GENERALIZED FUZZY VALUED θ-CHOQUET INTEGRALS AND THEIR DOUBLE-NULL ASYMPTOTIC ADDITIVITY

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Abstract. The generalized fuzzy valued θ-Choquet integrals will be established for the given μ-integrable fuzzy valued functions on a general fuzzy measure space, and the convergence theorems of this kind of fuzzy valued integral are being discussed. Furthermore, the whole of integrals is regarded as a fuzzy valued set function on measurable space, the double-null asymptotic additivity and pseudo-double-null asymptotic additivity of the fuzzy valued set functions formed are studied when the fuzzy measure satisfies autocontinuity from above (below).

1. Introduction

Since Sugeno [4] first introduced the concept of the fuzzy measure and the fuzzy integral in 1974, theory of fuzzy measure has made a tremendous development, and was widely applied to fields of subjective judge process and so on. It is well known, fuzzy measure and fuzzy integral do not satisfy general additivity, so that it is difficult to construct some theory systems similar to the ones in classical measure theory without general additional condition. To solve this problem, Wang [5] proposed some important concepts for the first time in 1984, which are autocontinuity and null-additivity with respect to the set functions. Meanwhile, some theoretical frameworks describing the convergence properties of the sequences of measurable functions and fuzzy integrals on fuzzy measure space were founded. Furthermore, he went on presenting the concepts of pseudo-autocontinuity and pseudo-null-additivity in [6].

In 1996, by means of classical Lebesgue integrals and Choquet integrals, Wang [7] defined a new monotone set function, which discussed systematically their some hereditary properties and structural characteristics with respect to fuzzy measure in the aspect of autocontinuity. In 1999, aiming at a class of μ-integral fuzzy valued functions, a new fuzzy valued functional (fuzzy integral) were established for the first time in [8], a series of convergence theorems were obtained. In recent years, Wang [9] defined the generalized fuzzy-valued Choquet integral for a given μ-integral fuzzy valued functions, and regarding the integral as fuzzy valued set

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function. Furthermore, some hereditary properties of this integrals, such as autocontinuity and uniform-autocontinuity etc. were studied in [10-11].

In this paper, we establish generalized fuzzy valued θ-Choquet integrals, and discuss some convergence theorems with respect to them. By regarding this kind of integral as a set function taken by fuzzy number, we study the double-null asymptotic additivity and pseudo-double-null asymptotic additivity of the set functions formed by this kind of fuzzy integral. There is no doubt that these achievements have an important effect on enriching the theory of fuzzy integrals.

2. Preliminaries

Let X be an arbitrary nonempty set and $\Re$ be a σ-algebra of measurable subsets of the set $X, (X, \Re)$ denotes an arbitrary given measurable space, $\mathrm{R}^+ = [0, +\infty)$, we define the interval number on R^+ as $I_{R^+} = \{\bar{a} = [a^-, a^+] \mid a^- \leq a^+, a^-, a^+ \in \mathrm{R}^+\}$.

Definition 2.1. Let $\widetilde{A}$ be a mapping from R^+ to $[0, 1], \widetilde{A}$ is called a fuzzy number on R^+, if the following conditions (1)-(2) are satisfied:

(1) There exists $x_0 \in \mathrm{R}^+$ such that $\widetilde{A}(x_0) = 1$;

(2) For an arbitrary $\lambda \in (0, 1]$, its cut-set $A_\lambda = \{x \in \mathrm{R}^+ \mid \widetilde{A}(x) \geq \lambda\}$ is a closed interval, which is denoted as $(\widetilde{A})_\lambda = A_\lambda = [A_\lambda^-, A_\lambda^+]$.

Throughout this paper, we always let $F(\mathrm{R}^+)$ denote the set of all fuzzy numbers on R^+. Especially, in the light of the decomposition theorem of fuzzy sets, let us denote the null fuzzy number by $\widetilde{0} = \bigcup_{\lambda \in (0,1]} \lambda \cdot \{0\} = \bigcup_{\lambda \in (0,1]} \lambda[0, 0] \in F(\mathrm{R}^+)$.

Definition 2.2. Let $\{\widetilde{A}_n\} \subseteq F(\mathrm{R}^+)$ be a sequence of fuzzy numbers, and $\widetilde{A} \in F(\mathrm{R}^+)$. If $\lim_{n\to\infty} (A_n)_\lambda^- = A_\lambda^-$ and $\lim_{n\to\infty} (A_n)_\lambda^+ = A_\lambda^+$ for all $\lambda \in (0, 1]$, then we say that $\{\widetilde{A}_n\}$ is convergent to $\widetilde{A}$, and denote it by $\lim_{n\to\infty} \widetilde{A}_n = \widetilde{A}$ or $\widetilde{A}_n \to \widetilde{A}(n \to \infty)$.

Definition 2.3. Let $(X, \Re)$ be an arbitrary measurable space, and a set function $\mu : \Re \to [0, +\infty]$. If the following conditions (1)-(4) are satisfied:

(1) $\mu(\emptyset) = 0$;

(2) If $A, B \in \Re$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ (monotonicity);

(3) If $A_n \uparrow A$ and $A_n \in \Re$, then $\mu(A_n) \uparrow \mu(A)$ (lower semi-continuity);

(4) If $A_n \downarrow A$ and $A_n \in \Re$, there exists n_0 such that $\mu(A_{n_0}) < +\infty$ implies $\mu(A_n) \downarrow \mu(A)$ (upper semi-continuity).

Then the set function μ is called a fuzzy measure, corresponding triplet $(X, \Re, \mu)$ is called a fuzzy measure space. In particular, μ is called a lower semicontinuous fuzzy measure if it satisfies (1), (2) and (3); μ is called an upper semicontinuous fuzzy measure if it satisfies (1), (2) and (4).

Definition 2.4. [2] Let $(X, \Re)$ be a measurable space, $\mu : \Re \to [0, +\infty]$ be a set function. For arbitrary $A, B \in \Re$, if $\mu(B) = 0$ implies $\mu(A \cup B) = \mu(A)$ (or $\mu(A - B) = \mu(A)$), then μ is said to be null-additive (or null-subtractive).

Definition 2.5. [2] Let $(X, \Re)$ be a measurable space. A set function $\mu : \Re \to [0, +\infty]$ is said to be autocontinuous from above (or autocontinuous from below) if $\lim_{n\to\infty} \mu(B_n) = 0$ implies $\lim_{n\to\infty} \mu(B_n \cup A) = \mu(A)$ (or $\lim_{n\to\infty} \mu(A - B_n) = \mu(A)$) whenever $A \in \Re, \{B_n\} \subseteq \Re$ with $A \cap B_n = \emptyset$(or $B_n \subseteq A, n = 1, 2, 3, \cdots$). If μ is both autocontinuous from above and below, then μ is called autocontinuous.

Definition 2.6. [3] Let $(X, \Re)$ be a measurable space, $\mu : \Re \to [0, +\infty]$ be a set function. If for any $\{B_n\} \subseteq \Re$ and $\lim_{n\to\infty} \mu(B_n) = 0$, there always exists a sequence of subsets $\{B_{n_i}\} \subseteq \{B_n\}$ such that $\mu(\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} B_{n_i}) = 0$ (or $\mu(A - \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} B_{n_i}) = \mu(A)$ for any $A \in \Re$), then μ is called S-property (or pseudo-S-property), respectively.

By reference [3], it is not difficult for us to define the double-null asymptotic additivity and pseudo-double-null asymptotic additivity with respect to fuzzy valued set functions.

Definition 2.7. Let $(X, \Re)$ be a measurable space, a fuzzy valued set function $\widetilde{\nu} : \Re \to F(\mathrm{R}^+)$. For any $\{A_n\}, \{B_m\} \subseteq \Re$, if $\lim_{n\to\infty} \widetilde{\nu}(A_n) = \widetilde{0}$ and $\lim_{m\to\infty} \widetilde{\nu}(B_m) = \widetilde{0}$ implies $\lim_{n,m\to\infty} \widetilde{\nu}(A_n \cup B_m) = \widetilde{0}$ (or $\lim_{n,m\to\infty} \widetilde{\nu}(A - A_n \cup B_m) = \widetilde{A}$ for all $A \in \Re$), we call the fuzzy valued set function $\widetilde{\nu}$ double-null asymptotic additive (or pseudo-double-null asymptotic additive), respectively.

3. Generalized Fuzzy Valued θ-Choquet Integrals

Based on the applied background, aiming at nonnegative real valued measurable functions, Wang [6] gave the series of definitions with respect to pseudo-autocontinuity and pseudo-null additive in 1985. These characteristics can be used well to describe the convergence properties of the sequences of the measurable functions and fuzzy integrals. In this section, as for a given class of μ-integrable fuzzy valued functions, we establish the Mathematics models of the generalized fuzzy valued θ-Choquet integral, furthermore,some of their convergence theorems are studied.

Definition 3.1. [6] Let $(X, \Re)$ be a measurable space, a mapping $\widetilde{f} : X \to F(\mathrm{R}^+)$. If for any $\lambda \in (0, 1]$ and $\theta \in (0, +\infty)$, the set $(f_\lambda^-)_\theta \triangleq \{x \in X \mid f_\lambda^-(x) \geq \theta\} \in \Re$ and $(f_\lambda^+)_\theta \triangleq \{x \in X \mid f_\lambda^+(x) \geq \theta\} \in \Re$, then $\widetilde{f}$ is called a measurable fuzzy valued function, where $(\widetilde{f}(x))_\lambda \triangleq [f_\lambda^-(x), f_\lambda^+(x)] \in I_{R^+}$ for arbitrary $x \in X$ and $(f_\lambda^-)_\theta \subseteq (f_\lambda^+)_\theta$ for all $\theta \in (0, +\infty)$.

Definition 3.2. [5] Let $\widetilde{f}$ be a measurable fuzzy valued function on fuzzy measure space $(X, \Re, \mu)$. For arbitrary $\lambda \in (0, 1]$, if its Choquet type Lebesgue integral $\int_0^{+\infty} \mu((f_\lambda^+)_\theta) \mathrm{d}\theta$ exists and its value is finite, then μ is called Lebesgue integrable with respect to μ on $(X, \Re, \mu)$, simply call it μ-integrable.

Remark 3.3. Obviously, for all $\lambda \in (0, 1]$ and $\theta \in (0, +\infty)$, we have $\mu((f_\lambda^-)_\theta) \leq \mu((f_\lambda^+)_\theta)$ hold. Hence, $\int_0^{+\infty} \mu((f_\lambda^+)_\theta)\mathrm{d}\theta < +\infty$ implies $\int_0^{+\infty} \mu((f_\lambda^-)_\theta)\mathrm{d}\theta < +\infty$.

Definition 3.4. Let $(X,\Re,\mu)$ be a fuzzy measure space, $\widetilde{f}$ be μ-integrable and $A\in\Re$. Putting $(C)\int_A \widetilde{f}\mathrm{d}\mu \triangleq \bigcup_{\lambda\in(0,1]}\lambda[\int_0^{+\infty}\theta\wedge\mu(A\cap(f_\lambda^-)_\theta)\mathrm{d}\theta, \int_0^{+\infty}\theta\wedge\mu(A\cap(f_\lambda^+)_\theta)\mathrm{d}\theta]$, then $(C)\int_A \widetilde{f}\mathrm{d}\mu$ is called a generalized fuzzy valued θ-Choquet integral of $\widetilde{f}$ with respect to μ on A, simply call it θ-Choquet integral, where the right integral is an integral in the sense of Lebesgue and $(C)\int_A \widetilde{f}\mathrm{d}\mu\in F(\mathrm{R}^+)$.

Remark 3.5. For any $\lambda\in(0,1]$, if $\widetilde{f}$ is a μ-integrable fuzzy valued function, by Definition 3.1 and the monotonicity of Lebesgue integrals, it is straightforward to see that $\int_0^{+\infty}\theta\wedge\mu(A\cap(f_\lambda^-)_\theta)\mathrm{d}\theta\le\int_0^{+\infty}\theta\wedge\mu(A\cap(f_\lambda^+)_\theta)\mathrm{d}\theta\le\int_0^{+\infty}\mu((f_\lambda^+)_\theta\mathrm{d}\theta)<+\infty$, where $\theta\wedge\mu(A\cap(f_\lambda^-)_\theta)=\frac{1}{2}(\theta+\mu(A\cap(f_\lambda^-)_\theta)-\mid\theta-\mu(A\cap(f_\lambda^-)_\theta)\mid)$, the "$\wedge$" is minimal operator, i.e., $a\wedge b=\min(a,b)$.

In addition, in order to save the length of writing, we will omit the discussion for some basic properties of this kind of generalized fuzzy valued θ-Choquet integrals in this paper, which will be specially studied in other papers, we will no longer to repeat them here.

Lemma 3.6. *Let real numbers $a,b,c\in R^+$, then $\mid a\wedge b-a\wedge c\mid\le a\wedge\mid b-c\mid$.*

Theorem 3.7. *(Upper monotone theorem). Let $(X,\Re,\mu)$ be an upper semicontinuous fuzzy measure space, $\{\widetilde{f}_n\}$ be a sequence of μ-integrable monotone decreasing fuzzy valued functions, and $\lim_{n\to\infty}\widetilde{f}_n(x)=\widetilde{f}(x)$ for all $x\in X, A\in\Re$, then fuzzy valued function $\widetilde{f}$ is also μ-integrable and $\lim_{n\to\infty}(C)\int_A\widetilde{f}_n\mathrm{d}\mu=(C)\int_A\widetilde{f}\mathrm{d}\mu$.*

Proof. First, as $\{\widetilde{f}_n\}$ is monotone decreasing, it is easy to see that $\{\widetilde{f}_n\}$ is convergent, and its limit function $\widetilde{f}$ certainly exists.

By hypothesis, for all $n\in\mathbf{N}$ and $x\in X$, we have $\widetilde{f}(x)\le\widetilde{f}_n(x)$, utilizing the definition of fuzzy numbers for all $\lambda\in(0,1]$, we get

$$f_\lambda^-(x)\le(f_n)_\lambda^-(x),\ f_\lambda^+(x)\le(f_n)_\lambda^+(x).$$

From the hypothesis and the monotonicity of Lebesgue integrals, we have

$$\int_0^{+\infty}\mu((f_\lambda^-)_\theta)\mathrm{d}\theta\le\int_0^{+\infty}\mu((f_\lambda^+)_\theta)\mathrm{d}\theta\le\int_0^{+\infty}\mu(((f_n)_\lambda^+)_\theta)\mathrm{d}\theta<+\infty.$$

In the light of Definition 3.1, we obtain that the $\widetilde{f}$ is μ-integrable.

Secondly, for arbitrary $\lambda\in(0,1]$ and $n=1,2,3,\cdots$, it is not difficult to see that

$$\int_0^{+\infty}\theta\wedge\mu((f_\lambda^-)_\theta)\mathrm{d}\theta\le\int_0^{+\infty}\theta\wedge\mu((f_\lambda^+)_\theta)\mathrm{d}\theta\le\int_0^{+\infty}\mu(((f_n)_\lambda^+)_\theta)\mathrm{d}\theta<+\infty.$$

On the other hand, as $\{\widetilde{f}_n\}$ is monotone decreasing, the sequence $\{(f_n)_\lambda^-\cap A\}$ of sets is monotone decreasing with respect to natural number $n\in\mathbf{N}$. Consequently, for all $\theta\in(0,+\infty)$, we have

$$\lim_{n\to\infty}(((f_n)_\lambda^-)_\theta\bigcap A)=\bigcap_{n=1}^{\infty}(((f_n)_\lambda^-)_\theta\bigcap A).$$

At the moment, for any $\lambda \in (0,1]$, we can derive from that $\bigcap_{n=1}^{\infty}((f_n)_\lambda^-)_\theta = (f_\lambda^-)_\theta$.

In fact, for arbitrary $x \in \bigcap_{n=1}^{\infty}((f_n)_\lambda^-)_\theta$, for all natural number $n \in$ N, we have $(f_n)_\lambda^-(x) \geq \theta$. Let $n \to \infty$, by the hypothesis and Definition 2.2, we obtain $f_\lambda^-(x) = \lim_{n\to\infty}(f_n)_\lambda^-(x) \geq \theta$. Hence, $x \in (f_\lambda^-)_\theta$, i.e., $\bigcap_{n=1}^{\infty}((f_n)_\lambda^-)_\theta \subseteq (f_\lambda^-)_\theta$.

On the contrary, for any $x \in (f_\lambda^-)_\theta$, i.e., $f_\lambda^-(x) \geq \theta$, by hypothesis, as the sequence $\{\widetilde{f}_n\}$ is decreasing and tend to $\widetilde{f}$, we have $(f_n)_\lambda^-(x) \geq f_\lambda^-(x) \geq \theta$, this implies that $x \in ((f_n)_\lambda^-)_\theta$. Therefore, $(f_\lambda^-)_\theta \subseteq \bigcap_{n=1}^{\infty}((f_n)_\lambda^-)_\theta$. Furthermore, we obtain

$$\lim_{n\to\infty}(((f_n)_\lambda^-)_\theta \bigcap A) = (\bigcap_{n=1}^{\infty}((f_n)_\lambda^-)_\theta) \bigcap A = (f_\lambda^-)_\theta \bigcap A.$$

For all $\theta \in (0,+\infty)$ and $\lambda \in (0,1]$, as μ is an upper semicontinuous fuzzy measure, we deduce

$$\lim_{n\to\infty}\mu(((f_n)_\lambda^-)_\theta \bigcap A) = \mu(\lim_{n\to\infty}(((f_n)_\lambda^-)_\theta \bigcap A)) = \mu((f_\lambda^-)_\theta \bigcap A).$$

In accordance with the convergence of the sequence of numbers, for any $\varepsilon > 0$,there exists a natural number $N \in \mathbf{N}$, whenever $n \geq N$, we have

$$\mid \mu(((f_n)_\lambda^-)_\theta \bigcap A) - \mu((f_\lambda^-)_\theta \bigcap A) \mid < \varepsilon.$$

Take advantage of Lemma 3.6, we get immediately that

$$\mid \theta \wedge \mu(((f_n)_\lambda^-)_\theta \bigcap A) - \theta \wedge \mu((f_\lambda^-)_\theta \bigcap A) \mid \leq$$

$$\theta \wedge \mid \mu(((f_n)_\lambda^-)_\theta \bigcap A) - \mu((f_\lambda^-)_\theta \bigcap A) \mid < \theta \wedge \varepsilon = \varepsilon.$$

Hence, we can obtain $\lim_{n\to\infty}(\theta \wedge \mu(((f_n)_\lambda^-)_\theta \bigcap A) = \theta \wedge \mu((f_\lambda^-)_\theta \bigcap A)$.

In addition, since $\widetilde{f}$ is μ-integrable, and $\theta \wedge \mu(((f_n)_\lambda^-)_\theta \bigcap A) \leq \mu((f_\lambda^-)_\theta \bigcap A) \leq \mu((f_\lambda^-)_\theta)$ for all $\lambda \in (0,1]$ and $\theta \in (0,+\infty)$, $n = 1,2,3,\cdots$, regarding $\mu((f_\lambda^-)_\theta)$ as a dominant function with respect to θ, use the dominant convergence theorem of Lebesgue integrals and the upper semicontinuity of fuzzy measure μ, it is easy to know that

$$\lim_{n\to\infty}\int_0^{+\infty}\theta \wedge \mu(A \bigcap ((f_n)_\lambda^-)_\theta)\mathrm{d}\theta = \int_0^{+\infty}\theta \wedge \mu(A \bigcap (f_\lambda^-)_\theta)\mathrm{d}\theta.$$

In the light of the similar method, for all $\lambda \in (0,1]$, aim at the sequence $\{(f_n)_\lambda^+\}$ of nonnegative real valued functions, we can infer

$$\lim_{n\to\infty}\int_0^{+\infty}\theta \wedge \mu(A \bigcap ((f_n)_\lambda^+)_\theta)\mathrm{d}\theta = \int_0^{+\infty}\theta \wedge \mu(A \bigcap (f_\lambda^+)_\theta)\mathrm{d}\theta.$$

By Definition2.2 and Definition 2.3, we immediately obtain

$$\lim_{n\to\infty}(C)\int_A \widetilde{f}_n \mathrm{d}\mu = (C)\int_A \widetilde{f}\mathrm{d}\mu.$$

□

Theorem 3.8. *(Lower monotone theorem). Let $(X, \Re, \mu)$ be a lower semicontinuous fuzzy measure space, $\{\widetilde{f}_n\}$ be a sequence of μ-integrable and monotone increasing fuzzy valued functions and $\lim_{n\to\infty} \widetilde{f}_n(x) = \widetilde{f}(x)$ for all $x \in X, A \in \Re$, then $\widetilde{f}$ is μ-integrable too, and $\lim_{n\to\infty}(C)\int_A \widetilde{f}_n \mathrm{d}\mu = (C)\int_A \widetilde{f}\mathrm{d}\theta$.*

Proof. Imitating the proof of Theorem 3.7, for any $\lambda \in (0, 1]$ and $\theta \in (0, +\infty)$, it is straightforward to see that

$$\lim_{n\to\infty}(((f_n)^-_\lambda)_\theta \bigcap A) = \bigcup_{n=1}^{\infty}(((f_n)^-_\lambda)_\theta \bigcap A).$$

Using the similar method of Theorem 3.7, we can immediately prove this conclusion holds, so we omit it. □

Theorem 3.9. *(Fatou's Lemma 3.6). Let $\{\widetilde{f}_n\}$ be a sequence of μ-integrable fuzzy valued functions on $(X, \Re, \mu), A \in \Re$, then $(C)\int_A \underline{\lim}_{n\to\infty} \widetilde{f}_n \mathrm{d}\mu \leq \underline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n \mathrm{d}\mu$.*

Proof. Putting $\widetilde{p}_n(x) = \inf_{i\geq n} \widetilde{f}_i(x)$ for every $x \in X$, $n = 1, 2, \cdots$. For arbitrary $\lambda \in (0, 1]$, using the definition of fuzzy valued functions and infimum, we have $(p_n)^-_\lambda(x) = \inf_{i\geq n}(f_i)^-_\lambda(x)$ and $(p_n)^+_\lambda(x) = \inf_{i\geq n}(f_i)^+_\lambda(x)$.

Evidently, $\{(p_n)^-_\lambda\}$ and $\{(p_n)^+_\lambda\}$ are the sequences of increasing nonnegative measurable functions with respect to $n \in \mathbf{N}$. Consequently, the sequence $\{\widetilde{p}_n\}$ of fuzzy valued functions is monotone increasing too, and satisfies

$$\lim_{n\to\infty}(p_n)^-_\lambda(x) = \underline{\lim}_{n\to\infty}(f_n)^-_\lambda(x), \ \lim_{n\to\infty}(p_n)^+_\lambda(x) = \underline{\lim}_{n\to\infty}(f_n)^+_\lambda(x).$$

Therefore, $\lim_{n\to\infty} \widetilde{p}_n(x) = \lim_{n\to\infty}\inf_{i\geq n} f_i(x) = \underline{\lim}_{n\to\infty} \widetilde{f}_n(x)$. By Theorem 3.7, we can get

$$\lim_{n\to\infty}(C)\int_A \widetilde{p}_n(x)\mathrm{d}\mu = (C)\int_A \underline{\lim}_{n\to\infty} \widetilde{f}_n(x)\mathrm{d}\mu.$$

On the other hand, as for sequence $\{\widetilde{p}_n\}$ of fuzzy valued functions, for arbitrary $\lambda \in (0, 1]$ and $x \in X$, obviously,$(p_n)^-_\lambda(x) \leq (f_i)^-_\lambda(x)$, $i = n, n+1, n+2, \cdots$. Furthermore, for every $\theta \in (0, +\infty)$, $A\bigcap((p_n)^-_\lambda)_\theta \subseteq A\bigcap((f_i)^-_\lambda)_\theta$, $i = n, n+1, n+2, \cdots$. Applying the decomposition theorem with respect to fuzzy sets and definition 3.4, we can derive from

$$\int_A (p_n)^-_\lambda \mathrm{d}\mu = \int_0^{+\infty} \theta \wedge \mu(A\bigcap((p_n)^-_\lambda)_\theta)\mathrm{d}\theta$$
$$\leq \int_0^{+\infty} \theta \wedge \mu(A\bigcap((f_i)^-_\lambda)_\theta)\mathrm{d}\theta = \int_A (f_i)^-_\lambda \mathrm{d}\mu.$$

Taking the infimum with respect to natural number $i \in \mathbf{N}$, for the two sides in the above formula, and let $n \to \infty$, consequently, we have

$$\lim_{n\to\infty}\int_A (p_n)^-_\lambda(x)\mathrm{d}\mu \leq \lim_{n\to\infty}\inf_{i\geq n}\int_A (f_i)^-_\lambda(x)\mathrm{d}\mu = \underline{\lim}_{n\to\infty}\int_A (f_n)^-_\lambda(x)\mathrm{d}\mu.$$

Similarly, for any $\lambda \in (0,1]$, it is straightforward to see that

$$\lim_{n\to\infty}\int_A (\widetilde{p}_n)^+_\lambda(x)\mathrm{d}\mu \leq \underline{\lim}_{n\to\infty}\int_A (f_n)^+_\lambda(x)\mathrm{d}\mu.$$

In accordance with Definition 2.2, Definition 3.4 and the definition of the order of fuzzy numbers, we can infer

$$\lim_{n\to\infty}(C)\int_A \widetilde{p}_n(x)\mathrm{d}\mu \leq \underline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n(x)\mathrm{d}\mu.$$

Furthermore, we obtain

$$(C)\int_A \underline{\lim}_{n\to\infty}\widetilde{f}_n\mathrm{d}\mu = (C)\int_A \lim_{n\to\infty}\widetilde{p}_n\mathrm{d}\mu = \lim_{n\to\infty}(C)\int_A \widetilde{p}_n\mathrm{d}\mu \leq \underline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n\mathrm{d}\mu.$$

□

Theorem 3.10. *(Fatou's Lemma 4.1). Let $(X,\Re,\mu)$ be a finite fuzzy measure space, $\{\widetilde{f}_n\}$ be a sequence of μ-integrable fuzzy valued functions, and $\mu(X) < +\infty$, then $\overline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n\mathrm{d}\mu \leq (C)\int_A \overline{\lim}_{n\to\infty}\widetilde{f}_n\mathrm{d}\mu$ for all $A\in\Re$*

Proof. Imitating the proof of Theorem 3.9, let $\widetilde{q}_n(x) = \sup_{i\geq n}\widetilde{f}_i(x)$, it is easy to prove, so we omit it. □

Theorem 3.11. *(Dominated convergence theorem). Let $(X,\Re,\mu)$ be a finite fuzzy measure space, $\{\widetilde{f}_n\}$ be a sequence of fuzzy valued functions and $\lim_{n\to\infty}\widetilde{f}_n(x) = \widetilde{f}(x)$ for all $x\in X$. If there exists a μ-integrable fuzzy valued function $\widetilde{g}$ such that $\widetilde{f}_n(x) \leq \widetilde{g}(x)$ for every $x\in X$, $n = 1,2,3,\cdots$, for any $A\in\Re$, then $\widetilde{f}$ is also μ-integrable and $\lim_{n\to\infty}(C)\int_A \widetilde{f}_n\mathrm{d}\mu = (C)\int_A \widetilde{f}\mathrm{d}\mu$.*

Proof. By hypothesis and the definition of the order of fuzzy numbers, letting $n\to\infty$, we have $\widetilde{f}(x) = \lim_{n\to\infty}\widetilde{f}_n(x) \leq \widetilde{g}(x)$. It is not hard to see that $\widetilde{f}$ and $\widetilde{f}_n$ are μ-integrable, $n = 1,2,3,\cdots$. For all $\lambda\in(0,1]$, it is straightforward to see that

$$\underline{\lim}_{n\to\infty}\int_A (f_i)^-_\lambda\mathrm{d}\mu \leq \overline{\lim}_{n\to\infty}\int_A (f_i)^-_\lambda\mathrm{d}\mu,\ \ \underline{\lim}_{n\to\infty}\int_A (f_i)^+_\lambda\mathrm{d}\mu \leq \overline{\lim}_{n\to\infty}\int_A (f_i)^+_\lambda\mathrm{d}\mu,$$

where

$$\int_A (f_i)^-_\lambda\mathrm{d}\mu = \int_0^{+\infty}\theta\wedge\mu(A\bigcap((f_i)^-_\lambda)_\theta)\mathrm{d}\theta$$

and

$$\int_A (f_i)^+_\lambda\mathrm{d}\mu = \int_0^{+\infty}\theta\wedge\mu(A\bigcap((f_i)^+_\lambda)_\theta)\mathrm{d}\theta.$$

Therefore, it follows that

$$\underline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n\mathrm{d}\mu \leq \overline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n\mathrm{d}\mu.$$

At the moment, according to Theorem 3.9 and Theorem 3.10, we can obtain

$$(C)\int_A \underline{\lim}_{n\to\infty}\widetilde{f}_n\mathrm{d}\mu \leq \underline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n\mathrm{d}\mu \leq \overline{\lim}_{n\to\infty}(C)\int_A \widetilde{f}_n\mathrm{d}\mu \leq (C)\int_A \overline{\lim}_{n\to\infty}\widetilde{f}_n\mathrm{d}\mu.$$

Since $\widetilde{f}_n \to \widetilde{f}(n \to \infty)$, it means that $\underline{\lim}_{n\to\infty} \widetilde{f}_n(x) = \overline{\lim}_{n\to\infty} \widetilde{f}_n(x) = \widetilde{f}(x)$ for all $x \in X$. And so,the above inequalities hold if and only if the equal sign hold. Therefore,

$$\underline{\lim}_{n\to\infty} (C)\int_A \widetilde{f}_n \mathrm{d}\mu = \overline{\lim}_{n\to\infty} (C)\int_A \widetilde{f}_n \mathrm{d}\mu = (C)\int_A \widetilde{f}\mathrm{d}\mu.$$

Consequently, $\lim_{n\to\infty} (C)\int_A \widetilde{f}_n \mathrm{d}\mu = (C)\int_A \widetilde{f}\mathrm{d}\mu$. □

4. Double-null Asymptotic Additivity

It is well known that, the double-null asymptotic additivity and pseudo-double-null asymptotic additivity are two-component limit forms to describe set functions. In this section, we regard the whole generalized fuzzy valued θ-Choquet integrals as a set function taken by fuzzy numbers, taking advantage of the concepts of the autocontinuity from above (below) and superior (inferior) limit with respect to fuzzy measures, then we discuss double-null asymptotic and pseudo-double-null asymptotic additivity of this kind of generalized fuzzy valued θ-Choquet integrals.

Lemma 4.1. [13] *Let $(X, \Re, \mu)$ be a fuzzy measure space. For any sequence of sets $\{A_n\} \subseteq \Re$, then $\mu(\underline{\lim}_{n\to\infty} A_n) \leq \underline{\lim}_{n\to\infty} \mu(A_n)$; if the condition $\mu(X) < +\infty$ is satisfied, then $\mu(\overline{\lim}_{n\to\infty} A_n) \geq \overline{\lim}_{n\to\infty} \mu(A_n)$, where the inferior limit is denoted as $\underline{\lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{m\geq n}^{\infty} A_m$, superior limit is denoted as $\overline{\lim}_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m\geq n}^{\infty} A_m$.*

Lemma 4.2. [13] *Let $(X, \Re, \mu)$ be a fuzzy measure space, for any sequence of sets $\{E_n\}$, if $\lim_{n\to\infty} \mu(E_n) = 0$ and μ is autocontinuous from above (or below), then there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that $\mu(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} E_{n_i}) = 0$ (or $\mu(A - \bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} E_{n_i}) = \mu(A)$), respectively.*

Theorem 4.3. *Let $(X, \Re, \mu)$ be a finite fuzzy measure space, $\mu(X) < +\infty, \widetilde{f}$ be a μ-integrable fuzzy number valued function, let $\widetilde{\nu}(A) \triangleq (C)\int_A \widetilde{f}\mathrm{d}\mu$ for any $A \in \Re$. If μ is autocontinuous from above, then $\widetilde{\nu}$ is double-null asymptotic additive.*

Proof. Take any sequences of sets $\{A_n\}, \{B_m\} \subseteq \Re$, satisfying $\lim_{n\to\infty} \widetilde{\nu}(A_n) = \widetilde{0}$ and $\lim_{m\to\infty} \widetilde{\nu}(B_m) = \widetilde{0}$, we are going to prove

$$\lim_{n,m\to\infty} \widetilde{\nu}(A_n \bigcup B_m) = \widetilde{0}. \quad (1)$$

First, as $\widetilde{f}$ is μ-integrable, for all $\lambda \in (0, 1]$ and $n, m = 1, 2, 3, \cdots$, we have

$$\int_0^{+\infty} \theta \wedge \mu(A_n \bigcap (f_\lambda^-)_\theta)\mathrm{d}\theta \leq \int_0^{+\infty} \mu((f_\lambda^+)_\theta)\mathrm{d}\theta < +\infty$$

and

$$\int_0^{+\infty} \theta \wedge \mu(B_m \bigcap (f_\lambda^-)_\theta)\mathrm{d}\theta \leq \int_0^{+\infty} \mu((f_\lambda^+)_\theta)\mathrm{d}\theta < +\infty.$$

Hence, these integrals are integrable in the sense of Lebesgue integrals.

On the other hand, it is not difficult to prove that $\widetilde{\nu}$ is a fuzzy measure. As μ is autocontinuous from above, by reference [11], we know that fuzzy valued set function $\widetilde{\nu}$ is also autocontinuous from above. Meanwhile, for arbitrary $\lambda \in (0,1]$, the set functions ν_λ^- and ν_λ^+ are autocontinuous from above, where $\widetilde{\nu}_\lambda = [\nu_\lambda^-, \nu_\lambda^-]$.

In fact, if (1) doesn't hold, i.e., $\lim\limits_{n,m\to\infty} \widetilde{\nu}(A_n \bigcup B_m) \neq \widetilde{0}$, by Definitions 2.2 and 2.7, there exist sequences of sets $\{A_n\}, \{B_m\} \subseteq \Re$ such that $\lim\limits_{n\to\infty} \widetilde{\nu}(A_n) = \widetilde{0}$ and $\lim\limits_{m\to\infty} \widetilde{\nu}(B_m) = \widetilde{0}$, respectively. Furthermore, there exist $\varepsilon_0 > 0$ and $\lambda_0 \in (0,1]$ such that

$$\nu_{\lambda_0}^-(A_n \bigcup B_m) = \int_0^{+\infty} \theta \wedge \mu((A_n \bigcup B_m) \bigcap (f_{\lambda_0}^-)_\theta) \mathrm{d}\theta \geq \varepsilon_0$$

or

$$\nu_{\lambda_0}^+(A_n \bigcup B_m) = \int_0^{+\infty} \theta \wedge \mu((A_n \bigcup B_m) \bigcap (f_{\lambda_0}^+)_\theta) \mathrm{d}\theta \geq \varepsilon_0.$$

Without loss of generality, as for $\lambda_0 \in [0,1]$, we let $\lim\limits_{n\to\infty} \nu_{\lambda_0}^-(A_n) = \lim\limits_{m\to\infty} \nu_{\lambda_0}^-(B_m) = 0$. Since set functions ν_λ^- and ν_λ^+ are autocontinuous from above, by Lemma 4.2, there exist sequences of sets $\{A_{n_i}\} \subseteq \{A_n\}$ and $\{B_{m_k}\} \subseteq \{B_m\}$ such that

$$\nu_{\lambda_0}^-(\bigcap_{j=1}^\infty \bigcup_{i=j}^\infty A_{n_i}) = 0,\ \nu_{\lambda_0}^-(\bigcap_{s=1}^\infty \bigcup_{k=s}^\infty B_{m_k}) = 0.$$

As set functions $\nu_{\lambda_0}^-$ and $\nu_{\lambda_0}^+$ are autocontinuous from above, it is clear to view that $\nu_{\lambda_0}^-$ and $\nu_{\lambda_0}^+$ are null-additive. As for $\nu_{\lambda_0}^-$, obviously,

$$\nu_{\lambda_0}^-((\bigcap_{j=1}^\infty \bigcup_{i=j}^\infty A_{n_i}) \bigcup (\bigcap_{s=1}^\infty \bigcup_{k=s}^\infty B_{m_k})) = \nu_{\lambda_0}^-(\bigcap_{j=1}^\infty \bigcup_{i=j}^\infty A_{n_i}) = 0.$$

On the other hand, by Lemma 4.1 and Theorem 3.10, we obtain

$$\begin{aligned}
&\nu_{\lambda_0}^-((\bigcap_{j=1}^\infty \bigcup_{i=j}^\infty A_{n_i}) \bigcup (\bigcap_{s=1}^\infty \bigcup_{k=s}^\infty B_{m_k})) \\
=&\int_0^{+\infty} \theta \wedge \mu\{((\bigcap_{j=1}^\infty \bigcup_{i=j}^\infty A_{n_i}) \bigcup (\bigcap_{s=1}^\infty \bigcup_{k=s}^\infty B_{m_k})) \bigcap (f_{\lambda_0}^-)_\theta\} \mathrm{d}\theta \\
=&\int_0^{+\infty} \theta \wedge \mu\{\bigcap_{j=1}^\infty \bigcup_{i=j}^\infty \bigcap_{s=1}^\infty \bigcup_{k=s}^\infty ((A_{n_i} \bigcup B_{m_k}) \bigcap (f_{\lambda_0}^-)_\theta)\} \mathrm{d}\theta \\
=&\int_0^{+\infty} \theta \wedge \mu\{\overline{\lim_{i\to\infty}}\, \overline{\lim_{k\to\infty}} ((A_{n_i} \bigcup B_{m_k}) \bigcap (f_{\lambda_0}^-)_\theta)\} \mathrm{d}\theta \\
\geq&\int_0^{+\infty} \theta \wedge \overline{\lim_{i\to\infty}}\, \overline{\lim_{k\to\infty}}\, \mu((A_{n_i} \bigcup B_{m_k}) \bigcap (f_{\lambda_0}^-)_\theta) \mathrm{d}\theta \\
=&\int_0^{+\infty} \overline{\lim_{i\to\infty}}\, \overline{\lim_{k\to\infty}} (\theta \wedge \mu((A_{n_i} \bigcup B_{m_k}) \bigcap (f_\lambda^-)_\theta)) \mathrm{d}\theta \\
\geq&\ \overline{\lim_{i\to\infty}}\, \overline{\lim_{k\to\infty}} \int_0^{+\infty} \theta \wedge \mu((A_{n_i} \bigcup B_{m_k}) \bigcap (f_{\lambda_0}^-)_\theta) \mathrm{d}\theta \\
=&\ \overline{\lim_{i\to\infty}}\, \overline{\lim_{k\to\infty}}\, \nu_{\lambda_0}^-(A_{n_i} \bigcup B_{m_k}) \geq \overline{\lim_{i\to\infty}}\, \overline{\lim_{k\to\infty}}\, \varepsilon_0 = \varepsilon_0.
\end{aligned}$$

Therefore, we can infer that $\varepsilon_0 \leq 0$, which is in contradiction with $\varepsilon_0 > 0$. Hence, for all $\lambda \in (0,1]$, we have $\lim_{n,m\to\infty} \nu_\lambda^-(A_n \bigcup B_m) = 0$.

Analogously, we may derive from $\lim_{n,m\to\infty} \nu_\lambda^+(A_n \bigcup B_m) = 0$. Consequently, in the light of the decomposition theorem of fuzzy sets, we get $\lim_{n,m\to\infty} \widetilde{\nu}(A_n \bigcup B_m) = \widetilde{0}$. By Definition 2.7, the set function $\widetilde{\nu}$ is double-null asymptotic additive. □

Corollary 4.4. *Let $(X, \Re, \mu)$ be a finite fuzzy measure space, $\widetilde{f}$ be a μ-integrable,and let $\widetilde{\nu}(A) = (C)\int_A \widetilde{f}\mathrm{d}\mu$ for any $A \in \Re$. If μ possesses S-property, then $\widetilde{\nu}$ is double-null asymptotic additive.*

Theorem 4.5. *Let $(X, \Re, \mu)$ be a fuzzy measure space, $\widetilde{f}$ be a μ-integrable fuzzy number valued function. Put $\widetilde{\nu}(A) = (C)\int_A \widetilde{f}\mathrm{d}\mu$ for any $A \in \Re$. If μ is autocontinuous from below, then $\widetilde{\nu}$ is pseudo-double-null asymptotic additive.*

Proof. Let $\{A_n\} \subseteq \Re$ and $\{B_m\} \subseteq \Re$, satisfy $\lim_{n\to\infty} \widetilde{\nu}(A_n) = \widetilde{0}$ and $\lim_{m\to\infty} \widetilde{\nu}(B_m) = \widetilde{0}$. We are going to prove

$$\lim_{n,m\to\infty} \widetilde{\nu}(A - A_n \bigcup B_m) = \widetilde{\nu}(A). \tag{2}$$

Since fuzzy measure μ is autocontinuous from below, by reference [11], we can view that the set function $\widetilde{\nu}$ is also autocontinuous from below. At the same time, for any $\lambda \in (0,1]$, the set functions ν_λ^- and ν_λ^+ are autocontinuous from below.

Clearly, $A - A_n \bigcup B_m \subseteq A, n, m = 1, 2, \cdots$ and $\lim_{n,m\to\infty} \widetilde{\nu}(A - A_n \bigcup B_m) \leq \widetilde{\nu}(A)$.

In fact, if (2) doesn't hold, it is straightforward to see that

$$\lim_{n,m\to\infty} \widetilde{\nu}(A - A_n \bigcup B_m) < \widetilde{\nu}(A).$$

By Definition 2.2, there exist sequences of sets $\{A_n\} \subseteq \Re$ and $\{B_m\} \subseteq \Re$ such that $\lim_{n\to\infty} \widetilde{\nu}(A_n) = \widetilde{0}$ and $\lim_{m\to\infty} \widetilde{\nu}(B_m) = \widetilde{0}$, but there exist $\varepsilon_0 > 0$ and $\lambda_0 \in (0,1]$ such that

$$\nu_{\lambda_0}^-(A - A_n \bigcup B_m) < \nu_{\lambda_0}^-(A) - \varepsilon_0$$

or

$$\nu_{\lambda_0}^+(A - A_n \bigcup B_m) < \nu_{\lambda_0}^+(A) - \varepsilon_0.$$

Here $\nu_{\lambda_0}^{\pm}(A - A_n \cup B_m) = \int_0^{+\infty} \theta \wedge (\mu(A - A_n \cup B_m) \cap (f_{\lambda_0}^{\pm})_\theta)\mathrm{d}\theta,\ n, m = 1, 2, \cdots$.

Without loss of generality, let $\lim_{n\to\infty} \nu_{\lambda_0}^-(A_n) = \lim_{m\to\infty} \nu_{\lambda_0}^-(B_m) = 0$. As set function $\nu_{\lambda_0}^-$ is autocontinuous from below, by Lemma 4.2, there certainly exist sequences of subsets $\{A_{n_i}\} \subseteq \{A_n\}$ and $\{B_{m_k}\} \subseteq \{B_m\}$ such that

$$\nu_{\lambda_0}^-(A - \bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} A_{n_i}) = \nu_{\lambda_0}^-(A),\ \nu_{\lambda_0}^-(A - \bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} B_{m_k}) = \nu_{\lambda_0}^-(A).$$

Since the set functions $\nu^-_{\lambda_0}$ and $\nu^+_{\lambda_0}$ are autocontinuous from below, it is not difficult to see that $\nu^-_{\lambda_0}$ and $\nu^+_{\lambda_0}$ are null-subtractive, aiming at $\nu^-_{\lambda_0}$, we have

$$\nu^-_{\lambda_0}(A - \bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} A_{n_i} - \bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} B_{m_k}) = \nu^-_{\lambda_0}(A - \bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} A_{n_i}) = \nu^-_{\lambda_0}(A).$$

On the other hand, by Lemma 4.1 and Theorem 3.9, we obtain

$$\nu^-_{\lambda_0}(A - \bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty} A_{n_i} - \bigcap_{s=1}^{\infty}\bigcup_{k=s}^{\infty} B_{m_k})$$

$$= \int_0^{+\infty} \theta \wedge \mu(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}\bigcup_{s=1}^{\infty}\bigcap_{k=s}^{\infty}(A \bigcap A'_{n_i} \bigcap B'_{m_k} \bigcap (f^-_{\lambda_0})_\theta))\mathrm{d}\theta$$

$$= \int_0^{+\infty} \theta \wedge \mu(\varliminf_{i\to\infty} \varliminf_{k\to\infty} ((A - A_{n_i} \bigcup B_{m_k}) \bigcap (f^-_{\lambda_0})_\theta)\mathrm{d}\theta$$

$$\leq \int_0^{+\infty} \varliminf_{i\to\infty} \varliminf_{k\to\infty} (\theta \wedge \mu((A - A_{n_i} \bigcup B_{m_k}) \bigcap (f^-_{\lambda_0})_\theta))\mathrm{d}\theta$$

$$\leq \varliminf_{i\to\infty} \varliminf_{k\to\infty} \int_0^{+\infty} \theta \wedge \mu((A - A_{n_i} \bigcup B_{m_k}) \bigcap (f^-_{\lambda_0})_\theta)\mathrm{d}\theta$$

$$= \varliminf_{i\to\infty} \varliminf_{k\to\infty} \nu^-_{\lambda_0}(A - A_{n_i} \bigcup B_{m_k}) \leq \nu^-_{\lambda_0}(A) - \varepsilon_0.$$

Consequently, $\nu^-_{\lambda}{}_0(A) \leq \nu^-_{\lambda}{}_0(A) - \varepsilon$. As $\widetilde{f}$ is μ-integrable, we immediately can get

$$\nu^-_{\lambda_0}(A) = \int_0^{+\infty} \theta \wedge \mu(A \bigcap (f^-_{\lambda_0})_\theta)\mathrm{d}\theta \leq \int_0^{+\infty} \mu((f^-_{\lambda_0})_\theta)\mathrm{d}\theta < +\infty.$$

It is not difficult to see that $\varepsilon_0 \leq 0$, which is in contradiction with $\varepsilon_0 > 0$.

Similarly, letting $\lim_{n\to\infty} \nu^+_{\lambda_0}(A_n) = \lim_{m\to\infty} \nu^+_{\lambda_0}(B_m) = 0$, we can infer $\varepsilon_0 \leq 0$. Hence, we obtain $\lim_{n,m\to\infty} \widetilde{\nu}(A - A_n \bigcup B_m) = \widetilde{\nu}(A)$.

By Definition 2.7, it is straightforward to see that the set function $\widetilde{\nu}$ is also pseudo-double-null asymptotic additive. □

Corollary 4.6. *Let $(X, \Re, \mu)$ be a fuzzy measure space, $\widetilde{f}$ be a fuzzy valued function which is μ-integrable. Let $\widetilde{\nu}(A) = (C)\int_A \widetilde{f}\mathrm{d}\mu$ for any $A \in \Re$. If μ possesses pseudo-S-property, then $\widetilde{\nu}$ is pseudo-double-null asymptotic additive.*

5. Conclusions

The construction of a fuzzy valued set function defined by the generalized fuzzy valued-Choquet integrals preserves almost all desirable structural characteristics of the original fuzzy measures, but the additivity is not necessarily preserved by the construction. Due to these reasons, the Choquet integral can be used for constructing sound fuzzy measures in various application areas. In fact, it is an essential premise for a fuzzy valued measure to construct a theory of fuzzy integrals. The double-null asymptotic additivity and pseudo-double-null asymptotic additivity are two-component limit forms to describe set functions, they are two important aspect of depicting set functions. From section 3, we can easily see that Theorems 3.7-3.11 and Theorems 4.3-4.5 are based on the assumption that a function $\widetilde{f}$ is $\mu-$integrable. The problem being suggested by this is worth thinking: can we replace it with a weaker condtion ? In our further research, aim at the above problem ,we will comtimve their studying. It is not difficult to see that these results have some important effects on the application of the theory of fuzzy integrals.

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