Example 1.9. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$. Then $\phi \in \Phi$.

2. Main Results

Theorem 2.1. *Let* (X_i, M_i, θ_i), $i = \overline{1,n}$ *be n fuzzy metric spaces,* $A_i : X_i \to X_{i+1}$, $i = \overline{1, n-1}$ *and* $A_n : X_n \to X_1$ *be n mappings satisfying*

$$\phi_1 \begin{pmatrix} M_1 \left(A_n A_{n-1}..A_2 x_2, A_n A_{n-1}..A_2 A_1 x_1, t\right), M_1 \left(x_1, A_n A_{n-1}..A_2 x_2, t\right), \\ M_1 \left(x_1, A_n A_{n-1}..A_2 A_1 x_1, t\right), M_2 \left(x_2, A_1 x_1, t\right), \\ M_2 \left(x_2, A_1 A_n A_{n-1}..A_2 x_2, t\right), M_2 \left(A_1 x_1, A_1 A_n A_{n-1}..A_2 x_2, t\right) \end{pmatrix} > 0 \tag{2.1}$$

for all $x_1 \in X_{\ 1}$, $x_2 \in X_{\ 2}$ *with* $x_2 \neq A_1 x_1$ *and for all* $t > 0$,

$$\phi_2 \begin{pmatrix} M_2 \left(A_1 A_n A_{n-1}..A_3 x_3, A_1 A_n A_{n-1}..A_3 A_2 x_2, t\right), \\ M_2 \left(x_2, A_1 A_n A_{n-1}..A_3 x_3, t\right), \\ M_2 \left(x_2, A_1 A_n A_{n-1}..A_2 x_2, t\right), M_3 \left(x_3, A_2 x_2, t\right), \\ M_3 \left(x_3, A_2 A_1 A_n A_{n-1}..A_3 x_3, t\right), \\ M_3 \left(A_2 x_2, A_2 A_1 A_n A_{n-1}..A_3 x_3, t\right) \end{pmatrix} > 0 \tag{2.2}$$

for all $x_2 \in X_{\ 2}$, $x_3 \in X_{\ 3}$ *with* $x_3 \neq A_2 x_2$ *and for all* $t > 0$,

$$\phi_3 \begin{pmatrix} M_3 \left(A_2 A_1 A_n A_{n-1}..A_4 x_4, A_2 A_1 A_n A_{n-1}..A_3 x_3, t\right), \\ M_3 \left(x_3, A_2 A_1 A_n A_{n-1}..A_4 x_4, t\right), \\ M_3 \left(x_3, A_2 A_1 A_n A_{n-1}..A_3 x_3, t\right), M_4 \left(x_4, A_3 x_3, t\right), \\ M_4 \left(x_4, A_3 A_2 A_1 A_n A_{n-1}..A_4 x_4, t\right), \\ M_4 \left(A_3 x_3, A_3 A_2 A_1 A_n A_{n-1}..A_4 x_4, t\right) \end{pmatrix} > 0$$

for all $x_3 \in X_{\ 2}$, $x_4 \in X_{\ 3}$ *with* $x_4 \neq A_3 x_3$ *and for all* $t > 0$

$$\phi_i \begin{pmatrix} M_i \left(A_{i-1} A_{i-2} ... A_1 A_n A_{n-1}..A_{i+1} x_{i+1}, A_{i-1} A_{i-2} ... A_1 A_n A_{n-1}..A_i x_i, t\right), \\ M_i \left(x_i, A_{i-1} A_{i-2} ... A_1 A_n A_{n-1}..A_{i+1} x_{i+1}, t\right), \\ M_i \left(x_i, A_{i-1} A_{i-2} ... A_1 A_n A_{n-1}..A_i x_i, t\right), M_{i+1} \left(x_{i+1}, A_i x_i, t\right), \\ M_{i+1} \left(x_{i+1}, A_i A_{i-1} ... A_1 A_n A_{n-1}..A_{i+1} x_{i+1}, t\right), \\ M_{i+1} \left(A_i x_i, A_i A_{i-1} ... A_1 A_n A_{n-1}..A_{i+1} x_{i+1}, t\right) \end{pmatrix} > 0 \tag{2.i}$$

for all $x_i \in X_{\ i}$, $x_{i+1} \in X_{\ i+1}$ *with* $x_{i+1} \neq A_i x_i$, $4 \leq i \leq n-1$ *and for all* $t > 0$,

$$\phi_n \begin{pmatrix} M_n \left(A_{n-1} A_{n-2} ... A_1 x_1, A_{n-1} A_{n-2} ... A_1 A_n x_n, t\right), \\ M_n \left(x_n, A_{n-1} A_{n-2} ... A_1 x_1, t\right), \\ M_n \left(x_n, A_{n-1} A_{n-2} ... A_1 A_n x_n, t\right), M_1 \left(x_1, A_n x_n, t\right), \\ M_1 \left(x_1, A_n A_{n-1} A_{n-2} ... A_1 x_1, t\right), \\ M_1 \left(A_n x_n, A_n A_{n-1} A_{n-2} ... A_1 x_1, t\right) \end{pmatrix} > 0 \tag{2.n}$$

for all $x_1 \in X_1$, $x_n \in X_n$ *with* $x_1 \neq A_n x_n$ *and for all* $t > 0$, *where* $\phi_i \in \Phi$, $i = 1, ..., n$.

Suppose that one of the following is true

(a_1) (X_1, M_1, θ_1) *is sequentially compact and* $A_n A_{n-1}..A_1$ *is continuous on* X_1

(a_2) (X_2, M_2, θ_2) *is sequentially compact and* $A_1 A_n A_{n-1}..A_2$ *is continuous on* X_2,

$(a_{i)}$ (X_i, M_i, θ_i) is sequentially compact and $A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_i$ is continuous on X_i, $i \geq 3$,

(a_n) (X_n, M_n, θ_n) is sequentially compact and $A_{n-1}A_{n-2}...A_1A_n$ is continuous on X_n.

Then $A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_i$ has a unique fixed point $w_i \in X_i$, for all $i = \overline{1,n}$.

Further, $A_iw_i = w_{i+1}$ for all $i = \overline{1, n-1}$ and $A_nw_n = w_1$.

Proof. Suppose that (X_1, M_1, θ_1) is sequentially compact and $A_nA_{n-1}..A_1$ is continuous on X_1.

For every $t > 0$, define

$$\phi(x_1) = M_1(x_1, A_nA_{n-1}..A_1x_1, t) \text{ for all } x_1 \in X_1.$$

Then, there exists $p_1 \in X_1$ such that

$$\phi(p_1) = M_1(p_1, A_nA_{n-1}..A_1p_1, t) = \max\{\phi(x_1) : x_1 \in X_1\},$$

there exists $p_2 \in X_2$ such that $p_2 = A_1p_1$, $p_3 \in X_3$ such that $p_3 = A_2p_2 = A_2A_1p_1$.

In the same manner, there exists $p_{i-1} \in X_{i-1}$ and $p_i \in X_i$ such that

$$p_i = A_{i-1}p_{i-1} = A_{i-1}A_{i-2}p_{i-2} = \cdots = A_{i-1}A_{i-2}\cdots A_2A_1p_1, i = \overline{2..n}$$

for $i = n$ we get

$$p_n = A_{n-1}p_{n-1} = A_{n-1}A_{n-2}p_{n-2} = \cdots = A_{n-1}A_{n-2}\cdots A_2A_1p_1$$

Therefore, $\phi(p_1) = M_1(p_1, A_np_n, t) = \max\{\phi(x_1) : x_1 \in X_1\}$.

Suppose that

$$A_1(A_nA_{n-1}..A_1)^{n-2}(p_1) \neq A_1(A_nA_{n-1}..A_1)^{n-1}(p_1), \text{ i.e.,}$$

$$(A_1A_nA_{n-1}..A_2)^{n-2}A_1(p_1) \neq (A_1A_nA_{n-1}..A_2)^{n-1}A_1(p_1)$$

as $A_1(p_1) = p_2$. Then

$$\begin{cases} (A_1A_nA_{n-1}..A_2)^{n-2}(p_2) \neq (A_1A_nA_{n-1}..A_2)^{n-1}(p_2) \\ (A_1A_nA_{n-1}..A_2)^{n-3}(p_1) \neq (A_1A_nA_{n-1}..A_2)^{n-2}(p_2) \\ \vdots \\ (A_1A_nA_{n-1}..A_2)(p_1) \neq (A_1A_nA_{n-1}..A_2)^2(p_2) \\ (A_nA_{n-1}..A_2p_2) \neq (A_nA_{n-1}..A_2)(A_1A_nA_{n-1}..A_2p_2) \\ (A_{n-1}..A_2p_2) \neq (A_{n-1}..A_2)(A_1A_nA_{n-1}..A_2p_2) \\ \vdots \\ p_2 \neq A_1A_nA_{n-1}..A_2p_2 \\ p_1 \neq A_nA_{n-1}..A_1p_1 \end{cases}$$

Putting $x_2 = A_1 (A_n A_{n-1}..A_1)^{n-2} (p_1)$ and $x_1 = (A_n A_{n-1}..A_1)^{n-1} (p_1)$ in (2.1) we have

$$\phi_1 \begin{pmatrix} M_1 \left((A_n A_{n-1}..A_1)^{n-1} (p_1), (A_n A_{n-1}..A_2 A_1)^n (p_1), t \right), \\ M_1 \left((A_n A_{n-1}..A_1)^{n-1} (p_1), (A_n A_{n-1}..A_1)^{n-1} (p_1), t \right), \\ M_1 \left((A_n A_{n-1}..A_1)^{n-1} (p_1), (A_n A_{n-1}..A_1)^n (p_1), t \right), \\ M_2 \left((A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), (A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), (A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), (A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), t \right) \end{pmatrix} > 0 \tag{3.1}$$

and so

$$\begin{aligned} & \phi \left((A_n A_{n-1}..A_1)^{n-1} (p_1) \right) && (3.2) \\ = \ & M_1 \left((A_n A_{n-1}..A_1)^{n-1} (p_1), (A_n A_{n-1}..A_2 A_1)^n (P_1), t \right) \\ > \ & M_2 \left((A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), (A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), t \right) \end{aligned}$$

Putting $x_2 = (A_1 A_n A_{n-1}..A_2)^{n-2} (p_2)$ and $x_3 = A_2 (A_1 A_n A_{n-1}..A_2)^{n-3} (p_2)$ in (2.2) we get

$$\phi_2 \begin{pmatrix} M_2 \left((A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), (A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), (A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), t \right), \\ M_2 \left((A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), (A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), t \right), \\ M_3 \left((A_2 A_1 A_n A_{n-1}..A_3)^{n-3} (p_3), (A_2 A_1 A_n A_{n-1}..A_3)^{n-2} (p_3), t \right), \\ M_3 \left((A_2 A_1 A_n A_{n-1}..A_3)^{n-3} (p_3), (A_2 A_1 A_n A_{n-1}..A_3)^{n-2} (p_3), t \right), \\ M_3 \left((A_2 A_1 A_n A_{n-1}..A_3)^{n-2} (p_3), (A_2 A_1 A_n A_{n-1}..A_3)^{n-2} (p_3), t \right) \end{pmatrix} > 0$$

and so

$$\begin{aligned} & M_2 \begin{pmatrix} (A_1 A_n A_{n-1}..A_2)^{n-2} (p_2), \\ (A_1 A_n A_{n-1}..A_2)^{n-1} (p_2), t \end{pmatrix} && (3.3) \\ > \ & M_3 \begin{pmatrix} (A_2 A_1 A_n A_{n-1}..A_3)^{n-3} (p_3), \\ (A_2 A_1 A_n A_{n-1}..A_3)^{n-2} (p_3), t \end{pmatrix} \end{aligned}$$

By the same manner we get for all $i = \overline{1, n-1}$

$$\begin{aligned} & M_i \begin{pmatrix} (A_{i-1} A_{i-2}..A_1 A_n A_{n-1}..A_i)^{n-i} (p_i), \\ (A_{i-1} A_{i-2}..A_1 A_n A_{n-1}..A_i)^{n-i+1} (p_i), t \end{pmatrix} && (3.i) \\ > \ & M_{i+1} \begin{pmatrix} (A_i A_{i-1}..A_1 A_n A_{n-1}..A_{i+1})^{n-i-1} (p_{i+1}), \\ ((A_i A_{i-1}..A_1 A_n A_{n-1}..A_{i+1}))^{n-i} (p_{i+1}), t \end{pmatrix}. \end{aligned}$$

For example if $i = n-1$ we have

$$\begin{aligned} & M_{n-1}\begin{pmatrix} (A_{n-2}A_{n-3}..A_1A_nA_{n-1})(p_{n-1}), \\ (A_{n-2}A_{n-3}..A_1A_nA_{n-1})^2(p_{n-1}), t \end{pmatrix} \\ > \quad & M_n(p_n, (A_{n-1}A_{n-2}..A_1A_n)(p_n), t) \end{aligned} \tag{3.n-1}$$

Now, putting $x_1 = p_1$ and $x_n = A_{n-1}A_{n-2}...A_1(p_1) = p_n$ in $(2.n)$ we obtain

$$\phi_n\begin{pmatrix} M_n(p_n, A_{n-1}A_{n-2}...A_1A_n(p_n), t), \\ M_n(p_n, p_n, t), \\ M_n(p_n, A_{n-1}A_{n-2}...A_1A_n(p_n), t), \\ M_1(p_1, A_np_n, t), \\ M_1(p_1, A_np_n, t), \\ M_1(A_np_n, A_np_n, t) \end{pmatrix} > 0$$

and so

$$M_n(p_n, A_{n-1}A_{n-2}...A_1A_n(p_n), t) > M_1(p_1, A_nA_{n-1}A_{n-2}...A_1, t) \tag{3.n}$$

From $(3.1), (3.2), (3.3), .., (3.i), .., (3.n-1)$ and $(3.n)$ we get

$$\phi\left((A_nA_{n-1}..A_1)^{n-1}(p_1)\right) > \phi(p_1) \tag{4.1}$$

which is a contradiction. Therefore

$$(A_1A_nA_{n-1}..A_2)^{n-2}(p_2) = (A_1A_nA_{n-1}..A_2)^{n-1}(p_2). \tag{4.2}$$

Using (4.2), we have

$$\begin{aligned} p_2 &= A_1A_nA_{n-1}..A_3A_2(p_2) = A_1A_nA_{n-1}..A_3(p_3) = ... \\ &= A_1A_nA_{n-1}..A_{i+1}A_i(p_i) = ... = A_1A_n(p_n) = A_1p_1. \end{aligned}$$

Also

$$\begin{aligned} p_1 &= A_nA_{n-1}..A_2A_1(p_1) = A_nA_{n-1}..A_2(p_2) = ... \\ &= A_nA_{n-1}..A_{i+1}A_i(p_i) = ... = A_n(p_n). \end{aligned}$$

For all $\overline{i = 1, n}$, there exists p_i in X_i such that

$$\begin{aligned} p_i &= A_{i-1}A_{i-2}..A_1A_nA_{n-1}..A_i(p_i) \\ &= A_{i-1}A_{i-2}..A_1A_nA_{n-1}..A_{i+1}(p_{i+1}) \\ &= ... \\ &= A_{i-1}A_{i-2}..A_1A_nA_{n-1}(p_{n-1}) \\ &= A_{i-1}A_{i-2}..A_1A_n(p_n) \\ &= A_{i-1}A_{i-2}..A_1(p_1). \end{aligned}$$

For the uniqueness of p_1, suppose that there exists z_1 in X_1 such that $A_nA_{n-1}..A_2A_1(z_1) = z_1$ with $z_1 \neq p_1$ and for all $i = \overline{2,n}$, $z_i = A_{i-1}z_{i-1}$in X_i. Then,

$$\begin{aligned} A_nA_{n-1}..A_2A_1(z_1) &\neq A_nA_{n-1}..A_2A_1(p_1) \\ A_{n-1}..A_2A_1(z_1) &\neq A_{n-1}..A_2A_1(p_1) \\ &\vdots \\ A_2A_1(z_1) &\neq A_2A_1(p_1) \\ A_1(z_1) &\neq A_1(p_1) \end{aligned}$$

Putting $x_1 = p_1$ and $x_2 = A_1z_1 = z_2$ in (2.1) we have

$$\phi_1\begin{pmatrix} M_1(A_nA_{n-1}..A_2A_1z_1, A_nA_{n-1}..A_2A_1p_1, t), \\ M_1(p_1, A_nA_{n-1}..A_2A_1z_1, t), \\ M_1(p_1, A_nA_{n-1}..A_2A_1p_1, t), M_2(A_1z_1, A_1p_1, t), \\ M_2(A_1z_1, A_1A_nA_{n-1}..A_2A_1z_1, t), \\ M_2(A_1p_1, A_1A_nA_{n-1}..A_2A_1z_1, t) \end{pmatrix} > 0$$

and so

$$M_1(z_1, p_1, t) > M_2(z_2, p_2, t) \tag{4.3}$$

Putting $x_3 = A_2A_1p_1 = A_2p_2 = p_3$, $x_2 = A_1z_1 = z_2$ in (2.2) we get

$$\phi_2\begin{pmatrix} M_2(A_1A_nA_{n-1}..A_3A_2p_2, A_1A_nA_{n-1}..A_3A_2z_2, t), \\ M_2(z_2, A_1A_nA_{n-1}..A_3A_2p_2, t), \\ M_2(z_2, A_1A_nA_{n-1}..A_2z_2, t), M_3(p_3, A_2z_2, t), \\ M_3(p_3, A_2A_1A_nA_{n-1}..A_3p_3, t), \\ M_3(A_2z_2, A_2A_1A_nA_{n-1}..A_3p_3, t) \end{pmatrix} > 0$$

and so

$$M_2(z_2, p_2, t) > M_3(z_3, p_3, t) \tag{4.4}$$

By the same manner, putting for all $i = \overline{3, n-1}$, $x_{i+1} = A_iA_{i-1}...A_2A_1p_1 = A_ip_i = p_{i+1}$ and $x_i = A_{i-1}z_{i-1} = z_i$ in $(2.i)$ we obtain

$$\phi_i\begin{pmatrix} M_i(A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}A_ip_i, A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_iz_i, t), \\ M_i(z_i, A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}A_ip_i, t), \\ M_i(z_i, A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_iz_i, t), M_{i+1}(p_{i+1}, A_iz_i, t), \\ M_{i+1}(p_{i+1}, A_iA_{i-1}...A_1A_nA_{n-1}..A_{i+1}p_{i+1}, t), \\ M_{i+1}(A_iz_i, A_iA_{i-1}...A_1A_nA_{n-1}..A_{i+1}p_{i+1}, t) \end{pmatrix} > 0$$

and so

$$M_i(z_i, p_i, t) > M_{i+1}(z_{i+1}, p_{i+1}, t) \tag{4.i}$$

Putting $x_1 = A_nA_{n-1}...A_2A_1p_1 = A_np_n = p_1$ and $x_n = A_{n-1}...A_2A_1A_nz_n = z_n$ in $(2.n)$ we get:

$$\phi_n(M_n(p_n, z_n, t), M_n(z_n, p_n, t), 1, M_1(p_1, z_1, t), 1, M_1(z_1, p_1, t)) > 0$$

and so

$$M_n\left(p_n, z_n, t\right) > M_1\left(p_1, z_1, t\right) \tag{4.n}$$

Using $(4.3), (4.4), (4.i)$ and $(4.n)$ we get

$$M_1\left(p_1, z_1, t\right) > M_1\left(p_1, z_1, t\right)$$

which is a contradiction. Hence, p_1 is the unique fixed point of $A_nA_{n-1}...A_2A_1$. Similarly, we can prove the uniqueness of fixed points of $A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_i$ for all $i = \overline{2,n}$.

In a similar manner, the Theorem 2.1 holds if one of (a_i), $i \geq 2$ is satisfied. □

The following example illustrates our Theorem 2.1.

Example 2.2. Let (M_I, X_i, θ_i), $i = \overline{1..n}$., be n fuzzy metric spaces such that $M_i\left(x_i, y_i, t\right) = \dfrac{t}{t + |x_i - y_i|}$ for $i = \overline{1..n}$ and $X_1 = [0,1]$, $X_i =]i-1, i[$ for all $i \geq 2$. Define $A_i : X_i \to X_{i+1}$ for $i = \overline{1..n-1}$ and $A_n : X_n \to X_1$ by

$$\begin{aligned}
A_1x_1 &= \begin{cases} \frac{5}{4} \text{ if } x_1 \in [0, \frac{1}{2}[\\ \frac{3}{2} \text{ if } x_1 \in [\frac{1}{2}, 1] \end{cases}, \\
A_ix_i &= \begin{cases} i + \frac{1}{4} \text{ if } x_i \in \left]i-1, i-\frac{3}{4}\right[\\ i + \frac{1}{2} \text{ if } x_i \in \left[i - \frac{3}{4}, i\right[\end{cases} \quad \text{for all } i = \overline{2..n-1}, \\
A_nx_n &= \begin{cases} \frac{3}{4} \text{ if } x_n \in]n-1, n-\frac{3}{4}[\\ 1 \text{ if } x_n \in [n - \frac{3}{4}, n[\end{cases}
\end{aligned}$$

Let $\phi_1 = \phi_2 = ... = \phi_n = \phi$ and $\phi\left(t_1, t_2, t_3, t_4, t_5, t_6\right) = t_1 - \min\left\{t_2, t_3, t_4, t_5, t_6\right\}$.

Note that there exists w_i in X_i such that $\left(A_{i-1}A_{i-2}..A_1A_n...A_i\right)w_i = w_i$, $\forall i = \overline{1..n}$.

(a) For $i = n$ we get $\left(A_{n-1}A_{n-2}..A_1A_n\right)w_n = w_n$ if $w_n = n - \dfrac{1}{2}$ because

$$\begin{aligned}
& \left(A_{n-1}A_{n-2}..A_1A_n\right)\left(n - \frac{1}{2}\right) \\
&= A_{n-1}A_{n-2}..A_1(1) = A_{n-1}A_{n-2}..A_2\left(\frac{3}{2}\right) \\
&= A_{n-1}A_{n-2}..A_{i+1}\left(i + \frac{1}{2}\right) = ... \\
&= A_{n-1}A_{n-2}\left(n - \frac{5}{2}\right) = A_{n-1}\left(n - \frac{3}{2}\right) \\
&= n - \frac{1}{2}
\end{aligned}$$

since $n - \dfrac{3}{2} \in [n - \dfrac{7}{4}, n-1[$.

(b) Remark that for all $i=\overline{1,n-1}$ and $x_i \in \left[i-\frac{3}{4},i\right[$; $A_ix_i \in \left[(i+1)-\frac{3}{4},i+1\right[$ and $A_nx_n \in [\frac{1}{2},1]$ for all $x_n \in (n-\frac{3}{4},n)$, then there exists $w_i=i+\frac{1}{2}$ such that

$(A_{i-1}A_{i-2}..A_1A_n...A_i)\left(i+\frac{1}{2}\right)=i+\frac{1}{2}$ for all $i=1,2,...,n-1$.

Further, $A_nA_{n-1}...A_3A_2A_1$ is continuous in X_1 because if $x=\frac{1}{2}$ is the point of discontinuity for A_1 we have

$$\begin{aligned}
\lim_{x\to\frac{1}{2}^-} A_nA_{n-1}...A_3A_2A_1x &= A_nA_{n-1}...A_3A_2\left(\frac{3}{2}\right)= \\
A_nA_{n-1}...A_3\left(2+\frac{1}{2}\right) &= A_nA_{n-1}...A_4\left(3+\frac{1}{2}\right)= \\
A_nA_{n-1}...A_{i+1}A_i\left(i-\frac{1}{2}\right) &= ...= \\
&= A_nA_{n-1}\left(n-\frac{3}{2}\right)=A_n\left(n-\frac{1}{2}\right)=1 \\
&\text{and} \\
\lim_{x\to\frac{1}{2}^+} A_nA_{n-1}...A_3A_2A_1x &= A_nA_{n-1}...A_3A_2\left(\frac{3}{2}\right)= \\
A_nA_{n-1}...A_3\left(2+\frac{1}{2}\right) &= A_nA_{n-1}...A_4\left(3+\frac{1}{2}\right)= \\
A_nA_{n-1}...A_{i+1}A_i\left(i-\frac{1}{2}\right) &= ...=A_nA_{n-1}\left(n-\frac{3}{2}\right)=A_n\left(n-\frac{1}{2}\right)=1
\end{aligned}$$

(c) The inequalities $(2.i)$ for all $i=\overline{1,n}$ are satisfied since the value of the left hand side of each inequality is 1. In fact,

$$M_i\left(A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}x_{i+1},A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_ix_i,t\right)=1$$

for all $i=\overline{1,n-1}$ because

(1) if $x_i \in \left]i-1,i-\frac{3}{4}\right[$ we have

$$\begin{aligned}
A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_ix_i &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}\left(i+\frac{1}{4}\right), \\
&= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+2}\left((i+1)+\frac{1}{2}\right) \\
&\vdots \\
&= A_{i-1}A_{i-2}...A_1A_nA_{n-1}\left((n-2)+\frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right) \\
&= A_{i-1}A_{i-2}...A_2A_1(1), \\
&= A_{i-1}A_{i-2}...A_2\left(1+\frac{1}{2}\right)= \\
&\vdots \\
A_{i-1}\left((i-2)+\frac{1}{2}\right) &= (i-1)+\frac{1}{2}=i-\frac{1}{2}.
\end{aligned}$$

(2) if $x_i \in [i - \frac{3}{4}, i[$ we have

$$\begin{aligned}
A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_ix_i &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}\left(i+\frac{1}{2}\right), \\
&\vdots \\
&= A_{i-1}A_{i-2}...A_1A_nA_{n-1}\left((n-2)+\frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}...A_2A_1(1) = \\
&\vdots \\
A_{i-1}\left((i-2)+\frac{1}{2}\right) &= (i-1)+\frac{1}{2} = i-\frac{1}{2}.
\end{aligned}$$

(3) if $x_{i+1} \in \left]i, i+\frac{1}{4}\right[$ we get

$$\begin{aligned}
A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}x_{i+1} &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+2}\left((i+1)+\frac{1}{4}\right), \\
&\vdots \\
&= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}...A_2A_1(1) = \\
&\vdots \\
A_{i-1}\left((i-2)+\frac{1}{2}\right) &= (i-1)+\frac{1}{2} = i-\frac{1}{2}.
\end{aligned}$$

(4) if $x_{i+1} \in [i + \frac{1}{4}, i+1[$ we obtain

$$\begin{aligned}
A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}x_{i+1} &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+2}\left((i+1)+\frac{1}{2}\right), \\
&\vdots \\
&= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right), \\
&= A_{i-1}A_{i-2}...A_2A_1(1) = \\
&\vdots \\
A_{i-1}\left((i-2)+\frac{1}{2}\right) &= (i-1)+\frac{1}{2} = i-\frac{1}{2}.
\end{aligned}$$

Hence, all conditions of Theorem 2.1 are satisfied.

If we take $n = 5$ in Theorem 2.1, we get the following Corollary 2.3.

Corollary 2.3. *Let* (X_i, M_i, θ_i), $i = \overline{1,5}$ *be* 5 *fuzzy metric spaces,* $A_i : X_i \to X_{i+1}$, $i = \overline{1,4}$ *and* $A_5 : X_5 \to X_1$ *be* 5 *mappings satisfying*

$$\phi_1 \left(\begin{array}{c} M_1\left(A_5A_4A_3A_2x_2, A_5A_4A_3A_2A_1x_1, t\right), M_1\left(x_1, A_5A_4A_3A_2x_2, t\right), \\ M_1\left(x_1, A_5A_4A_3A_2A_1x_1, t\right), M_2\left(x_2, A_1x_1, t\right), \\ M_2\left(x_2, A_1A_5A_4A_3A_2x_2, t\right), M_2\left(A_1x_1, A_1A_5A_4A_3A_2x_2, t\right) \end{array} \right) > 0 \tag{5.1}$$

for all $x_1 \in X_{\ 1}$, $x_2 \in X_{\ 2}$ *with* $x_2 \neq A_1x_1$ *and for all* $t > 0$,

$$\phi_2 \left(\begin{array}{c} M_2\left(A_1A_5A_4A_3x_3, A_1A_5A_4A_3A_2x_2, t\right), \\ M_2\left(x_2, A_1A_5A_4A_3x_3, t\right), \\ M_2\left(x_2, A_1A_5A_4A_3A_2x_2, t\right), M_3\left(x_3, A_2x_2, t\right), \\ M_3\left(x_3, A_2A_1A_5A_4A_3x_3, t\right), \\ M_3\left(A_2x_2, A_2A_1A_5A_4A_3x_3, t\right) \end{array} \right) > 0 \tag{5.2}$$

for all $x_2 \in X_{\ 2}$, $x_3 \in X_{\ 3}$ *with* $x_3 \neq A_2x_2$ *and for all* $t > 0$,

$$\phi_3 \left(\begin{array}{c} M_3\left(A_2A_1A_5A_4x_4, A_2A_1A_5A_4A_3x_3, t\right), \\ M_3\left(x_3, A_2A_1A_5A_4x_4, t\right), \\ M_3\left(x_3, A_2A_1A_5A_4A_3x_3, t\right), M_4\left(x_4, A_3x_3, t\right), \\ M_4\left(x_4, A_3A_2A_1A_5A_4x_4, t\right), \\ M_4\left(A_3x_3, A_3A_2A_1A_5A_4x_4, t\right) \end{array} \right) > 0 \tag{5.3}$$

for all $x_3 \in X_{\ 2}$, $x_4 \in X_{\ 3}$ *with* $x_4 \neq A_3x_3$ *and for all* $t > 0$

$$\phi_4 \left(\begin{array}{c} M_4\left(A_3A_2A_1A_5x_5, A_3A_2A_1A_5A_4x_4, t\right), \\ M_4\left(x_4, A_3A_2A_1A_5x_5, t\right), \\ M_4\left(x_4, A_3A_2A_1A_5A_4x_4, t\right), M_5\left(x_5, A_4x_4, t\right), \\ M_5\left(x_5, A_4A_3A_2A_1A_5x_5, t\right), \\ M_5\left(A_4x_4, A_4A_2A_2A_1A_5x_5, t\right) \end{array} \right) > 0 \tag{5.4}$$

for all $x_4 \in X_{\ 4}$, $x_5 \in X_{\ 5}$ *with* $x_5 \neq A_4x_4$ *and for all* $t > 0$.

$$\phi_5 \left(\begin{array}{c} M_5\left(A_4A_3A_2A_1x_1, A_4A_3A_2A_1A_5x_5, t\right), \\ M_5\left(x_5, A_4A_3A_2A_1x_1, t\right), \\ M_5\left(x_5, A_4A_3A_2A_1A_5x_5, t\right), M_1\left(x_1, A_5x_5, t\right), \\ M_1\left(x_1, A_5A_4A_3A_2A_1x_1, t\right), \\ M_1\left(A_5x_5, A_5A_4A_3A_2A_1x_1, t\right) \end{array} \right) > 0 \tag{5.5}$$

for all $x_1 \in X_1$, $x_5 \in X_5$ *with* $x_1 \neq A_5x_5$ *and for all* $t > 0$.

Suppose that one of the following is true
(a_1) (X_1, M_1, θ_1) *is sequentially compact and* $A_5A_4A_3A_2A_1$ *is continuous on* X_1,

(a_2) (X_2, M_2, θ_2) is sequentially compact and $A_1A_5A_4A_3A_2$ is continuous on X_2,
(a_3) (X_3, M_3, θ_3) is sequentially compact and $A_2A_1A_5A_4A_3$ is continuous on X_3,
(a_4) (X_4, M_4, θ_4) is sequentially compact and $A_3A_2A_1A_5A_4$ is continuous on X_4,
(a_5) (X_5, M_5, θ_5) is sequentially compact and $A_4A_3A_2A_1A_5$ is continuous on X_5.
Then
(b_1) $A_5A_4A_3A_2A_1$ has a unique fixed point $w_1 \in X_1$,
(b_2) $A_1A_5A_4A_3A_2$ has a unique fixed point $w_2 \in X_2$,
(b_3) $A_2A_1A_5A_4A_3$ has a unique fixed point $w_3 \in X_3$,
(b_4) $A_3A_2A_1A_5A_4$ has a unique fixed point $w_4 \in X_4$,
(b_5) $A_4A_3A_2A_1A_5$ has a unique fixed point $w_5 \in X_5$,
Further, $A_1w_1 = w_2$, $A_2w_2 = w_3$, $A_3w_3 = w_4$, $A_4w_4 = w_5$ and $A_5w_5 = w_1$.

If we take $n = 4$ in Theorem 2.1, we obtain Theorem 2.1 of [15]
If we take $n = 3$ in Theorem 2.1, we obtain Theorem 2.3 of [14].
If we take $n = 2$ in Theorem 2.1, we obtain Theorem 2.1 of [2].
The following example illustrates our Corollary 2.3.

Example 2.4. Let $X_1 = [0,1]$, $X_2 = [1,2[$, $X_3 =]2,3]$, $X_4 = [3,4[$, $X_5 =]4,5]$ and let $M_i(x_i, x_{i+1}, t) = \dfrac{t}{t + |x_{i+1} - x_i|}$ for all $i = \overline{1,4}$ and $M_5(x_5, x_1, t) = \dfrac{t}{t + |x_5 - x_1|}$. Define $A_1 : X_1 \to X_2$, $A_2 : X_2 \to X_3$, $A_3 : X_3 \to X_4$, $A_4 : X_4 \to X_5$ and $A_5 : X_5 \to X_1$ by

$$A_1x_1 = \begin{cases} 1 \text{ if } x_1 \in [0, \frac{3}{4}[\\ \frac{3}{2} \text{ if } x_1 \in [\frac{3}{4}, 1] \end{cases}, \quad A_2x_2 = \begin{cases} \frac{5}{2} \text{ if } x_2 \in [1, \frac{3}{2}[\\ 3 \text{ if } x_2 \in [\frac{3}{2}, 2[\end{cases}$$

$$A_3x_3 = \begin{cases} \frac{13}{4} \text{ if } x_3 \in]2, \frac{5}{2}[\\ \frac{7}{2} \text{ if } x_3 \in [\frac{5}{2}, 3] \end{cases}, \quad A_4x_4 = \begin{cases} \frac{17}{4} \text{ if } x_4 \in [3, \frac{7}{2}[\\ \frac{9}{2} \text{ if } x_4 \in [\frac{7}{2}, 4[\end{cases}$$

$$A_5x_5 = \begin{cases} \frac{3}{4} \text{ if } x_5 \in]4, \frac{9}{2}[\\ 1 \text{ if } x_5 \in [\frac{9}{2}, 5] \end{cases}$$

Let $\phi_1(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5$.

Here X_1 is compact, but the others spaces are not compact. Further the inequalities $(5.1), (5.2), (5.3), (5.4), (5.5)$ are satisfied since the left hand side of each inequality is 1 and $A_5A_4A_3A_2A_1$ is continuous in X_1 because if $x = \frac{3}{4}$, the point of discontinuity for A_1, we get

$$\begin{aligned}\lim_{x\to\frac{3}{4}^-} A_5A_4A_3A_2A_1x &= A_5A_4A_3A_2(1) = A_5A_4A_3\left(\frac{5}{2}\right)\\ &= A_5A_4\left(\frac{7}{2}\right) = A_5\left(\frac{9}{2}\right) = 1\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\frac{3}{4}^+} A_5A_4A_3A_2A_1x &= A_5A_4A_3A_2\left(\frac{3}{2}\right) = A_5A_4A_3(3)\\ &= A_5A_4\left(\frac{7}{2}\right) = A_5\left(\frac{9}{2}\right) = 1\end{aligned}$$

We have

$$\begin{aligned}A_5A_4A_3A_2A_1(1) &= 1\\ A_1A_5A_4A_3A_2\left(\frac{3}{2}\right) &= \frac{3}{2}\\ A_2A_1A_5A_4A_3\left(\frac{5}{2}\right) &= \frac{5}{2}\\ A_3A_2A_1A_5A_4\left(\frac{7}{2}\right) &= \frac{7}{2}\\ A_4A_3A_2A_1A_5\left(\frac{9}{2}\right) &= \frac{9}{2}.\end{aligned}$$

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Faycel Merghadi, Department of Mathematics, University of Tebessa, 12000, Algeria
E-mail address: faycel_mr@yahoo.fr

Abdelkrim Aliouche*, Department of Mathematics, University of Larbi Ben M'Hidi, Oum-El-Bouaghi, 04000, Algeria
E-mail address: alioumath@yahoo.fr

*Corresponding author