L-ORDERED FUZZIFYING CONVERGENCE SPACES

W. WU AND J. FANG

ABSTRACT. Based on a complete Heyting algebra, we modify the definition of lattice-valued fuzzifying convergence space using fuzzy inclusion order and construct in this way a Cartesian-closed category, called the category of L-ordered fuzzifying convergence spaces, in which the category of L-fuzzifying topological spaces can be embedded. In addition, two new categories are introduced, which are called the category of principal L-ordered fuzzifying convergence spaces and that of topological L-ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of L-fuzzifying neighborhood spaces and that of L-fuzzifying topological spaces respectively.

1. Introduction

Convergence structures are more general than topological structures. If a convergence structure additionally satisfies proper conditions, it is equivalent to a topological structure. Lowen [12] constructed convergence systems using prefilters, through which Min [13] proposed fuzzy limit structures. Xu [14] proved that topological L-fuzzifying convergence structures and L-fuzzifying topologies [17] are equivalent, where classical filters play a crucial role. By stratified L-filters [7], Jäger [8] introduced stratified L-fuzzy convergence spaces in the many-valued case. The category of these spaces was developed to a significant extent in the recent years [1,2,4,5,9-11,14,15].

In 2009, Yao [16] defined L-fuzzifying convergence spaces, and showed the category of L-fuzzifying topological spaces [17] could be embedded in the category of L-fuzzifying convergence spaces as a reflective subcategory and the latter is Cartesian-closed. L-fuzzifying convergence spaces were based on L-filters of ordinary subsets.

This paper can be seen as a further step towards [16]. It proposes a new latticevalued fuzzifying convergence structure, called L-ordered fuzzifying convergence structure, which is compatible with the fuzzy inclusion order of L-filters of ordinary subsets. The category of L-fuzzifying topological spaces [17] can be embedded in the resulting category. As a matter of fact, it is easier for a bigger category to be Cartesian-closed, and it makes sense to establish a smaller Cartesian-closed category. Note that the category of L-ordered fuzzifying convergence spaces is

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"smaller" than that of L-fuzzifying convergence spaces [16], and it is Cartesianclosed. In addition, two new categories are introduced, which are called the category of principal L-ordered fuzzifying convergence spaces and that of topological L-ordered fuzzifying convergence spaces, and it is shown that they are isomorphic to the category of L-fuzzifying neighborhood spaces and that of L-fuzzifying topological spaces respectively.

2. Preliminaries

Let (L, \lor, \land) be a complete lattice. If the finite meets are distributive over arbitrary joins, i.e. for all $a, b_i \in L, (i \in J)$

$$a \wedge (\bigvee_{i \in J} b_i) = \bigvee_{i \in J} (a \wedge b_i)$$

L is called a complete Heyting algebra. For L, we define an implication operator $\to:L\times L\to L$ as follows:

$$\forall a, b \in L, a \to b = \bigvee \{c \in L | a \land c \leq b\}.$$

Then it is the right adjoint for \wedge , i.e.,

$$\forall a, b, c \in L, a \land c \leq b \Leftrightarrow c \leq a \to b.$$

Theorem 2.1. [7] Let L be a complete Heyting algebra. For all $a, b, c, d, a_i, b_i \in L, (i \in J)$, the following holds:

$$\begin{array}{l} (H1) \ a \leq (b \rightarrow c) \Leftrightarrow a \wedge b \leq c, \ and \ a \leq b \Leftrightarrow (a \rightarrow b) = 1, \\ (H2) \ a \rightarrow (\bigwedge_{i \in J} b_i) = \bigwedge_{i \in J} (a \rightarrow b_i), \ (\bigvee_{i \in J} b_i) \rightarrow a = \bigwedge_{i \in J} (b_i \rightarrow a), \\ (H3) \ (b \rightarrow c) \leq (a \rightarrow b) \rightarrow (a \rightarrow c), \ (a \rightarrow c) \wedge (b \rightarrow d) \leq (a \wedge b) \rightarrow (c \wedge d), \\ (H4) \ a \rightarrow b \geq b, \ a \leq (a \rightarrow b) \rightarrow b, \\ (H5) \ a \wedge b = a \wedge (a \rightarrow b), \ therefore, \ b = 1 \rightarrow b, \\ (H6) \ a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c, \\ (H7) \ \bigwedge_{i \in J} (a_i \rightarrow b_i) \leq (\bigwedge_{i \in J} a_i) \rightarrow (\bigwedge_{i \in J} b_i). \end{array}$$

In what follows, we consider X a nonempty set and L a complete Heyting algebra unless otherwise stated.

For a given set X, L^X denotes the set of all L-subsets on X. Define a binary mapping $S(-,-): L^X \times L^X \to L$ by $S(U,V) = \bigwedge_{x \in X} (U(x) \to V(x))$ for each pair $(U,V) \in L^X \times L^X$.

Definition 2.2. [6] A map $\mathscr{F} : 2^X \to L$ is called an *L*-filter of ordinary subsets of X if it satisfies $\forall x \in X, A, B \in 2^X$,

- (F1) $\mathscr{F}(\emptyset) = 0, \mathscr{F}(X) = 1,$
- (F2) $A \subseteq B \Rightarrow \mathscr{F}(A) \leq \mathscr{F}(B),$
- (F3) $\mathscr{F}(A \cap B) \ge \mathscr{F}(A) \land \mathscr{F}(B).$

The family of all L-filters of ordinary subsets on X will be denoted by $\mathscr{F}_L(X)$. An order on $\mathscr{F}_L(X)$ is defined as follows: $\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X), \mathscr{F} \leq \mathscr{G} \Leftrightarrow \forall U \in 2^X, \mathscr{F}(U) \leq \mathscr{G}(U)$.

For every $x \in X$, $[x] \in \mathscr{F}_L(X)$ is defined by $\forall A \in 2^X$,

$$[x](A) = \begin{cases} 1, & x \in A, \\ 0, & otherwise \end{cases}$$

Let \mathscr{F} be a filter of ordinary subsets on X and $f: X \to Y$ be a mapping. Then the mapping $f^{\Rightarrow}(\mathscr{F}): 2^Y \to L$, where $\forall B \in 2^Y, f^{\Rightarrow}(\mathscr{F})(B) = \mathscr{F}(f^{\leftarrow}(B))$, is an L-filter of ordinary subsets on Y and is called the image of \mathscr{F} under f.

For every $\mathscr{F} \in \mathscr{F}_L(X)$, $\mathscr{G} \in \mathscr{F}_L(Y)$, $\mathscr{F} \times \mathscr{G} \in \mathscr{F}_L(X \times Y)$ is defined as follows: $\forall C \in 2^{X \times Y}$, $(\mathscr{F} \times \mathscr{G})(C) = \bigvee_{A \times B \subseteq C} \mathscr{F}(A) \wedge \mathscr{G}(B)$.

Definition 2.3. [18] An L-fuzzifying neighborhood structure on a set X is a family of functions $N = \{N_x : 2^X \to L \mid x \in X\}$ with the following conditions: For all $x \in X$, U, $V \in 2^X$,

- (LN1) $N_x(X) = 1$,
- (LN2) $N_x(U) > 0$ implies $x \in U$,

(LN3) $N_x(U \cap V) = N_x(U) \wedge N_x(V).$

The pair (X, N) is called an *L*-fuzzifying neighborhood space, and it will be called topological if it satisfies moreover: For all $x \in X$, $U \in 2^X$,

(LN4) $N_x(U) = \bigvee_{x \in V \subset U} \bigwedge_{y \in V} N_y(V).$

A continuous function between L-fuzzifying neighborhood spaces (X, N^1) and (Y, N^2) is a map $f : X \to Y$ such that for all $x \in X$, $U \in 2^Y$, $N_x^1(f^{\leftarrow}(U)) \ge N_{f(x)}^2(U)$.

Let $L-\mathbf{NGH}$ denote the category of L-fuzzifying neighborhood spaces with continuous maps, and $L-\mathbf{TNGH}$ the full subcategory of $L-\mathbf{NGH}$ consisting of topological L-fuzzifying neighborhood spaces.

Definition 2.4. [17] An *L*-fuzzifying topology on *X* is a function $\tau : 2^X \to L$ which satisfies

- (FO1) $\tau(\emptyset) = \tau(X) = 1$,
- (FO2) $\tau(A \cap B) \ge \tau(A) \land \tau(B)$,
- (FO3) $\tau(\bigcup_{i \in J} A_j) \ge \bigwedge_{i \in J} \tau(A_j).$

For an L-fuzzifying topology τ on X, the pair (X, τ) is called an L-fuzzifying topological space. A map $f: X \to Y$ is called continuous with respect to the given L-fuzzifying topological spaces (X, τ_1) and (Y, τ_2) iff $\forall B \in 2^Y$, $\tau_1(f^{\leftarrow}(B)) \geq$ $\tau_2(B)$. The category of L-fuzzifying topological spaces with continuous maps as morphisms will be denoted by L-**FYS**.

It was proved in [20] that for any completely distributive lattice L, topological L-fuzzifying neighborhood systems and L-fuzzifying topologies are conceptually equivalent with transferring process $N_x(U) = \bigvee_{x \in V \subseteq U} \tau(V)$ and $\tau(U) = \bigwedge_{x \in U} N_x(U)$. **Theorem 2.5.** [19] Let $\varphi : (X, \tau_1) \to (Y, \tau_2)$ be a mapping. If L is a completely distributive lattice, then φ is continuous iff $N_x^{\tau_1}(\varphi^{\leftarrow}(U)) \ge N_{\varphi(x)}^{\tau_2}(U), \forall x \in X, U \in 2^Y$.

3. L-ordered Fuzzifying Convergence Structure

In [16], the author developed lattice-valued convergence structure lim : $\mathscr{F}_L(X) \to L^X$ as follows:

Definition 3.1. [16] A mapping $\lim : \mathscr{F}_L(X) \to L^X$, subject to the conditions (LY1) $\forall x \in X$, $\limx = 1$,

(LY2) $\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X), \ \mathscr{F} \leq \mathscr{G} \Rightarrow \forall x \in X, \lim \mathscr{F}(x) \leq \lim \mathscr{G}(x),$

is called an L-fuzzifying convergence structure on X, and (X, \lim) an L-fuzzifying convergence space.

The set of all L-fuzzifying convergence structures on X is denoted by $\lim_{ly}(X)$. An order on $\lim_{ly}(X)$ can be defined by $\lim_{1 \leq lim_{2}}$ iff for all $\mathscr{F} \in \mathscr{F}_{L}(X), x \in X$, $\lim_{1 \leq \ell \leq r} \mathscr{F}(x) \leq \lim_{1 \leq \ell \leq r} \mathscr{F}(x)$.

In Definition 3.1, the *L*-filters in the axiom (LY2) are in nature *L*-sets on 2^X . We use the method in [3] and define an *L*-partial order $S_F(-,-)$ on $\mathscr{F}_L(X)$ as follows: $S_F(-,-):\mathscr{F}_L(X) \times \mathscr{F}_L(X) \to L$

$$\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X), S_F(\mathscr{F}, \mathscr{G}) = \bigwedge_{U \in 2^X} (\mathscr{F}(U) \to \mathscr{G}(U)).$$

In this case, we can redefine the axiom (LY2) in Definition 3.1, proposing the following new lattice-valued convergence structure.

Definition 3.2. An *L*-fuzzifying convergence structure $\lim : \mathscr{F}_L(X) \to L^X$, satisfying the following condition:

OLY2)
$$\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L(X), \ S_F(\mathscr{F}, \mathscr{G}) \leq S(\lim \mathscr{F}, \lim \mathscr{G}),$$

is called an L-ordered fuzzifying convergence structure, and the pair (X, \lim) an L-ordered fuzzifying convergence space.

A function $\varphi : (X, \lim^X) \to (Y, \lim^Y), (X, \lim^X), (Y, \lim^Y)$ *L*-ordered fuzzifying convergence spaces, is called continuous iff for all $\mathscr{F} \in \mathscr{F}_L(X), x \in X, \lim^X \mathscr{F}(x) \leq \lim^Y \varphi^{\Rightarrow}(\mathscr{F})(\varphi(x)).$

We do not go into details here, but only remark that (OLY2) implies (LY2).

The next example shows there exists an L-fuzzifying convergence structure lim which is not an L-ordered fuzzifying convergence structure.

Example 3.3. Let $X = \{x, y\}$, $L = \{0, \alpha, 1\}$ be a chain. Define a map lim : $\mathscr{F}_L(X) \to L^X, \forall \mathscr{F} \in \mathscr{F}_L(X), z \in X,$

$$\lim \mathscr{F}(z) = \left\{ \begin{array}{ll} 1, & \mathscr{F} \geq [z], \\ 0, & otherwise. \end{array} \right.$$

It is obvious that lim is an L-fuzzifying convergence structure. Define a mapping $\mathscr{F}^*: 2^X \to L$ as follows: $\forall A \in 2^X$,

$$\mathscr{F}^{*}(A) = \begin{cases} 1, & A = X, \\ \alpha, & A = \{x\}, \\ 0, & A = \{y\} \text{ or } A = \emptyset \end{cases}$$

It can be verified that \mathscr{F}^* is an *L*-filter of ordinary subsets on *X*. Then

$$S_{F}([x], \mathscr{F}^{*}) = \bigwedge_{A \in 2^{X}} ([x](A) \to \mathscr{F}^{*}(A))$$

$$= ([x](\emptyset) \to \mathscr{F}^{*}(\emptyset)) \bigwedge ([x](\{x\}) \to \mathscr{F}^{*}(\{x\}))$$

$$\bigwedge ([x](\{y\}) \to \mathscr{F}^{*}(\{y\})) \bigwedge ([x](X) \to \mathscr{F}^{*}(X))$$

$$= 1 \land \alpha \land 1 \land 1$$

$$= \alpha$$

And

$$S(\lim[x], \lim \mathscr{F}^*) = \bigwedge_{z \in X} \left(\lim[x](z) \to \lim \mathscr{F}^*(z) \right)$$
$$= \left(\limx \to \lim \mathscr{F}^*(x) \right) \bigwedge \left(\lim[x](y) \to \lim \mathscr{F}^*(y) \right)$$
$$= (1 \to 0) \land (0 \to 0)$$
$$= 0$$

We can see that $S_F([x], \mathscr{F}^*) \nleq S(\lim[x], \lim \mathscr{F}^*)$, hence lim is not an *L*-ordered fuzzifying convergence structure.

Example 3.4. Let $(X, \tau) \in L$ -**FYS** and define a mapping $\lim_{\tau} : \mathscr{F}_L(X) \to L^X$, $\forall \mathscr{F} \in \mathscr{F}_L(X), x \in X, \lim_{\tau} \mathscr{F}(x) = S_F(N^x_{\tau}, \mathscr{F})$. Here, the *L*-fuzzifying neighborhood system N^x_{τ} of $x \in X$ is defined by $N^x_{\tau}(A) = \bigvee_{x \in B \subseteq A} \tau(B)$. Then $\lim_{\tau} \tau$ is an *L*-ordered fuzzifying convergence structure.

From Example 3.4, we see that an L- fuzzifying topology can induce an L-ordered fuzzifying convergence structure. The following theorem shows that the induced L-ordered fuzzifying convergence structure from the L-fuzzifying topology can determine the induced L-fuzzifying neighborhood structure from the L-fuzzifying topology. This idea has been presented in [8].

Theorem 3.5. Let $(X, \tau) \in L$ -**FYS**. Then the following holds:

$$N^x_\tau(U) = \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} (\lim_\tau \mathscr{F}(x) \to \mathscr{F}(U)), \forall x \in X, U \in 2^X.$$

Let L-**FYCS** [16] denote the category of L-fuzzifying convergence spaces with continuous maps and L-**OFYC** the full subcategory of L-**FYCS** formed by all L-ordered fuzzifying convergence spaces.

The set of all *L*-ordered fuzzifying convergence structures on *X* is denoted by $\lim_{loy}(X)$. An order on $\lim_{loy}(X)$ can be defined by $\lim_{1 \to \infty} \lim_{1 \to \infty} \lim_{1 \to \infty} \inf_{x \to \infty} \int_{0}^{\infty} \mathbb{F}(x)$. For $\lim_{loy}(X)$ here, we immediately

obtain that there are a maximum element and a minimum element in $(\lim_{loy}(X), \leq)$, denoted by \lim_{sm} and \lim_m respectively: $\forall \mathscr{F} \in \mathscr{F}_L(X), x \in X, \lim_{sm} \mathscr{F} = 1_X;$ $\lim_m \mathscr{F}(x) = S_F([x], \mathscr{F})$. The supremum element of a family of *L*-ordered fuzzifying convergence structures $(\lim_j)_{j \in J} \subseteq \lim_{loy}(X)$ is defined by $(\sup_{j \in J} \lim_{j \in J} \mathscr{F}(x)) = 1$

 $\bigwedge_{j\in J} \lim_{j\notin J} \mathscr{F}(x), \forall \mathscr{F} \in \mathscr{F}_L(X), \ x \in X. \text{ Obviously, } \sup_{j\in J} \lim_{loy} (X). \text{ Therefore,}$ the following proposition holds.

Proposition 3.6. $(\lim_{loy}(X), \leq)$ is a complete lattice.

We will next address the result that the category of L-ordered fuzzifying convergence spaces is a topological category. To this end, we note the following proposition.

Proposition 3.7. The category L-**OFYC** is a full reflective subcategory in the category L-**FYCS**.

Proof. Let $(X, \overline{\lim}) \in L$ -**FYCS** and $E_{\overline{\lim}} = \{ \lim | (X, \lim) \in L$ -**OFYC**, $\lim \leq \overline{\lim} \}$. Note that $E_{\overline{\lim}}$ is not empty because it always contains \lim_{sm} . Then with Proposition 3.6, we can construct an L-ordered fuzzifying convergence structure $\overline{\lim}_* : \mathscr{F}_L(X) \to L^X$ as follows: For all $\mathscr{F} \in \mathscr{F}_L(X), x \in X$, $\overline{\lim}_* \mathscr{F}(x) = \bigwedge_{\lim \in E_{\overline{\lim}}} \lim \mathscr{F}(x)$. From this, we have

(1) $id_X: (X, \overline{\lim}) \to (X, \overline{\lim}_*)$ is trivially continuous;

(2) For an *L*-ordered fuzzifying convergence space (Y, \lim^Y) , if $f : (X, \overline{\lim}) \to (Y, \lim^Y)$ is a continuous mapping, then $f : (X, \overline{\lim}) \to (Y, \lim^Y)$ is also continuous. We leave the above check to the reader.

From the above facts, we immediately obtain that $L-\mathbf{OFYC}$ is a full reflective subcategory in $L-\mathbf{FYCS}$.

In [16] Yao proved that the category L-**FYCS** is topological. By Proposition 3.7, we have the following main result.

Theorem 3.8. The category of L-ordered fuzzifying convergence spaces L-**OFYC** is topological.

4. The Relations Between Categories of *L*-FYS and *L*-OFYC

This section is motivated by reference [8]. In this section, we will resolve the embedding of L-**FYS** into L-**OFYC**. By Example 3.4 and Theorem 3.5, we see that L-ordered convergence structures can be induced from L-fuzzifying topologies. Moreover, they are unique. In order to show that L-**FYS** can be embedded in the category of L-**OFYC**, the following theorem is necessary.

Theorem 4.1. Let L be a completely distributive lattice. Then the map f: $(X, \tau_1) \rightarrow (Y, \tau_2)$ between two L-fuzzifying topological spaces is continuous iff $f: (X, \lim_{\tau_1}) \rightarrow (Y, \lim_{\tau_2})$ is continuous. *Proof.* Suppose that $f: (X, \tau_1) \to (Y, \tau_2)$ is continuous, by Theorem 2.5, we have for all $\mathscr{F} \in \mathscr{F}_L(X), x \in X$,

$$\lim_{\tau_2} \varphi^{\Rightarrow}(\mathscr{F})(\varphi(x)) = \bigwedge_{V \in 2^Y} (N_{\tau_2}^{\varphi(x)}(V) \to \varphi^{\Rightarrow}(\mathscr{F})(V))$$

$$\geq \bigwedge_{V \in 2^Y} (N_{\tau_1}^x(\varphi^{\leftarrow}(V)) \to \mathscr{F}(\varphi^{\leftarrow}(V)))$$

$$\geq \bigwedge_{U \in 2^X} (N_{\tau_1}^x(U) \to \mathscr{F}(U))$$

$$= \lim_{\tau_1} \mathscr{F}(x).$$

Hence, $f: (X, \lim_{\tau_1}) \to (Y, \lim_{\tau_2})$ is continuous.

Conversely, if $f:(X, \lim_{\tau_1}) \to (Y, \lim_{\tau_2})$ is continuous, by Theorem 3.5, we have $\forall x \in X, U \in 2^Y$,

$$\begin{split} N^x_{\tau_1}(\varphi^{\leftarrow}(U)) &= \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)} (\lim_{\tau_1}\mathscr{F}(x) \to \mathscr{F}(\varphi^{\leftarrow}(U))) \\ &\geq \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)} (\lim_{\tau_2}(\varphi^{\Rightarrow}(\mathscr{F}))(\varphi(x)) \to (\varphi^{\Rightarrow}(\mathscr{F}))(U)) \\ &\geq \bigwedge_{\mathscr{G}\in\mathscr{F}_L(Y)} (\lim_{\tau_2}\mathscr{G}(\varphi(x)) \to \mathscr{G}(U)) \\ &= N^{\varphi(x)}_{\tau_2}(U). \end{split}$$

Therefore, by Theorem 2.5, $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ is continuous.

As a consequence of the above theorems, we have the following result.

Theorem 4.2. Let L be a completely distributive lattice. L-**FYS** can be embedded in the category of L-**OFYC**.

In Theorem 3.8 we know that $L-\mathbf{OFYC}$ is topological. So, in order to show that it is Cartesian-closed, the following results are necessary. Similar to the definition of product spaces in $L-\mathbf{FYCS}$, it can be shown that there are also product spaces in $L-\mathbf{OFYC}$. We refer the reader to [16]. Here, we only present the main results. Note that for two L-ordered fuzzifying convergence spaces $(X, \lim_X), (Y, \lim_Y)$, let $[X \to Y]$ denote the set of all continuous maps from (X, \lim_X) to (Y, \lim_Y) .

Lemma 4.3. [16] Let $g: X \to Y$ and $\mathscr{G} \in \mathscr{F}_L(X)$, then $g^{\Rightarrow}(\mathscr{G}) \leq ev^{\Rightarrow}([g] \times \mathscr{G})$, where $ev: [X \to Y] \times X \to Y$ is the evaluation map.

Theorem 4.4. Let $(X, \lim_X), (Y, \lim_Y)$ be L-ordered fuzzifying convergence spaces, then $\lim_{[X \to Y]} : F_L([X \to Y]) \to L^{[X \to Y]}, \forall \mathscr{F} \in F_L([X \to Y]), \forall f \in [X \to Y], \lim_{[X \to Y]} \mathscr{F}(f) = \bigwedge_{(\mathscr{G}, x) \in \mathscr{F}_L(X) \times X} (\lim_X \mathscr{G}(x) \to \lim_Y ev^{\Rightarrow} (\mathscr{F} \times \mathscr{G})(f(x)))$ is an L-ordered fuzzifying convergence structure on $[X \to Y]$. Proof. For (LY1), $\forall g \in [X \to Y]$,

$$\lim_{[X \to Y]} g = \bigwedge_{(\mathscr{G}, x) \in \mathscr{F}_L(X) \times X} \lim_{X} \mathscr{G}(x) \to \lim_{Y} (ev^{\Rightarrow}([g] \times \mathscr{G}))(g(x))$$

$$\geq \bigwedge_{(\mathscr{G}, x) \in \mathscr{F}_L(X) \times X} \lim_{X} \mathscr{G}(x) \to \lim_{Y} (g^{\Rightarrow}(\mathscr{G}))(g(x))$$

$$= 1.$$

For (OLY2), $\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L([X \to Y])$,

 $= S_F(\mathscr{F}, \mathscr{G}).$

$$\begin{split} &S(\lim_{[X \to Y]} \mathscr{F}, \lim_{[X \to Y]} \mathscr{G}) \\ &= \bigwedge_{g \in [X \to Y]} \left(\left(\bigwedge_{(\mathscr{E}, x) \in \mathscr{F}_{L}(X) \times X} \lim_{X} \mathscr{E}(x) \to \lim_{Y} (ev^{\Rightarrow}(\mathscr{F} \times \mathscr{E}))(g(x)) \right) \\ &\to \left(\bigwedge_{(\mathscr{H}, x) \in \mathscr{F}_{L}(X) \times X} \lim_{X} \mathscr{H}(x) \to \lim_{Y} (ev^{\Rightarrow}(\mathscr{G} \times \mathscr{H}))(g(x)) \right) \right) \\ &\geq \bigwedge_{g \in [X \to Y]} \bigwedge_{(\mathscr{H}, x) \in \mathscr{F}_{L}(X) \times X} \left(\left(\lim_{X} \mathscr{H}(x) \to \lim_{Y} (ev^{\Rightarrow}(\mathscr{G} \times \mathscr{H}))(g(x)) \right) \right) \\ &\to \left(\lim_{X} \mathscr{H}(x) \to \lim_{Y} (ev^{\Rightarrow}(\mathscr{G} \times \mathscr{H}))(g(x)) \right) \right) \\ &\geq \bigwedge_{\mathscr{H} \in \mathscr{F}_{L}(X)} S(\lim_{Y} (ev^{\Rightarrow}(\mathscr{F} \times \mathscr{H})), \lim_{Y} (ev^{\Rightarrow}(\mathscr{G} \times \mathscr{H})))) \\ &\geq \bigwedge_{\mathscr{H} \in \mathscr{F}_{L}(X)} S_{F}(ev^{\Rightarrow}(\mathscr{F} \times \mathscr{H}), ev^{\Rightarrow}(\mathscr{G} \times \mathscr{H})). \\ &\forall \mathscr{H} \in \mathscr{F}_{L}(X), \\ &\forall \mathscr{H} \in \mathscr{F}_{L}(X), \\ &= \bigwedge_{U \in 2^{Y}} \left((\mathscr{F} \times \mathscr{H})(ev^{\leftarrow}(U)) \to (\mathscr{G} \times \mathscr{H})(ev^{\leftarrow}(U)) \right) \\ &= \bigwedge_{U \in 2^{Y}} \left((\mathscr{F} \times \mathscr{H})(ev^{\leftarrow}(U)) \to (\mathscr{G} \times \mathscr{H})(ev^{\leftarrow}(U)) \right) \\ &\geq \bigwedge_{U \in 2^{Y}} \bigwedge_{A \times B \subseteq ev^{\leftarrow}(U)} (\mathscr{F}(A) \land \mathscr{H}(B)) \to (\mathscr{G}(A) \land \mathscr{H}(B))) \\ &\geq \bigwedge_{U \in 2^{Y} A \times B \subseteq ev^{\leftarrow}(U)} \left(\mathscr{F}(A) \to \mathscr{G}(A) \right) \\ &\geq \bigwedge_{U \in 2^{Y} A \times B \subseteq ev^{\leftarrow}(U)} (\mathscr{F}(A) \to \mathscr{G}(A)) \\ &\geq \bigwedge_{U \in 2^{Y} A \times B \subseteq ev^{\leftarrow}(U)} (\mathscr{F}(A) \to \mathscr{G}(A)) \\ &\geq \bigwedge_{U \in 2^{Y} A \times B \subseteq ev^{\leftarrow}(U)} (\mathscr{F}(A) \to \mathscr{G}(A)) \end{aligned}$$

Therefore, the above completes the proof. In other words, $\lim_{[X \to Y]} \text{is an } L - \text{ordered}$ fuzzifying convergence structure on $[X \to Y]$. \Box

Remark 4.5. The evaluation map $ev : [X \to Y] \times X \to Y$ mentioned above is continuous. Let $f : X \times Y \to Z$ be a map, $\forall x \in X$, define a map $f_x : Y \to Z$, $\forall y \in Y$, $f_x(y) = f(x, y)$, $f^* : X \to Z^Y$, $\forall x \in X$, $f^*(x) = f_x$, and $\varphi : Z^{(X \to Y)} \to (Z^Y)^X$, $\forall f \in Z^{(X \to Y)}, \varphi(f) = f^*$. Then it can be proved that

(1) If $f: (X, \lim_X) \times (Y, \lim_Y) \to (Z, \lim_Z)$ is continuous, then for each $x \in X$, $f_x: (Y, \lim_Y) \to (Z, \lim_Z)$ is continuous.

(2) For all $\mathscr{F} \in \mathscr{F}_L(X), \ \mathscr{G} \in \mathscr{F}_L(Y), \ ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathscr{F}) \times \mathscr{G}) = f^{\Rightarrow}(\mathscr{F} \times \mathscr{G}).$

(3) If $f: X \times Y \to Z$ is continuous, then $\varphi(f): X \to [Y \to Z]$ is continuous. (We refer to [16] for a detail proof of the above results.)

We collect our findings in the following theorem.

Theorem 4.6. *L***-OFYC** *is a Cartesian-closed category.*

5. The Relations Between *L*-fuzzifying Neighborhood Spaces and Principle *L*-ordered Fuzzifying Convergence Spaces

In this section, we define a subcategory of the category of L-ordered fuzzifying convergence spaces: the category of principle L-ordered fuzzifying convergence spaces and show that the new category and that of L-fuzzifying neighborhood spaces are isomorphic. Furthermore, each fibre on a fixed set of the category of L-fuzzifying neighborhood spaces and that of the category of principal L-ordered fuzzifying convergence spaces are isomorphic. At the end of the section, we propose that the category of principle L-ordered fuzzifying convergence spaces is a reflective subcategory of L-**OFYC** and it is a topological category. Again, this section is mostly motivated by reference [8].

Proposition 5.1. Let $(X, \lim) \in L$ -**OFYC**. The structure $\{N_{\lim}^x : 2^X \to L\}_{x \in X}$ defined by: For $x \in X$, $\forall U \in 2^X, N_{\lim}^x(U) = \bigwedge_{\mathscr{F} \in \mathscr{F}_L(X)} (\lim \mathscr{F}(x) \to \mathscr{F}(U))$ is an L-fuzzifying neighborhood structure. We call it the induced L-fuzzifying neighborhood structure of (X, \lim) .

Theorem 3.5 suggests for $(X, \lim) \in L$ -**OFYC** the following definition.

Definition 5.2. Let lim be an L-ordered fuzzifying convergence structure. If in addition the following condition (LYP) holds,

$$(LYP) \ \forall \mathscr{F} \in \mathscr{F}_L(X), \ x \in X, \ \lim \mathscr{F}(x) = S_F(N_{\lim}^x, \mathscr{F}),$$

then lim is called a principal L-ordered fuzzifying convergence structure, and the pair (X, \lim) is called a principle L-ordered fuzzifying convergence space.

The full subcategory of L-**OFYC** consisting of all principle L-ordered fuzzifying convergence spaces is denoted by L-**POFYC**.

If an L-ordered fuzzifying convergence spaces satisfies (LYP), then a nice characterization of principle L-ordered convergence spaces in terms of L-fuzzifying neighborhood spaces is possible. We first need three theorems for preparation. **Theorem 5.3.** Let (X, N) be an L-fuzzifying neighborhood space. Then there exists a principle L-ordered fuzzifying convergence structure lim on X satisfying $\forall x \in X, N_{\lim}^x = N^x$.

Proof. For the *L*-fuzzifying neighborhood space (X, N), define $\lim_N : \mathscr{F}_L(X) \to L^X$

$$\forall \mathscr{F} \in \mathscr{F}_L(X), \ x \in X, \ \lim_N \mathscr{F}(x) = \bigwedge_{A \in 2^X} (N^x(A) \to \mathscr{F}(A)) = S_F(N^x, \mathscr{F}).$$

It is then readily checked that for (X, \lim_N) , the axiom (LY1), (OLY2), (LYP) hold. The properties of the residual implication of Theorem 2.1 are used.

(LY1):
$$\forall x \in X$$
, $\lim_N x = \bigwedge_{A \in 2^X} (N^x(A) \to [x](A)) = 1$.

(OLY2): In fact,

$$\begin{split} S(\lim_{N} \mathscr{F}, \lim_{N} \mathscr{G}) &= \bigwedge_{x \in X} \left(S_{F}(N^{x}, \mathscr{F}) \to S_{F}(N^{x}, \mathscr{G}) \right) \\ &= \bigwedge_{x \in X} \left(\bigwedge_{A \in 2^{X}} \left(N^{x}(A) \to \mathscr{F}(A) \right) \to \bigwedge_{B \in 2^{X}} \left(N^{x}(B) \to \mathscr{G}(B) \right) \right) \\ &= \bigwedge_{x \in X} \bigwedge_{B \in 2^{X}} \left(\bigwedge_{A \in 2^{X}} \left(N^{x}(A) \to \mathscr{F}(A) \right) \to \left(N^{x}(B) \to \mathscr{G}(B) \right) \right) \\ &\geq \bigwedge_{x \in X} \bigwedge_{B \in 2^{X}} \left(\left(N^{x}(B) \to \mathscr{F}(B) \right) \to \left(N^{x}(B) \to \mathscr{G}(B) \right) \right) \\ &\geq \bigwedge_{x \in X} \bigwedge_{B \in 2^{X}} \left(\left(\mathscr{F}(B) \to \mathscr{G}(B) \right) \right) \\ &= S_{F}(\mathscr{F}, \mathscr{G}). \end{split}$$

(LYP): For all $\mathscr{F} \in \mathscr{F}_L(X)$, we prove $\lim_N \mathscr{F}(x) = S_F(N_{\lim_N}^x, \mathscr{F})$. By the definition of $\lim_N, \lim_N \mathscr{F}(x) = S_F(N^x, \mathscr{F})$. It remains to verify that $N_{\lim_N}^x = N^x$. On one hand, for all $A \in 2^X$,

$$N_{\lim_{N \to \infty}}^{x}(A) = \bigwedge_{\mathscr{F} \in \mathscr{F}_{L}(X)} (\lim_{N} \mathscr{F}(x) \to \mathscr{F}(A))$$

$$\leq \lim_{N \to \infty} N^{x}(x) \to N^{x}(A)$$

$$= N^{x}(A).$$

On the other hand,

$$N_{\lim_{N}}^{x}(A) = \bigwedge_{\mathscr{F} \in \mathscr{F}_{L}(X)} (\lim_{N} \mathscr{F}(x) \to \mathscr{F}(A))$$
$$= \bigwedge_{\mathscr{F} \in \mathscr{F}_{L}(X)} \left(\bigwedge_{B \in 2^{X}} \left(N^{x}(B) \to \mathscr{F}(B) \right) \to \mathscr{F}(A) \right)$$
$$\ge \bigwedge_{\mathscr{F} \in \mathscr{F}_{L}(X)} (N^{x}(A) \to \mathscr{F}(A) \to \mathscr{F}(A))$$
$$\ge N^{x}(A).$$

From this, the result follows by a standard argument.

In view of the above theorem, if N is an L-fuzzifying neighborhood structure, then there exists a principle L-ordered fuzzifying convergence structure \lim_N on X. Moreover, N_{\lim_N} is also an L-fuzzifying neighborhood structure and $N_{\lim_N} = N$. Conversely, we have the following theorem.

Theorem 5.4. If lim is a principle L-ordered fuzzifying convergence structure on X, then $\lim_{N_{\text{lim}}} = \lim$.

Proof. For all
$$\mathscr{F} \in \mathscr{F}_L(X), x \in X$$
, by (LYP), we have,

$$\lim_{N_{\mathrm{lim}}} \mathscr{F}(x) = \bigwedge_{A \in 2^X} \left(N^x_{\mathrm{lim}}(A) \to \mathscr{F}(A) \right) = S_F(N^x_{\mathrm{lim}}, \mathscr{F}) = \lim \mathscr{F}(x).$$

With respect to Theorem 5.3 and Theorem 5.4, we have a one-one correspondence between the objects of L-**NGH** and L-**POFYC**. The following theorem is about the relation between morphisms of them.

Theorem 5.5. Let $(X, \lim^X), (Y, \lim^Y)$ be principle L-ordered fuzzifying convergence spaces, $(X, N_1), (Y, N_2)$ be L-fuzzifying neighborhood spaces, then we have

(1) If $f:(X, \lim^X) \to (Y, \lim^Y)$ is continuous, then $f:(X, N_{\lim}x) \to (Y, N_{\lim}x)$ is also continuous;

(2) If $f:(X, N_1) \to (Y, N_2)$ is continuous, then $f:(X, \lim_{N_1}^X) \to (Y, \lim_{N_2}^Y)$ is also continuous.

Proof. (1) By the fact that $f : (X, \lim^X) \to (Y, \lim^Y)$ is continuous, we have $\forall x \in X, U \in 2^Y$,

$$N_{\lim^X}^x(f^{\leftarrow}(U)) = \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)} \left(\lim^X \mathscr{F}(x) \to \mathscr{F}(f^{\leftarrow}(U))\right)$$
$$\geq \bigwedge_{\mathscr{F}\in\mathscr{F}_L(X)} \left(\lim^Y f^{\Rightarrow}(\mathscr{F})(f(x)) \to f^{\Rightarrow}(\mathscr{F})(U)\right)$$
$$\geq \bigwedge_{\mathscr{G}\in\mathscr{F}_L(Y)} \left(\lim^Y \mathscr{G}(f(x)) \to \mathscr{G}(U)\right)$$
$$= N_{\lim^Y}^{f(x)}(U),$$

as desired.

(2) Conversely, by the fact that $f: (X, N_1) \to (Y, N_2)$ is continuous, we have $\forall \mathscr{F} \in \mathscr{F}_L(X), x \in X$,

$$\begin{split} \lim_{N_2}^{Y} f^{\Rightarrow}(\mathscr{F})(f(x)) &= \bigwedge_{B \in 2^Y} \left(N_2^{f(x)}(B) \to f^{\Rightarrow}(\mathscr{F})(B) \right) \\ &\geq \bigwedge_{B \in 2^Y} \left(N_1^x(f^{\leftarrow}(B)) \to \mathscr{F}(f^{\leftarrow}(B)) \right) \\ &\geq \bigwedge_{A \in 2^X} \left(N_1^x(A) \to \mathscr{F}(A) \right) \\ &= \lim_{N_1}^{X} \mathscr{F}(x), \end{split}$$

as desired.

By Theorems 5.3, 5.4 and 5.5, we actually have proved the following comprehensive theorem.

Theorem 5.6. *L*-**NGH** *is isomorphic to L*-**POFYC**.

Let X be a set. A fibre on X of the category of L-fuzzifying neighborhood spaces is denoted by $\mathbf{PrFN}_{\mathbf{L}}(\mathbf{X})$. An order " \leq " on $\mathbf{PrFN}_{\mathbf{L}}(\mathbf{X})$ can be defined by

$$N^1 \le N^2 \Leftrightarrow \forall x \in X, A \in 2^X, N^1_x(A) \le N^2_x(A).$$

A fibre on X of the category of principle L-ordered fuzzifying convergence spaces is denoted by $\mathbf{PFYC}_{\mathbf{L}}(\mathbf{X})$. An order " \leq " on $\mathbf{PFYC}_{\mathbf{L}}(\mathbf{X})$ can be defined as follows:

$$\lim_{1 \le \infty} \lim_{1 \le \infty} \Leftrightarrow \forall \mathscr{F} \in \mathscr{F}_{L}(X), x \in X, \lim_{1 \le \infty} \mathscr{F}(x) \le \lim_{1 \le \infty} \mathscr{F}(x).$$

Then we have the following result.

Theorem 5.7. (PFYC_L(X), \leq) and (PrFN_L(X), \leq) are isomorphic.

Proof. Define a mapping: $h: \mathbf{PFYC}_{\mathbf{L}}(\mathbf{X}) \to \mathbf{PrFN}_{\mathbf{L}}(\mathbf{X}), \forall \lim \in \mathbf{PFYC}_{\mathbf{L}}(\mathbf{X}), h(\lim) = N_{\lim}$, and a mapping: $k: \mathbf{PrFN}_{\mathbf{L}}(\mathbf{X}) \to \mathbf{PFYC}_{\mathbf{L}}(\mathbf{X}), \forall N \in \mathbf{PrFN}_{\mathbf{L}}(\mathbf{X}), k(N) = \lim_{N \to \infty} I$. It has been verified in Theorems 5.3 and 5.4 that $h \circ k = id_{\mathbf{PrFN}_{\mathbf{L}}(\mathbf{X}), k \circ h = id_{\mathbf{PFYC}_{\mathbf{L}}(\mathbf{X})}$. So h and k are both bijective. Furthermore, $k = h^{-1}$.

(1) For all $\lim_{1, \dots, \infty} \in \mathbf{PFYC}_{\mathbf{L}}(\mathbf{X})$, if $\lim_{1, \dots, \infty} \le \lim_{1, \dots, \infty} \lim_{1, \dots, \infty} \sup_{1, \dots, \infty} \sup_{1, \dots, \infty} \lim_{1, \dots, \infty} \sup_{1, \dots, \infty} \sup_{1, \dots, \infty} \lim_{1, \dots, \infty} \sup_{1, \dots, \infty} \sup_{1, \dots, \infty} \lim_{1, \dots, \infty} \sup_{1, \dots, \infty} \lim_{1, \dots, \infty} \sup_{1, \dots, \infty} \lim_{1, \dots,$

$$N_{\lim_{1}}^{x}(A) = \bigwedge_{\mathscr{F} \in \mathscr{F}_{L}(X)} \left(\lim_{1} \mathscr{F}(x) \to \mathscr{F}(A) \right)$$
$$\leq \bigwedge_{\mathscr{F} \in \mathscr{F}_{L}(X)} \left(\lim_{2} \mathscr{F}(x) \to \mathscr{F}(A) \right)$$
$$= N_{\lim_{1}}^{x}(A).$$

So, $N_{\lim_1} \leq N_{\lim_2}$. i.e. $h(\lim_1) \leq h(\lim_2)$. Therefore, h is an order preserving map. (2) For all $N_1, N_2 \in \mathbf{PrFN}_{\mathbf{L}}(\mathbf{X})$, if $N_1 \leq N_2$, then for all $\mathscr{F} \in \mathscr{F}_L(X), x \in X$,

$$\lim_{N_1} \mathscr{F}(x) = \bigwedge_{A \in 2^X} \left(N_1^x(A) \to \mathscr{F}(A) \right)$$
$$\geq \bigwedge_{A \in 2^X} \left(N_2^x(A) \to \mathscr{F}(A) \right)$$
$$= \lim_{N_2} \mathscr{F}(x).$$

Hence, $\lim_{N_1} \leq \lim_{N_2}$, i.e. $h^{-1}(N_1) \leq h^{-1}(N_2)$. So h^{-1} is also an order preserving mapping.

From the above proof, we conclude that $(\mathbf{PFYC}_{\mathbf{L}}(\mathbf{X}), \leq)$ and $(\mathbf{PrFN}_{\mathbf{L}}(\mathbf{X}), \leq)$ are isomorphic.

At the end of this section, we propose the following results.

Theorem 5.8. The category L-**POFYC** is a reflective subcategory of L-**OFYC**.

Proof. Let $(X, \overline{\lim}) \in L-\mathbf{OFYC}$ and $E_{\lim} = \{ \lim | (X, \lim) \in L-\mathbf{POFYC} , \lim \leq \overline{\lim} \}$. Note that $E_{\overline{\lim}}$ is not empty because it always contains \lim_{sm} . Then we can construct a principal L-ordered fuzzifying convergence structure $\overline{\lim}_* : \mathscr{F}_L(X) \to L^X$ as follows: For all $\mathscr{F} \in \mathscr{F}_L(X), x \in X, \overline{\lim}_* \mathscr{F}(x) = \bigwedge_{\lim \in E_{\overline{\lim}}} \lim \mathscr{F}(x)$. From this, we have

(1) $id_X: (X, \overline{\lim}) \to (X, \overline{\lim}_*)$ is trivially continuous;

(2) For a principal *L*-ordered fuzzifying convergence space (Y, \lim^Y) , if $f : (X, \overline{\lim}) \to (Y, \lim^Y)$ is a continuous mapping, then $f : (X, \overline{\lim}) \to (Y, \lim^Y)$ is also continuous.

From the above facts, we immediately obtain that L-**POFYC** is a full reflective subcategory in L-**OFYC**.

Corollary 5.9. The category L-POFYC is topological.

6. The Relations Between *L*-fuzzifying Topological Spaces and Topological *L*-ordered Fuzzifying Convergence Spaces

In this section, we define another important subcategory of L-OFYC: the category of topological L-ordered fuzzifying convergence spaces. We will find out that the category mentioned above is isomorphic to L-FYS and to L-TNGH in case of a completely distributive lattice L. Furthermore, each fibre on X of the category of topological L-ordered fuzzifying convergence spaces is isomorphic to that of L-fuzzifying topological spaces and that of topological L-fuzzifying neighborhood spaces.

Definition 6.1. Let $(X, \lim) \in L$ -**POFYC**, if in addition the mapping $\lim : \mathscr{F}_L(X) \to L^X$ satisfies the following axiom:

$$(LYT) \ \forall U \in 2^X, \ N_{\lim}^x(U) \le \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N_{\lim}^y(V),$$

then lim is called a topological L-ordered fuzzifying convergence structure, and (X, \lim) is a topological L-ordered fuzzifying convergence space. The full subcategory of L-**OFYC** consisting of all topological L-ordered fuzzifying convergence spaces is denoted by L-**TOFYC**.

If lim is a topological L-ordered fuzzifying convergence structure, then a nice characterization of L-fuzzifying topologies is possible. We need two lemmas for preparation.

Lemma 6.2. Let (X, N) be a topological L-fuzzifying neighborhood space, then (X, \lim_N) is a topological L-ordered fuzzifying convergence space.

Proof. As for (X, N) is a topological L-fuzzifying neighborhood space, (X, N) is an L-fuzzifying neighborhood space. By Theorem 5.3, we see $N = N_{\lim_N}$. For N is a topological L-fuzzifying neighborhood structure, we know for all $x \in$

 $X, \forall U \in 2^X, N^x(U) \leq \bigvee_{x \in V \subseteq U} \bigwedge_{y \in V} N^y(V)$. Therefore, N_{\lim_N} satisfies (LYT). With Definition 6.1, the lemma holds.

Lemma 6.3. Let $(X, \lim) \in L$ -**TOFYC**, then there exists an L-fuzzifying topology τ on X such that $\lim_{\tau} = \lim_{\tau \to \infty} dt$.

Proof. Firstly, by Definition 6.1, $(X, \lim) \in L$ -**TOFYC** implies that N_{\lim} satisfies (N1) - (N4).

Secondly, let $\tau : 2^X \to L, \forall A \in 2^X, \tau(A) = \bigwedge_{x \in A} N_{\lim}^x(A)$. It can be easily proved that τ is an *L*-fuzzifying topology on *X*. Moreover, $N_{\tau} = N_{\lim}$. In fact, for all $x \in X, A \in 2^X$, we have by the axiom (LYT),

$$N^{x}_{\tau}(A) = \bigvee_{x \in B \subseteq A} \tau(B)$$
$$= \bigvee_{x \in B \subseteq A} \bigwedge_{y \in B} N^{y}_{\lim}(B)$$
$$= N^{x}_{\lim}(A). \quad (LYT)$$

With this and (LYP), we obtain for all $\mathscr{F} \in \mathscr{F}_L(X), x \in X$, $\lim_{\tau} \mathscr{F}(x) = S_F(N^x_{\tau}, \mathscr{F}) = S_F(N^x_{\lim}, \mathscr{F}) = \lim_{\tau \to \infty} \mathscr{F}(x)$. Therefore, $\lim_{\tau} = \lim_{\tau \to \infty} \operatorname{holds}$.

Lemmas 6.2, 6.3 together with the relations between topological L-fuzzifying neighborhood spaces and L-fuzzifying topological spaces in case that L is a completely distributive lattice show the following result.

Theorem 6.4. If L is a completely distributive lattice, then L-**FYS**, L-**TOFYC** and L-**TNGH** are isomorphic to each other.

We denote a fibre on X of the category of L-fuzzifying topological spaces by $\mathbf{FY}_{\mathbf{L}}(\mathbf{X})$, and an order " \leq " on it is defined as follows: $\forall \tau_1, \tau_2 \in \mathbf{FY}_{\mathbf{L}}(\mathbf{X})$,

$$\tau_1 \le \tau_2 \Leftrightarrow \forall A \in 2^X, \tau_1(A) \le \tau_2(A).$$

Denote a fibre on X of the category of topological L-fuzzifying neighborhood spaces by $\mathbf{FN}_{\mathbf{L}}(\mathbf{X})$ and a fibre on X of the category of topological L-ordered fuzzifying convergence spaces by $\mathbf{TFYC}_{\mathbf{L}}(\mathbf{X})$. In the same way as in the proof of Theorem 5.7, we obtain the following theorem trivially, and leave the straightforward proof for the interested reader.

Theorem 6.5. $(\mathbf{FN}_{\mathbf{L}}(\mathbf{X}), \leq), \ (\mathbf{FY}_{\mathbf{L}}(\mathbf{X}), \leq), \ (\mathbf{TFYC}_{\mathbf{L}}(\mathbf{X}), \leq) \ are \ isomorphic.$

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WENCHAO WU*, DEPARTMENT OF MATHEMATICS, OCEAN UNIVERSITY OF CHINA, 266100 QING-DAO, P. R. CHINA

E-mail address: wuwenchao107@163.com

JINMING FANG, DEPARTMENT OF MATHEMATICS, OCEAN UNIVERSITY OF CHINA, 266100 QING-DAO, P. R. CHINA

E-mail address: jinming-fang@163.com

*Corresponding Author