STATISTICAL CONVERGENCE AND STRONG $p-$CESÀRO SUMMABILITY OF ORDER β IN SEQUENCES OF FUZZY NUMBERS

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ABSTRACT. In this study we introduce the concepts of statistical convergence of order β and strong $p-$Cesàro summability of order β for sequences of fuzzy numbers. Also, we give some relations between the statistical convergence of order β and strong $p-$Cesàro summability of order β and construct some interesting examples.

1. Introduction

The theory of sequences of fuzzy numbers was first introduced by Matloka [15]. Matloka [15] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. Since that time, there has been increasing interest in the study of sequences of fuzzy numbers. (see [1, 2, 3, 4, 7, 8, 12, 13, 14, 15, 16, 17, 19, 20]). The notion of statistical convergence was introduced by Fast [9] and Schoenberg [18], independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Connor [5], Fridy [10], Tripathy [20], and many others. However, the order of statistical convergence for a sequence of positive linear operators was given by Gadjiev and Orhan [11]. Subsequently, for $\alpha \in (0,1]$, Çolak [6] introduced the strong $p-$Cesàro summability of order α for sequences of complex numbers. The statistical convergence and Cesàro summability in sequences of fuzzy numbers are very important topics in fuzzy mathematics. The purpose of this paper is to introduce statistical convergence of order β and strong $p-$Cesàro summability of order β in sequences of fuzzy numbers for $\beta \in (0,1]$, and is to give some relations between these concepts.

2. Preliminaries

In this section we give the basic notions related to fuzzy numbers and a brief information about statistical convergence and strong $p-$Cesàro summability of a sequence $X = (X_k)$ of fuzzy numbers. In addition, for $\beta \in (0,1]$ we define the

notions of statistical convergence of order β and strong $p-$Cesàro summability of order β for sequences of fuzzy numbers.

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

i) u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \le \lambda \le 1, u(\lambda x + (1 - \lambda)y) \ge \min[u(x), u(y)]$;

iii) u is upper semicontinuous;

iv) $\operatorname{supp} u = cl\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]^0$, is compact.

$\alpha-$level set $[u]^\alpha$ of a fuzzy number u is defined by

$$[u]^\alpha = \left\{ \begin{array}{ll} \{x \in \mathbb{R} : u(x) \ge \alpha\}, & \text{if } \alpha \in (0, 1] \\ \operatorname{supp} u, & \text{if } \alpha = 0. \end{array} \right.$$

It is clear that u is a fuzzy number if and only if $[u]^\alpha$ is a closed interval for each $\alpha \in [0, 1]$ and $[u]^1 \ne \emptyset$.

A real number r can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(x) = \left\{ \begin{array}{ll} 1, & x = r \\ 0, & x \ne r \end{array} \right. .$$

If $u \in L(\mathbb{R})$, then u is called a fuzzy number, and $L(\mathbb{R})$ is said to be a fuzzy number space.

Let $u, v \in L(\mathbb{R})$ and the $\alpha-$level sets of fuzzy numbers u and v be $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ and $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$, $\alpha \in (0, 1]$. Then, a partial ordering " $\le$ " in $L(\mathbb{R})$ is defined by $u \le v \Leftrightarrow \underline{u}^\alpha \le \underline{v}^\alpha$ and $\overline{u}^\alpha \le \overline{v}^\alpha$ for all $\alpha \in (0, 1]$.

In order to measure the distance between two fuzzy numbers u and v, we use the metric

$$d(u, v) = \sup_{0 \le \alpha \le 1} d_H([u]^\alpha, [v]^\alpha)$$

where d_H is the Hausdorff metric defined as

$$d_H([u]^\alpha, [v]^\alpha) = \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}.$$

It is known that d is a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), d)$ is a complete metric space [7].

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. Thus, a sequence of fuzzy numbers (X_k) is a correspondence between the set of positive integers and the set of fuzzy numbers, i.e., to each positive integer k there corresponds a fuzzy number $X(k)$. It is more common to write X_k rather than $X(k)$ and to denote the sequence by (X_k) rather than X. The fuzzy number X_k is called the k-th term of the sequence.

The sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if there exist fuzzy numbers u and v such that $u \le X_k \le v$ for each $k \in \mathbb{N}$ and convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $d(X_k, X_0) < \varepsilon$ for $k > k_0$. Let $\ell_\infty(\mathcal{F})$ and $c(\mathcal{F})$ denote

the set of all bounded sequences and all convergent sequences of fuzzy numbers, respectively [15].

Nuray and Savaş [17] defined the notion of statistical convergence for sequences of fuzzy numbers:

Let $X = (X_k)$ be a sequence of fuzzy numbers. Then (X_k) is said to be statistically convergent to the fuzzy number X_0, if

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : d(X_k, X_0) \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$, where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim X_k = X_0$ or $X_k \overset{s}{\to} X_0$.

Kwon [14] defined the concept of strong $p-$Cesàro summability for sequences of fuzzy numbers and examined relations between statistical convergence and strong $p-$Cesàro summability:

Let p be a positive real number. A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly $p-$Cesàro summable to the fuzzy number X_0, if there is a fuzzy number X_0 such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [d(X_k, X_0)]^p = 0.$$

Çolak [6] introduced $\beta-$density of subset A of the set $\mathbb{N}$ as follows:

Let β be any real number such that $\beta \in (0, 1]$. Then, $\beta-$density of subset A of the set $\mathbb{N}$ of all positive integers is defined by

$$\delta_\beta(A) = \lim_{n \to \infty} \frac{1}{n^\beta} |\{k \leq n : k \in A\}|$$

provided the limit exists, where $|\{k \leq n : k \in A\}|$ denotes the number of elements of A not exceeding n.

Obviously, any finite subset of $\mathbb{N}$ has zero $\beta-$density and $\delta_\beta(A^c) = 1 - \delta_\beta(A)$ for $\beta = 1$, but equality does not hold for $\beta \in (0, 1)$ in general. Also, the $\beta-$density of any set reduces to the natural density of the set for $\beta = 1$.

Definition 2.1. Let $\beta \in (0, 1]$ and $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent of order β, to fuzzy number X_0 if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n^\beta} |\{k \leq n : d(X_k, X_0) \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $S^\beta(\mathcal{F}) - \lim X_k = X_0$. We denote the set of all statistically convergent sequences of order β by $S^\beta(\mathcal{F})$.

It is clear that a sequence (X_k) of fuzzy numbers is levelwise statistically convergent of order β, to the fuzzy number X_0 if and only if $S^\beta(\mathcal{F}) - \lim_{k \to \infty} [\underline{X}_k]^\alpha = [\underline{X}_0]^\alpha$ and $S^\beta(\mathcal{F}) - \lim_{k \to \infty} [\overline{X}_k]^\alpha = [\overline{X}_0]^\alpha$ for all $\alpha \in [0, 1]$ and $\beta \in (0, 1]$, i.e. $\frac{1}{n^\beta} |\{k \leq n : d([X_k]^\alpha, [X_0]^\alpha) \geq \varepsilon\}| \to 0$ as $n \to \infty$ for all $\alpha \in [0, 1]$.

The statistical convergence of order β is the same with the statistical convergence for $\beta = 1$. Although the statistical convergence of order β is well defined for $\beta \in (0, 1]$, but it is not well defined for $\beta > 1$. (See Example 2.2).

Example 2.2. Define the sequence $X = (X_k)$ of fuzzy numbers as follows:

$$X_k(x) = \left\{ \begin{array}{ll} \left. \begin{array}{ll} x - 1, & \text{for } 1 \le x \le 2 \\ -x + 3, & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{array} \right\} := l_1, & \text{if } k \text{ is odd} \\ \left. \begin{array}{ll} x - 4, & \text{for } 4 \le x \le 5 \\ -x + 6, & \text{for } 5 \le x \le 6 \\ 0, & \text{otherwise} \end{array} \right\} := l_2, & \text{if } k \text{ is even} \end{array} \right.$$

Then, we calculate $\alpha-$level set of this sequence as follows:

$$[X_k]^\alpha = \left\{ \begin{array}{ll} [1 + \alpha, 3 - \alpha], & \text{if } k \text{ is odd} \\ [4 + \alpha, 6 - \alpha], & \text{if } k \text{ is even} \end{array} \right.$$

From this, we obtain

$$\lim_{n \to \infty} \frac{1}{n^\beta} \left| \{ k \le n : d([X_k]^\alpha, [l_1]^\alpha) \ge \varepsilon \} \right| \le \lim_{n \to \infty} \frac{n}{2n^\beta} = 0$$

and

$$\lim_{n \to \infty} \frac{1}{n^\beta} \left| \{ k \le n : d([X_k]^\alpha, [l_2]^\alpha) \ge \varepsilon \} \right| \le \lim_{n \to \infty} \frac{n}{2n^\beta} = 0$$

for $\beta > 1$, where $[l_1]^\alpha = [1 + \alpha, 3 - \alpha]$ and $[l_2]^\alpha = [4 + \alpha, 6 - \alpha]$ and hence sequence $X = (X_k)$ statistically converges of order β, both to fuzzy number l_1 and l_2. But this is not true for statistical convergence. (See Figure 1)
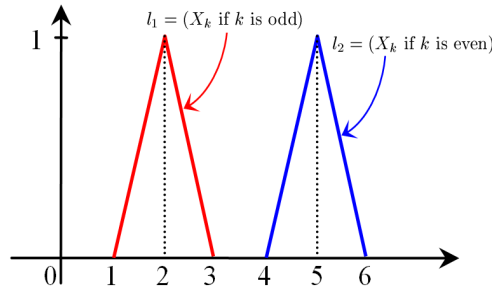

FIGURE 1. (X_k) is Statistically Convergent of Order β, Both to
Fuzzy Numbers l_1 and l_2 for $\beta > 1$

Theorem 2.3. *Let* $X = (X_k)$ *and* $Y = (Y_k)$ *be two sequences of fuzzy numbers and* $\beta \in (0, 1]$. *Then*

(i) If $S^\beta(\mathcal{F}) - \lim X_k = X_0$ *and* $c \in \mathbb{R}$, *then* $S^\beta(\mathcal{F}) - \lim cX_k = cX_0$.

(ii) If $S^\beta(\mathcal{F}) - \lim X_k = X_0$ and $S^\beta(\mathcal{F}) - \lim Y_k = Y_0$, then $S^\beta(\mathcal{F}) - \lim(X_k + Y_k) = X_0 + Y_0$.

Proof. (i) It follows from the inequality

$$\frac{1}{n^\beta}\left|\{k \leq n : d(cX_k, cX_0) \geq \varepsilon\}\right| \leq \frac{1}{n^\beta}\left|\left\{k \leq n : d(X_k, X_0) \geq \frac{\varepsilon}{|c|}\right\}\right|.$$

(ii) It follows from the inequality

$$\frac{1}{n^\beta}\left|\{k \leq n : d(X_k + Y_k, X_0 + Y_0) \geq \varepsilon\}\right| \leq \frac{1}{n^\beta}\left|\{k \leq n : d(X_k, X_0) + d(Y_k, Y_0) \geq \varepsilon\}\right|$$

$$\leq \frac{1}{n^\beta}\left|\left\{k \leq n : d(X_k, X_0) \geq \frac{\varepsilon}{2}\right\}\right|$$

$$+ \frac{1}{n^\beta}\left|\left\{k \leq n : d(Y_k, Y_0) \geq \frac{\varepsilon}{2}\right\}\right|.$$

$\square$

It is easy to see that every convergent sequence $X = (X_k)$ of fuzzy numbers is statistically convergent of order β. But the converse is not true, for this we can give the following example.

Example 2.4. Consider the sequence $X = (X_k)$ of fuzzy numbers as follows:

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} x - 4, & \text{for } 4 \leq x \leq 5 \\ -x + 6, & \text{for } 5 \leq x \leq 6 \\ 0, & \text{otherwise} \end{array}\right\} & \begin{array}{c} \text{if } k = m^3 \\ (m = 1, 2, 3, ...) \end{array} \\ \left.\begin{array}{cc} x - 1, & \text{for } 1 \leq x \leq 2 \\ -x + 3, & \text{for } 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{array}\right\} := X_0 & \text{if } k \neq m^3 \end{cases}$$

Then, we calculate $\alpha-$level set of this sequence as follows:

$$[X_k]^\alpha = \begin{cases} [4 + \alpha, 6 - \alpha], & \text{if } k = m^3 \\ [1 + \alpha, 3 - \alpha], & \text{if } k \neq m^3 \end{cases}$$

and so we have

$$\lim_{n \to \infty} \frac{1}{n^\beta}\left|\{k \leq n : d([X_k]^\alpha, [X_0]^\alpha) \geq \varepsilon\}\right| \leq \lim_{n \to \infty} \frac{\sqrt[3]{n}}{n^\beta}$$

Thus, $X = (X_k)$ is statistically convergent of order β, to fuzzy number X_0, where $[X_0]^\alpha = [1 + \alpha, 3 - \alpha]$ for $\beta \in \left(\frac{1}{3}, 1\right]$, but not convergent.

Definition 2.5. Let $\beta \in (0, 1]$ and p be a positive real number. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be strongly $p-$Cesàro summable of order β, to fuzzy number X_0 if there is a fuzzy number X_0 such that

$$\lim_{n \to \infty} \frac{1}{n^\beta} \sum_{k=1}^{n} [d(X_k, X_0)]^p = 0.$$

The strongly $p-$Cesàro summability of order β reduces to the strongly $p-$Cesàro summability in case $\beta = 1$. We denote the set of all strongly $p-$Cesàro summable sequences of order β by $w^\beta(\mathcal{F}, p)$.

3. **Main Results**

In this section, we give the relation between statistical convergence of order β and statistical convergence of order γ, and the relation between strong $p-$Cesàro summability of order β and strong $p-$Cesàro summability of order γ for $\beta, \gamma \in (0, 1]$. Also, we show that if a sequence $X = (X_k)$ of fuzzy numbers is statistically convergent of order β, and $X = (X_k)$ is bounded, then it doesn't need to be strongly $p-$Cesàro summable of order β and give an example of sequence of triangular fuzzy numbers.

Theorem 3.1. *Let $0 < \beta \leq \gamma \leq 1$. Then, if a sequence $X = (X_k)$ of fuzzy numbers is statistically convergent of order β, to the fuzzy number X_0, then it is statistically convergent of order γ, to the fuzzy number X_0, i.e. $S^\beta(\mathcal{F}) \subseteq S^\gamma(\mathcal{F})$ and the inclusion is strict for some β and γ such that $\beta < \gamma$.*

Proof. Let $0 < \beta \leq \gamma \leq 1$. Then we have

$$\frac{1}{n^\gamma} |\{k \leq n : d(X_k, X_0) \geq \varepsilon\}| \leq \frac{1}{n^\beta} |\{k \leq n : d(X_k, X_0) \geq \varepsilon\}|$$

for every $\varepsilon > 0$ and so we get $S^\beta(\mathcal{F}) \subseteq S^\gamma(\mathcal{F})$.

Now, we show that the inclusion is strict. For this, consider the sequence $X = (X_k)$ of fuzzy numbers as follows

$$X_k(x) = \begin{cases} \left. \begin{cases} x - 3, & \text{for } 3 \leq x \leq 4 \\ -x + 5, & \text{for } 4 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases} \right\} & \begin{array}{c} \text{if } k = m^2 \\ (m = 1, 2, 3, ...) \end{array} \\ \left. \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ -x + 2, & \text{for } 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases} \right\} := X_0 & \text{if } k \neq m^2 \end{cases}$$

Then, the $\alpha-$level set of sequence (X_k) is

$$[X_k]^\alpha = \begin{cases} [3 + \alpha, 5 - \alpha], & \text{if } k = m^2 \\ [\alpha, 2 - \alpha], & \text{if } k \neq m^2. \end{cases}$$

It can be easily seen that $X = (X_k)$ is statistically convergent of order γ, to fuzzy number X_0, for $\gamma \in \left(\frac{1}{2}, 1\right]$, where $[X_0]^\alpha = [\alpha, 2 - \alpha]$, but not statistically convergent of order β for $\beta \in \left(0, \frac{1}{2}\right]$, i.e. $X \in S^\gamma(\mathcal{F})$, but $X \notin S^\beta(\mathcal{F})$. This completes the proof. $\qquad\square$

Corollary 3.2. *If a sequence $X = (X_k)$ of fuzzy numbers is statistically convergent of order β, to fuzzy number X_0 for some $\beta \in (0, 1]$, then it is statistically convergent to fuzzy number X_0, i.e. $S^\beta(\mathcal{F}) \subset S(\mathcal{F})$ and the inclusion is strict.*

Theorem 3.3. *Let $0 < \beta \leq \gamma \leq 1$ and p be a positive real number. If a sequence $X = (X_k)$ of fuzzy numbers is strongly $p-$Cesàro summable of order β, to fuzzy number X_0, then it is strongly $p-$Cesàro summable of order γ, to X_0, i.e. $w^\beta(\mathcal{F}, p) \subseteq w^\gamma(\mathcal{F}, p)$ and the inclusion is strict for some β and γ such that $\beta < \gamma$.*

Proof. Suppose that the sequence $X = (X_k)$ is strongly $p-$Cesàro summable of order β, to X_0 such that $0 < \beta \leq \gamma \leq 1$. Then we can write

$$\frac{1}{n^\gamma} \sum_{k=1}^{n} [d\,(X_k, X_0)]^p \leq \frac{1}{n^\beta} \sum_{k=1}^{n} [d\,(X_k, X_0)]^p$$

for $p > 0$. It follows that $X = (X_k)$ is strongly $p-$Cesàro summable of order γ, to X_0 since right side of the inequality tends to zero as $n \to \infty$. i.e. $w^\beta\,(\mathcal{F}, p) \subseteq w^\gamma\,(\mathcal{F}, p)$.

In order to show that the inclusion is strict, consider the sequence $X = (X_k)$ of fuzzy numbers defined in the proof of Theorem 3.1. Then, it is easy to see that

$$\frac{1}{n^\gamma} \sum_{k=1}^{n} [d\,([X_k]^\alpha, [X_0]^\alpha)]^p \leq \frac{3\sqrt{n}}{n^\gamma}$$

for $p = 1$. Now, it follows that the sequence $X = (X_k)$ is strongly $p-$Cesàro summable of order γ, to fuzzy number X_0 for $\gamma \in \left(\frac{1}{2}, 1\right]$, i.e. $X \in w^\gamma\,(\mathcal{F}, p)$, but since

$$\frac{3\sqrt{n} - 3}{n^\beta} \leq \frac{1}{n^\beta} \sum_{k=1}^{n} [d\,([X_k]^\alpha, [X_0]^\alpha)]^p$$

for $p = 1$, we find that the sequence $X = (X_k)$ is not strongly $p-$Cesàro summable of order β as $n \to \infty$ for $\beta \in \left(0, \frac{1}{2}\right)$, i.e. $X \notin w^\beta\,(\mathcal{F}, p)$. Hence, the converse of the theorem does not hold. $\square$

Corollary 3.4. *Let $0 < \beta \leq \gamma \leq 1$ and $p > 0$. Then,*

(i) $w^\beta\,(\mathcal{F}, p) = w^\gamma\,(\mathcal{F}, p)$ *if and only if $\beta = \gamma$.*

(ii) $w^\beta\,(\mathcal{F}, p) \subseteq w\,(\mathcal{F}, p)$ *for each $\beta \in (0, 1]$.*

Theorem 3.5. *Let $0 < \beta \leq \gamma \leq 1$ and p be a positive real number. Then, if a sequence $X = (X_k)$ of fuzzy numbers is strongly $p-$Cesàro summable of order β, to fuzzy number X_0, then it is statistically convergent of order γ, to fuzzy number X_0. i.e. $w^\beta\,(\mathcal{F}, p) \subseteq S^\gamma\,(\mathcal{F})$.*

Proof. Let $p > 0$ and $0 < \beta \leq \gamma \leq 1$. Given $\varepsilon > 0$, for any sequence $X = (X_k)$ of fuzzy numbers, we may write

$$\sum_{k=1}^{n} [d\,(X_k, X_0)]^p \geq |\{k \leq n : [d\,(X_k, X_0)]^p \geq \varepsilon\}| \,.\varepsilon^p$$

and so

$$\frac{1}{n^\beta} \sum_{k=1}^{n} [d\,(X_k, X_0)]^p \geq \frac{1}{n^\beta} |\{k \leq n : [d\,(X_k, X_0)]^p \geq \varepsilon\}| \,.\varepsilon^p$$

$$\geq \frac{1}{n^\gamma} |\{k \leq n : [d\,(X_k, X_0)]^p \geq \varepsilon\}| \,.\varepsilon^p.$$

Hence, if $X = (X_k)$ is strongly $p-$Cesàro summable of order β, to fuzzy number X_0, then it is statistically convergent of order γ, to fuzzy number X_0. $\square$

If we take $\gamma = \beta$ in this theorem, we obtain the following result.

Corollary 3.6. *Let $\beta \in (0,1]$ and $0 < p < \infty$. If a sequence of fuzzy numbers is strongly $p-$Cesàro summable of order β, to fuzzy number X_0, then it is statistically convergent of order β, to fuzzy number X_0, i.e. $w^\beta(\mathcal{F}, p) \subset S^\beta(\mathcal{F})$ and the inclusion is strict.*

To show that the inclusion is strict, we give an example as follows:

Example 3.7. Consider the sequence

$$X_k(x) = \begin{cases} \begin{cases} \frac{x}{k^2} + 1, & \text{for } -k^2 \le x \le 0 \\ -\frac{x}{k^2} + 1, & \text{for } 0 \le x \le k^2 \\ 0, & \text{otherwise} \end{cases} & \text{if } k = m^3 \\ & (m = 1, 2, 3, ...) \\ \begin{cases} x - 3, & \text{for } 3 \le x \le 4 \\ -x + 5, & \text{for } 4 \le x \le 5 \\ 0, & \text{otherwise} \end{cases} := X_0 & \text{if } k \ne m^3 \end{cases}$$

Then, the $\alpha-$level set of sequence (X_k) is

$$[X_k]^\alpha = \begin{cases} \left[-k^2(1-\alpha), k^2(1-\alpha)\right], & \text{if } k = m^3 \\ [3+\alpha, 5-\alpha], & \text{if } k \ne m^3 \end{cases}.$$

Hence we conclude that the sequence (X_k) is statistically convergent of order β, to the fuzzy number X_0 for $\beta \in \left(\frac{1}{3}, 1\right]$, where $[X_0]^\alpha = [3+\alpha, 5-\alpha]$ (See Figure 2), but it is not strongly $p-$Cesàro summable for $p = 1$.
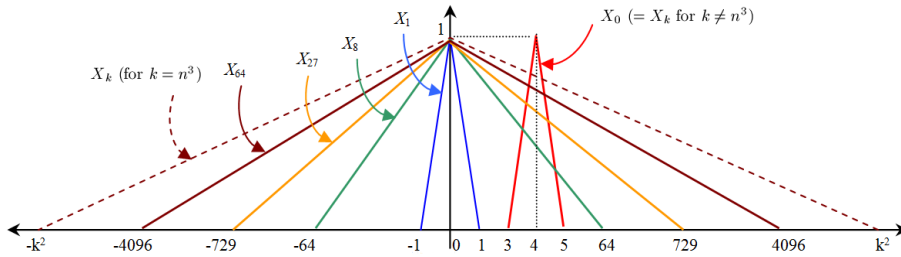

FIGURE 2. (X_k) is Statistical Convergent of Order β, to
Fuzzy Number X_0

Corollary 3.8. *Let $\beta \in (0,1]$ and $0 < p < \infty$. Then, if a sequence $X = (X_k)$ of fuzzy numbers is strongly $p-$Cesàro summable of order β, to the fuzzy number X_0, then it is statistically convergent to the fuzzy number X_0, i.e. $w^\beta(\mathcal{F}, p) \subset S(\mathcal{F})$.*

Proof. If a sequence $X = (X_k)$ of fuzzy numbers is strongly $p-$Cesàro summable of order β, to the fuzzy number X_0, then it is statistically convergent to fuzzy number

X_0 from Corollary 3.6 and Corollary 3.2. We show that the converse is not true. For this consider the sequence

$$X_k(x) = \left\{ \begin{array}{l} \left. \begin{array}{ll} x-3, & \text{for } 3 \le x \le 4 \\ -x+5, & \text{for } 4 \le x \le 5 \\ 0, & \text{otherwise} \end{array} \right\} \quad \begin{array}{l} \text{if } k = m^3 \\ (m = 1,2,3,...) \end{array} \\ \left. \begin{array}{ll} x, & \text{for } 0 \le x \le 1 \\ -x+2, & \text{for } 1 \le x \le 2 \\ 0, & \text{otherwise} \end{array} \right\} := X_0 \qquad \text{if } k \ne m^3 \end{array} \right.$$

Then, the $\alpha-$level set of the sequence (X_k) is

$$[X_k]^\alpha = \left\{ \begin{array}{ll} [3+\alpha, 5-\alpha], & \text{if } k = m^3 \\ [\alpha, 2-\alpha], & \text{if } k \ne m^3 \end{array} \right. .$$

Now, it is easy to see that the sequence $X = (X_k)$ is statistically convergent to the fuzzy number X_0, where $[X_0]^\alpha = [\alpha, 2-\alpha]$, but it is not strongly $p-$Cesàro summable of order β for $\beta \in \left(0, \frac{1}{3}\right]$ and $p = 1$. Really, we can write

$$\frac{3\sqrt[3]{n}-3}{n^\beta} \le \frac{1}{n^\beta} \sum_{k=1}^n [d([X_k]^\alpha, [X_0]^\alpha)]^p.$$

Hence, it follows that the sequence (X_k) is not strongly $p-$Cesàro summable of order β for $\beta \in \left(0, \frac{1}{3}\right]$ and $p = 1$, since $\frac{3\sqrt[3]{n}-3}{n^\beta} \to \infty$ as $n \to \infty$. This completes the proof. $\qquad\square$

Remark 3.9. It is known that if a sequence (X_k) of fuzzy numbers is bounded and statistical convergent, then it is strongly $p-$Cesàro summable. But, we see that if the sequence (X_k) of fuzzy numbers is bounded and statistical convergent of order β, then it cannot be strongly $p-$Cesàro summable of order β for $\beta \in (0,1)$. For this, define the sequence (X_k) of fuzzy numbers as follows:

$$X_k(x) = \left\{ \begin{array}{l} \left. \begin{array}{ll} \frac{2\sqrt{k}-2\sqrt{k}\sqrt{k+1}x}{\sqrt{k}-\sqrt{k+1}}, & \text{for } \frac{1}{\sqrt{k+1}} \le x \le \frac{\sqrt{k}+\sqrt{k+1}}{2\sqrt{k}\sqrt{k+1}} \\ \frac{2\sqrt{k}\sqrt{k+1}x-2\sqrt{k+1}}{\sqrt{k}-\sqrt{k+1}}, & \text{for } \frac{\sqrt{k}+\sqrt{k+1}}{2\sqrt{k}\sqrt{k+1}} \le x \le \frac{1}{\sqrt{k}} \\ 0, & \text{otherwise} \end{array} \right\}, \quad \begin{array}{l} \text{if } k \ne m^2 \\ (m=1,2,3,...) \end{array} \\ \left. \begin{array}{ll} x-2, & \text{for } 2 \le x \le 3 \\ -x+4, & \text{for } 3 \le x \le 4 \\ 0, & \text{otherwise} \end{array} \right\}, \qquad \text{if } k = m^2 \end{array} \right.$$

Then, the $\alpha-$level set of sequence (X_k) is

$$[X_k]^\alpha = \left\{ \begin{array}{ll} \left[\frac{-\alpha(\sqrt{k}-\sqrt{k+1})+2\sqrt{k}}{2\sqrt{k}\sqrt{k+1}}, \frac{\alpha(\sqrt{k}-\sqrt{k+1})+2\sqrt{k+1}}{2\sqrt{k}\sqrt{k+1}} \right], & \text{if } k \ne m^2 \\ [2+\alpha, 4-\alpha], & \text{if } k = m^2 \end{array} \right.$$

It is easy to see that the sequence (X_k) is statistical convergent of order β, to the fuzzy number $\bar{0} = [0,0]$ for $\beta \in (0,1]$ and bounded. Now we show that (X_k) is not strongly $p-$Cesàro summable.
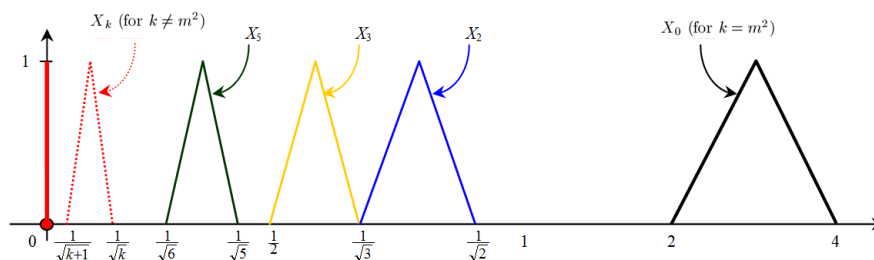
FIGURE 3. (X_k) is Statistical Convergent of Order β, to Fuzzy

Number $\bar{0}$ and Bounded

Firstly, we define the set $K_n = \left\{ k \leq n : k \neq m^2, \ m \in \mathbb{N} \right\}$. Hence we may write

$$\frac{1}{n^\beta} \sum_{k=1}^n \left[d\left([X_k]^\alpha, [0,0]\right)\right]^p = \frac{1}{n^\beta} \sum_{k \in K_n} \left[d\left([X_k]^\alpha, [0,0]\right)\right]^p + \frac{1}{n^\beta} \sum_{k \notin K_n} \left[d\left([X_k]^\alpha, [0,0]\right)\right]^p$$

$$= \frac{1}{n^\beta} \sum_{k \in K_n} \frac{1}{\sqrt{k}} + \frac{1}{n^\beta} \sum_{k \notin K_n} 4 > \frac{1}{n^\beta} \sum_{k \in K_n} \frac{1}{\sqrt{k}} + \frac{1}{n^\beta} \sum_{k \notin K_n} \frac{1}{\sqrt{k}}$$

$$= \frac{1}{n^\beta} \sum_{k=1}^n \frac{1}{\sqrt{k}} > \frac{1}{n^\beta} \sqrt{n}$$

for $p = 1$. Therefore, it follows that the sequence (X_k) is not strongly $p-$Cesàro summable for $\beta \in \left(0, \frac{1}{2}\right]$.

4. **Conclusions**

In this study, we develope the concepts of statistical convergence and strong $p-$Cesàro summability for sequences of fuzzy numbers by taking $\beta \in (0,1]$ and give some relations between these two concepts for sequences of fuzzy numbers. However, boundedness and statistical convergence of a sequence (X_k) of fuzzy numbers implies strongly $p-$Cesàro summability in case $\beta = 1$, but it is demonstrated that if the sequence (X_k) of fuzzy numbers is bounded and statistically convergent of order β, then it can not be strongly $p-$Cesàro summable of order β for $0 < \beta < 1$ (Remark 3.9).

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