

## BEST SIMULTANEOUS APPROXIMATION IN FUZZY NORMED SPACES

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ABSTRACT. The main purpose of this paper is to consider the  $t$ -best simultaneous approximation in fuzzy normed spaces. We develop the theory of  $t$ -best simultaneous approximation in quotient spaces. Then, we discuss the relationship in  $t$ -proximality and  $t$ -Chebyshevity of a given space and its quotient space.

### 1. Introduction

The concept of fuzzy topology has important applications in quantum particle physics, in particular in connection with both string and E-infinity theory see MS EL Naschie ([3],[4],[5],[13]). One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric and fuzzy normed spaces. The problem of fuzzy metric spaces has been introduced by Kramosil and Michalek [8] and improved by George and Veeramani[6]. Many mathematicians have considered the notion of fuzzy normed spaces from different points of view see ([1],[2],[12]). S. M. Vaezpour and F. Karimi have introduced the concept of  $t$ -best approximation in fuzzy normed spaces in [14]. In this paper we consider the set of all  $t$ -best simultaneous approximation in fuzzy normed spaces and we use the concept of simultaneous  $t$ -proximality and simultaneous  $t$ -Chebyshevity to introduce the theory of  $t$ -best simultaneous approximation in quotient spaces.

**Definition 1.1.** A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is said to be a continuous  $t$ -norm if  $([0, 1], *)$  is a topological monoid with unit 1 such that  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  ( $a, b, c, d \in [0, 1]$ ). We call  $*_1 \geq *_2$  if  $a *_1 b \geq a *_2 b$  for all  $a, b \in [0, 1]$ .

**Definition 1.2.** [12] The 3-tuple  $(X, N, *)$  is said to be a fuzzy normed space if  $X$  is a vector space,  $*$  is a continuous  $t$ -norm and  $N$  is a fuzzy set on  $X \times (0, \infty)$  satisfying the following conditions for every  $x, y \in X$  and  $s, t > 0$

- (i)  $N(x, t) > 0$
- (ii)  $N(x, t) = 1 \iff x = 0$

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Received: August 2009; Revised: November 2009; Accepted: December 2009

*Key words and phrases:*  $t$ -best simultaneous approximation,  $t$ -proximality,  $t$ -Chebyshevity, Quotient spaces.

(iii)  $N(\alpha x, t) = N(x, t/|\alpha|)$  for all  $\alpha \neq 0$

(iv)  $N(x, t) * N(y, s) \leq N(x + y, t + s)$

(v)  $N(x, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous

(vi)  $\lim_{t \rightarrow \infty} N(x, t) = 1$ .

**Lemma 1.3.** [12] Let  $N$  be a fuzzy norm. Then:

(i)  $N(x, t)$  is non decreasing with respect to  $t$  for each  $x \in X$ .

(ii)  $N(y - x, t) = N(x - y, t)$ .

**Example 1.4.** [12] Let  $(X, \|\cdot\|)$  be a normed space. We define  $a * b = ab$  or  $a * b = \min\{a, b\}$  and

$$N(x, t) = \frac{kt^n}{kt^n + m\|x\|}, \quad k, m, n \in \mathbb{R}^+.$$

Then  $(X, N, *)$  is a fuzzy normed space. In particular if  $k = m = n = 1$  we have

$$N(x, t) = \frac{t}{t + \|x\|},$$

which is called the standard fuzzy norm induced by the norm  $\|\cdot\|$ .

**Remark 1.5.** In [12], it was shown that every fuzzy norm induces a fuzzy metric and so every fuzzy normed space is a topological space.

**Definition 1.6.** [12] Let  $(X, N, *)$  be a fuzzy normed space. The open and closed ball  $B(x, r, t)$  and  $B[x, r, t]$  with the center  $x \in X$ , radius  $0 < r < 1$  and  $t > 0$  are defined as follows:

$$B(x, r, t) = \{y \in X : N(x - y, t) > 1 - r\}.$$

$$B[x, r, t] = \{y \in X : N(y - x, t) \geq 1 - r\}.$$

**Proposition 1.7.** [9] Let  $(X, M, *)$  be a fuzzy metric space. Then  $M$  is a continuous function on  $X \times X \times (0, \infty)$ .

**Corollary 1.8.** Let  $(X, N, *)$  be a fuzzy normed space. Then  $N : X \times (0, \infty) \rightarrow [0, 1]$  is continuous.

**Remark 1.9.** [6] For any  $r_1 > r_2$ , we can find  $r_3$  such that  $r_1 * r_3 \geq r_2$  and for any  $r_4$  we can find  $r_5$  such that  $r_5 * r_5 \geq r_4$ ,  $(r_1, r_2, r_3, r_4, r_5 \in (0, 1))$ .

## 2. t-best Simultaneous Approximation

**Definition 2.1.** Let  $(X, N, *)$  be a fuzzy normed space. A subset  $A \subseteq X$  is called F-bounded if there exists  $t > 0$  and  $0 < r < 1$  such that  $N(x, t) > 1 - r$  for all  $x \in A$ .

**Definition 2.2.** Let  $(X, N, *)$  be a fuzzy normed space,  $W$  be a subset of  $X$  and  $M$  be a F-bounded subset in  $X$ . For  $t > 0$ , we define,

$$d(M, W, t) = \sup_{w \in W} \inf_{m \in M} N(m - w, t).$$

An element  $w_0 \in W$  is called a t-best simultaneous approximation to  $M$  from  $W$  if for  $t > 0$ ,

$$d(M, W, t) = \inf_{m \in M} N(m - w_0, t).$$

The set of all t-best simultaneous approximations to  $M$  from  $W$  will be denoted by  $S_W^t(M)$  and we have,

$$S_W^t(M) = \left\{ w \in W : \inf_{m \in M} N(m - w, t) = d(M, W, t) \right\}.$$

**Definition 2.3.** Let  $W$  be a subset of  $(X, N, *)$ . It is called a simultaneous t-poximinal subset of  $X$  if for each F-bounded set  $M$  in  $X$ , there exists at least one t-best simultaneous approximation from  $W$  to  $M$ . Also it is called a simultaneous t-Chebyshev subset of  $X$  if for each F-bounded set  $M$  in  $X$  there exists a unique simultaneous t-best approximation from  $W$  to  $M$ .

**Definition 2.4.** Let  $(X, N, *)$  be a fuzzy normed space. A subset  $E$  of  $X$  is said to be convex if  $(1 - \lambda)x + \lambda y \in E$  whenever  $x, y \in E$  and  $0 < \lambda < 1$ .

As an example of a convex set we have the following lemma.

**Lemma 2.5.** Every open ball in  $(X, N, Min)$  is convex.

**Theorem 2.6.** Suppose that  $W$  is a subset of  $(X, N, *)$  and  $M$  is F-bounded in  $X$ . Then  $S_W^t(M)$  is a F-bounded subset of  $X$  and if  $W$  is convex and is a closed subset of  $X$  and  $*$  has the condition  $a * b \geq a$  for all  $a, b \in [0, 1]$ , then  $S_W^t(M)$  is closed and is convex for each F-bounded subspace  $M$  of  $X$ .

*Proof.* Since  $M$  is F-bounded, there exist  $t > 0$  and  $0 < r < 1$  such that  $N(x, t) > 1 - r$ , for all  $x \in M$ . If  $w \in S_W^t(M)$ , then

$$\inf_{m \in M} N(m - w, t) = d(M, W, t).$$

Now, for all  $m \in M$  and  $w \in S_W^t(M)$ ,

$$\begin{aligned} N(w, 2t) &\geq N(w - m, t) * N(m, t) \\ &\geq \inf_{m \in M} N(w - m, t) * N(m, t) \\ &\geq d(M, W, t) * (1 - r) \\ &\geq 1 - r_0, \end{aligned}$$

for some  $0 < r_0 < 1$ . Then  $S_W^t(M)$  is F-bounded. Suppose that  $W$  is convex and is a closed subset of  $X$ . We show that  $S_W^t(M)$  is convex and closed. Let  $x, y \in S_W^t(M)$  and  $0 < \lambda < 1$ . Since  $W$  is convex, there exists  $z_\lambda \in W$  such that  $z_\lambda = \lambda x + (1 - \lambda)y$ , for each  $0 < \lambda < 1$ . Now for  $t > 0$  we have,

$$\begin{aligned} \inf_{m \in M} N((\lambda x + (1 - \lambda)y) - m, t) &= \inf_{m \in M} N(z_\lambda - m, t) \\ &\leq d(M, W, t). \end{aligned}$$

On the other hand, for a given  $t > 0$ , take the natural number  $n$  such that  $t > \frac{1}{n}$ . By assumptions and Definition 1.2, we have,

$$\begin{aligned} \inf_{m \in M} N(\lambda x + (1 - \lambda)y - m, t) &= \inf_{m \in M} N(\lambda(x - y) + y - m, t) \\ &\geq \inf_{m \in M} N(x - y, \frac{1}{\lambda n}) * N(y - m, t - \frac{1}{n}) \\ &= N(x - y, \frac{1}{\lambda n}) * \inf_{m \in M} N(y - m, t - \frac{1}{n}) \\ &\geq \lim_{n \rightarrow \infty} (\inf_{m \in M} N(y - m, t - \frac{1}{n})) \\ &= d(M, W, t). \end{aligned}$$

So  $S_W^t(M)$  is convex. Finally let  $\{w_n\} \subset S_W^t(M)$  and suppose  $\{w_n\}$  converges to some  $w$  in  $X$ . Since  $\{w_n\} \subset W$  and  $W$  is closed so  $w \in W$ . Therefore by Corollary 1.8 for  $t > 0$  we have,

$$\begin{aligned} \inf_{m \in M} N(m - w, t) &= \inf_{m \in M} N(\lim_{n \rightarrow \infty} w_n - m, t), \\ &= \lim_{n \rightarrow \infty} \inf_{m \in M} N(w_n - m, t), \\ &= d(M, W, t). \end{aligned}$$

□

**Theorem 2.7.** *The following assertions are hold for  $t > 0$ ,*

- (i)  $d(M + x, W + x, t) = d(M, W, t), \forall x \in \mathbb{C}$ ,
- (ii)  $d(\lambda M, \lambda W, t) = d(M, W, \frac{t}{|\lambda|}), \forall \lambda \in \mathbb{C}$ ,
- (iii)  $S_{W+x}^t(M + x) = S_W^t(M) + x, \forall x \in X$ ,
- (iv)  $S_{\lambda W}^{|\lambda|t}(\lambda M) = \lambda S_W^t(M), \forall \lambda \in \mathbb{C}$ .

*Proof.* (i)

$$\begin{aligned} d(M + x, W + x, t) &= \sup_{w \in W} \inf_{m \in M} N(m + x - (w + x), t) \\ &= \sup_{w \in W} \inf_{m \in M} N(m - w, t) \\ &= d(M, W, t). \end{aligned}$$

(ii) Clearly equality holds for  $\lambda = 0$ , so suppose that  $\lambda \neq 0$ . Then,

$$\begin{aligned} d(\lambda M, \lambda W, t) &= \sup_{w \in W} \inf_{m \in M} N(\lambda(m - t), t) \\ &= \sup_{w \in W} \inf_{m \in M} N(m - w, \frac{t}{|\lambda|}) \\ &= d(M, W, t). \end{aligned}$$

(iii)  $x + W \in S_{W+x}^t(M + x)$  if and only if,

$$\inf_{m+x \in M+x} N(m + x - w - x, t) = d(M + x, W + x, t),$$

and by (i), the above equality holds if and only if,

$$\inf_{m \in M} N(m - w, t) = d(M, W, t),$$

for all  $w \in W$  and this shows that  $w \in S_W^t(M)$ . So  $x + w \in S_W^t(M) + x$ .

(iv)  $y_0 \in S_{\lambda W}^{|\lambda|t}(\lambda M)$  if and only if  $y_0 \in \lambda W$  and,

$$\begin{aligned} d(\lambda W, \lambda M, |\lambda|t) &= \inf_{\lambda m \in \lambda M} N(y_0 - \lambda M, |\lambda|t) \\ &= \inf_{m \in M} N(\frac{y_0}{\lambda} - m, t). \end{aligned}$$

But by(ii),

$$d(\lambda M, \lambda W, |\lambda|t) = d(W, M, t).$$

So we have  $\frac{y_0}{\lambda} \in W$  and  $d(M, W, t) = \inf_{m \in M} N(\frac{y_0}{\lambda} - m, t)$  or equivalently  $\frac{y_0}{\lambda} \in S_W^t(M)$  and the proof is completed.  $\square$

**Corollary 2.8.** *Let  $A$  be a non empty subset of a fuzzy normed space  $(X, N, *)$ . The following statements are hold.*

- (i)  *$A$  is simultaneous  $t$ -proximinal (resp. simultaneous  $t$ -Chebyshev) if and only if  $A + y$  is simultaneous  $t$ -proximinal (resp. simultaneous  $t$ -Chebyshev), for each  $y \in X$ ,*
- (ii)  *$A$  is simultaneous  $t$ -proximinal (resp. simultaneous  $t$ -Chebyshev) if and only if  $\alpha A$  is simultaneous  $|\alpha|$ - $t$ -proximinal (resp.  $|\alpha|$ - $t$ -Chebyshev), for each  $\alpha \in \mathbb{C}$ .*

**Corollary 2.9.** *Let  $M$  be a nonempty subspace of  $X$  and  $N$  be a  $F$ -bounded subset of  $X$ . Then for  $t > 0$ ,*

- (i)  *$d(M, N + y, t) = d(M, N, t), \forall y \in M$ ,*
- (ii)  *$S_M^t(N + y) = S_M^t(N) + y, \forall y \in M$ ,*
- (iii)  *$d(M, \alpha N, |\alpha| t) = d(M, N, t)$ , for  $0 \neq \alpha \in \mathbb{C}$ ,*
- (iv)  *$S_M^{|\alpha|t}(\alpha N) = \alpha S_M^t(N)$ , for  $0 \neq \alpha \in \mathbb{C}$ .*

### 3. Simultaneous $t$ -proximality and Simultaneous $t$ -Chebishevity in Quotient Spaces

In this section we give characterizations of simultaneous  $t$ -proximality and simultaneous  $t$ -Chebishevity in quotient spaces. First we remind that if  $(X, N, *)$  is a fuzzy normed space and  $M$  is a linear manifold in  $X$ , for  $t > 0$ , we define,

$$N(x + M, t) = \sup_{y \in M} N(x + y, t).$$

It has been proved in [12] that  $N$  is a fuzzy norm on  $X/M$ . Also  $Q : X \rightarrow X/M$  is the natural map,  $Qx = x + M$  and the followings hold,

- (i)  $N(Qx, t) \geq N(x, t)$ .
- (ii) If  $(X, N, *)$  is a fuzzy Banach space then  $(X/M, N, *)$  is a fuzzy Banach space.

Let  $A$  be a nonempty set in a fuzzy normed space  $(X, N, *)$ . For  $x \in X$  and  $t > 0$ , we shall denote  $P_A^t(x)$  the set of all elements of  $t$ -best approximation to  $x$  from  $A$ . i.e.

$$P_A^t(x) = \{y \in A : d(A, x, t) = N(y - x, t)\},$$

where,

$$d(A, x, t) = \sup_{y \in A} N(y - x, t).$$

If each  $x \in X$  has at least (resp. exactly) one  $t$ -best approximation in  $A$ , then  $A$  is called a  $t$ -proximinal (resp.  $t$ -Chebyshev) set.

**Lemma 3.1.** *Let  $(X, N, *)$  be a fuzzy normed space and  $M$  be a  $t$ -proximinal subspace of  $X$ . For each nonempty  $F$ -bounded set  $S$  in  $X$  and  $t > 0$ ,*

$$d(S, M, t) = \inf_{s \in S} \sup_{m \in M} N(s - m, t).$$

*Proof.* Since  $M$  is  $t$ -proximal it follows that for each  $s \in S$  there exists  $m_s \in P_M^t(S)$  such that for  $t > 0$ ,

$$N(s - m_s, t) = \sup_{m \in M} N(s - m, t).$$

So,

$$\begin{aligned} d(S, M, t) &= \sup_{m \in M} \inf_{s \in S} N(s - m, t) \\ &\geq \inf_{s \in S} N(s - m_s, t), \\ &= \inf_{s \in S} \sup_{m \in M} N(s - m, t) \\ &\geq \sup_{m \in M} \inf_{s \in S} N(s - m, t) \\ &= d(S, M, t). \end{aligned}$$

This implies that,

$$d(S, M, t) = \inf_{s \in S} \sup_{m \in M} N(s - m, t).$$

□

**Example 3.2.** Let  $(X = \mathbb{R}^2, \|\cdot\|)$  be the ordinary normed space and consider  $(X, N, *)$  as its standard induced fuzzy normed space (Example 1.4), where  $a * b = ab$  for each  $a, b \in [0, 1]$ . According to Lemma 1.18 of [7], a nonempty subset  $S$  of  $X$  is  $F$ -bounded if and only if  $S$  is bounded in  $(X, \|\cdot\|)$ . If we take  $M = \mathbb{R}$  we can easily prove that  $M$  is proximal in  $(X, \|\cdot\|)$ . Now perception of details in the above Lemma is straightforward.

**Lemma 3.3.** Let  $(X, N, *)$  be a fuzzy normed space,  $M$  a  $t$ -proximal subspace of  $X$  and  $S$  be an arbitrary subset of  $X$ . The following assertions are equivalent:

- (i)  $S$  is a  $F$ -bounded subset of  $X$ .
- (ii)  $S/M$  is a  $F$ -bounded subset of  $X/M$ .

*Proof.* Suppose that  $S$  be a  $F$ -bounded subset of  $X$ . Then there exist  $t > 0$ ,  $0 < r < 1$  such that,

$$N(x, t) > 1 - r. (\forall x \in S)$$

But,

$$\begin{aligned} N(x + M, t) &= \sup_{y \in M} N(x + y, t) \\ &\geq N(x, t) \\ &\geq 1 - r. \end{aligned}$$

So (i)  $\rightarrow$  (ii) is proved. (ii)  $\rightarrow$  (i). Let  $S/M$  be a  $F$ -bounded subset of  $X/M$ . Since  $M$  is  $t$ -proximal, then for each  $s \in S$  there exists  $m_s \in M$  such that,  $m_s \in P_M^t(S)$ . So for each  $s \in S$ ,

$$N(s - m_s, t) = \sup_{m \in M} N(s - m, t). \tag{1}$$

Now from Lemma 3.1 we conclude that for  $t > 0$ ,

$$\begin{aligned} \inf_{s \in S} N(s - m_s, t) &= \inf_{s \in S} \sup_{m \in M} N(s - m, t) \\ &= \sup_{m \in M} \inf_{s \in S} N(s - m, t). \end{aligned}$$

Then for  $0 < r < 1$  such that  $\inf_{s \in S} N(s - m_s, t) \geq r$  and  $t > 0$  there exists  $m_r \in M$  such that,

$$\begin{aligned} \inf_{s \in S} N(s - m_r, t) &\geq \inf_{s \in S} N(s - m_s, t) - r \\ &\geq 0. \end{aligned}$$

So by (1), for all  $s \in S$  we have,

$$\begin{aligned}
 N(s, t) &\geq N(s - m_r, \frac{t}{2}) * N(m_r, \frac{t}{2}) \\
 &\geq \inf_{s \in S} N(s - m_r, \frac{t}{2}) * N(m_r, \frac{t}{2}) \\
 &\geq (\inf_{s \in S} N(s - m_s, \frac{t}{2}) - r) * N(m_r, \frac{t}{2}) \\
 &= (\inf_{s \in S} \sup_{m \in M} N(s - m, \frac{t}{2}) - r) * N(m_r, \frac{t}{2}) \\
 &= (\inf_{s \in S} N(s + M, \frac{t}{2}) - r) * N(m_r, \frac{t}{2}).
 \end{aligned} \tag{2}$$

Since  $S/M$  is F-bounded, by its definition and remark 1.9, we can find  $0 < r_0 < 1$  such that the last equation in the right hand side of (2) be greater than or equal to  $1 - r_0$  and this completes the proof.  $\square$

**Lemma 3.4.** *Let  $M$  be a  $t$ -proximal subspace of  $(X, N, *)$  and  $W \supseteq M$  a subspace of  $X$ . Let  $K$  be F-bounded in  $X$ . If  $w_0 \in S_W^t(K)$ , then  $w_0 + M \in S_{W/M}^t(K/M)$ .*

*Proof.* Since  $K$  is F-bounded by Lemma 3.3,  $K/M$  is F-bounded in  $X/M$ . Assume that  $w_0 \in S_W^t(K)$  and  $w_0 + M$  not in  $S_{W/M}^t(K/M)$ . Thus there exists  $w' \in W$  such that for  $t > 0$ ,

$$\begin{aligned}
 \inf_{k \in K} N(k - (w' + M), t) &> \inf_{k \in K} N(k - (w_0 + M), t) \\
 &\geq \inf_{k \in K} N(k - w_0, t) \\
 &= d(K, W, t).
 \end{aligned} \tag{3}$$

On the other hand for each  $k \in K$  and for  $t > 0$ ,

$$N(k - (w' + M), t) = \sup_{m \in M} N(k - (w' + m), t).$$

Then for each  $0 < \varepsilon < 1$  and  $k \in K$  there exists  $m_k \in M$  such that for  $t > 0$ ,

$$N(k - (w' + m_k), t) \geq N(k - (w' + M), t) - \varepsilon.$$

Since  $w' + m_k \in W$  we conclude that

$$\begin{aligned}
 d(K, W, t) &\geq \inf_{k \in K} N(k - (w' + m_k), t) \\
 &\geq \inf_{k \in K} N(k - (w' + M), t) - \varepsilon.
 \end{aligned}$$

Thus,

$$d(K, W, t) \geq \inf_{k \in K} N(k - (w' + M), t). \tag{4}$$

By (3) and (4),

$$\begin{aligned}
 d(K, W, t) &\geq \inf_{k \in K} N(k - (w' + M), t) \\
 &> d(K, W, t),
 \end{aligned}$$

and this is a contradiction. Therefore,  $w_0 + M \in S_{W/M}^t(K/M)$  and the proof is completed.  $\square$

The mentioned Lemma has two following corollaries:

**Corollary 3.5.** *Let  $M$  be a  $t$ -proximal subspace of  $(X, N, *)$  and  $W \supseteq M$  a subspace of  $X$ . If  $W$  is simultaneous  $t$ -proximal then  $W/M$  is a simultaneous  $t$ -proximal subspace of  $X/M$ .*

**Corollary 3.6.** *Let  $M$  be a  $t$ -proximal subspace of  $(X, N, *)$  and  $W \supseteq M$  subspace of  $X$ . If  $W$  is simultaneous  $t$ -proximal then for each  $F$ -bounded set  $K$  in  $X$ ,*

$$Q(S_W^t(K)) \subseteq S_{W/M}^t(K/M).$$

**Theorem 3.7.** *Let  $M$  be a  $t$ -proximal subspace of  $(X, N, *)$  and  $W \supseteq M$  subspace of  $X$ . If  $K$  is a  $F$ -bounded set in  $X$  such that  $w_0 + M \in S_{W/M}^t(K/M)$  and  $m_0 \in S_M^t(K - w_0)$ , then  $w_0 + m_0 \in S_M^t(K)$ .*

*Proof.* In view of Lemma 3.1 for  $t > 0$  we have,

$$\begin{aligned} \inf_{k \in K} N((k - w_0) - m_0, t) &= \sup_{m \in M} \inf_{k \in K} N((k - w_0) - m, t) \\ &= \inf_{k \in K} \sup_{m \in M} N(k - (w_0 + m), t) \\ &= \inf_{k \in K} N(k - (w_0 + M), t) \\ &\geq \inf_{k \in K} N(k - (w + M), t), \forall w \in W \\ &\geq \inf_{k \in K} N(k - w, t), \forall w \in W. \end{aligned}$$

Hence,

$$\inf_{k \in K} N((k - (w_0 + m_0)), t) \geq \inf_{k \in K} N(k - w, t), \forall w \in W.$$

But  $w_0 + m_0 \in W$ . Then  $w_0 + m_0 \in S_W^t(K)$  and so the proof is complete.  $\square$

**Theorem 3.8.** *Let  $M$  be a  $t$ -proximal subspace of  $(X, N, *)$ ,  $W \supseteq M$  a simultaneous  $t$ -proximal subspace of  $X$ . Then for each  $F$ -bounded set  $K$  in  $X$ ,*

$$Q(S_W^t(K)) = S_{W/M}^t(K/M).$$

*Proof.* By Corollary 3.6 we obtain,

$$Q(S_W^t(K)) \subseteq S_{W/M}^t(K/M).$$

Also by Lemma 3.3,  $W/M$  is simultaneous  $t$ -proximal in  $X/M$ . Now let,

$$w_0 + M \in S_{W/M}^t(K/M),$$

where  $w_0 \in W$ . By simultaneous  $t$ -proximality of  $M$  there exists  $m_0 \in M$  such that  $m_0 \in S_M^t(K - w_0)$ . Then in view of Theorem 3.7 we conclude that  $w_0 + m_0 \in S_W^t(K)$ . Therefore  $w_0 + M \in Q(S_W^t(K))$  and the proof is complete.  $\square$

**Corollary 3.9.** *Let  $W$  and  $M$  be subspaces of  $(X, N, *)$ . If  $M$  is simultaneous  $t$ -proximal then the following assertions are equivalent:*

- (i)  $W/M$  is simultaneous  $t$ -proximal in  $X/M$ .
- (ii)  $W+M$  is simultaneous  $t$ -proximal in  $X$ .

*Proof.* (i)  $\rightarrow$  (ii). Let  $K$  be an arbitrary  $F$ -bounded set in  $X$ . Then by Lemma 3.3,  $K/M$  is a  $F$ -bounded set in  $X/M$ . Since  $(W + M)/M = W/M$  and  $M$  are simultaneous  $t$ -proximal it follows that there exists  $w_0 + M \in (W + M)/M$  and  $m_0 \in M$  such that  $w_0 + M \in S_{(W+M)/M}^t(K/M)$  and  $m_0 \in S_M^t(K - w_0)$ . By Theorem 3.7  $w_0 + m_0 \in S_{W+M}^t(K)$ . This shows that  $W + M$  is simultaneous  $t$ -proximal in  $X$ .

(ii)  $\rightarrow$  (i). Since  $W + M$  is simultaneous  $t$ -proximal and  $W + M \supseteq M$ , by Corollary 3.5  $(W + M)/M = W/M$  is simultaneous  $t$ -proximal.  $\square$



**Theorem 3.10.** *Let  $W$  and  $M$  be subspaces of  $(X, N, *)$ . If  $M$  is simultaneous  $t$ -Chebyshev then the following assertions are equivalent:*

- (i)  $W/M$  is simultaneous  $t$ -Chebyshev in  $X/M$ .
- (ii)  $W+M$  is simultaneous  $t$ -Chebyshev in  $X$ .

*Proof.* (i)  $\rightarrow$  (ii). By hypothesis  $(W+M)/M = W/M$  is simultaneous  $t$ -Chebyshev. Assume that (ii) is false. Then some  $F$ -bounded subset  $K$  of  $X$  has two distinct simultaneous  $t$ -best approximations such as  $l_0$  and  $l_1$  in  $W + M$ . Thus we have,

$$l_0, l_1 \in S_{W+M}^t(K). \tag{5}$$

Since  $W + M \supseteq M$  by Lemma 3.3,

$$l_0 + M, l_1 + M \in S_{(W+M)/M}^t(K/M) = S_{W/M}^t(K/M).$$

Since  $W/M$  is simultaneous  $t$ -Chebyshev,  $l_0 + M = l_1 + M$ . So there exists  $0 \neq m_0 \in M$  such that  $l_1 = l_0 + m_0$ . By (5) for all  $t > 0$ ,

$$\begin{aligned} \inf_{k \in K} N((k - l_0) - m_0, t) &= \inf_{k \in K} N(k - l_1, t) \\ &= \inf_{k \in K} N(k - l_0, t) \\ &= d(K, W + M, t) \\ &= d(K - l_0, W + M, t) \\ &\geq d(K - l_0, M, t). \end{aligned}$$

This shows that both  $m$  and zero are simultaneous  $t$ -best approximations to  $S - l_0$  form  $M$  and this is a contradiction.

(ii)  $\rightarrow$  (i). Assume that (i) does not hold. Then for some  $F$ -bounded subset  $K$  of  $X$ ,  $K/M$  has two distinct simultaneous  $t$ -best approximations such as  $w + M$  and  $w' + M$  in  $W/M$ . Thus  $w - w'$  is not in  $M$ . Since  $M$  is simultaneous  $t$ -proximal there exists simultaneous  $t$ -best approximations  $m$  and  $m'$  to  $K - w$  and  $K - w'$  form  $M$ , respectively. Therefore  $m \in S_M^t(K - w)$  and  $m' \in S_M^t(K - w')$ . Since  $W + M \supseteq M$ ,  $w + M$  and  $w' + M$  are in  $S_{W/M}^t(K/M) = S_{(K+M)/M}^t(K/M)$ , by Theorem 3.7,  $w + m$  and  $w' + m' \in S_{W+M}^t(K)$ . But  $W + M$  is simultaneous  $t$ -Chebyshev. Thus  $w + m = w' + m'$  and so  $w - w'$  belongs to  $M$ , which is a contradiction.  $\square$

**Corollary 3.11.** *Let  $M$  be simultaneous  $t$ -Chebyshev subspace of  $(X, N, *)$ . If  $W \supseteq M$  is a simultaneous  $t$ -Chebyshev subspace in  $X$ , then the following assertions are equivalent:*

- (i)  $W$  is simultaneous  $t$ -Chebyshev in  $X$ .
- (ii)  $W/M$  is simultaneous  $t$ -Chebyshev in  $X/M$ .

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