# A FUZZY VERSION OF HAHN-BANACH EXTENSION THEOREM

#### L. ZEDAM

ABSTRACT. In this paper, a fuzzy version of the analytic form of Hahn-Banach extension theorem is given. As application, the Hahn-Banach theorem for r-fuzzy bounded linear functionals on r-fuzzy normed linear spaces is obtained.

## 1. Introduction

Hahn-Banach theorem is one of the most famous and useful result in functional analysis. Ramakrishnan [15] established the norm-preserving fuzzy completion of a fuzzy normed algebra and gave a fuzzy extension of Hahn-Banach theorem. In the same year Rhie and Hwang [16] investigated the relation between fuzzy seminorms and crisp seminorms on a linear space X and extended the analytic form of the Hahn-Banach theorem with the notion of fuzzy seminorm. In recent years, a fuzzy version of Hahn-Banach theorem on a vector space over the set of fuzzy real numbers and some related applications were proved by Binimol and Sunny Kuriakose [6, 7]. There are also many other fuzzy versions of Hahn-Banach theorem for fuzzy bounded linear operators on fuzzy normed spaces (see e.g. [2, 9, 12, 19] etc...).

In this paper, using the definition of fuzzy order due to L. A. Zadeh (see [21]), we assume that the set of real numbers  $\mathbb{R}$  endowed with a fuzzy order r instead of the natural order  $\leq$  and prove a new fuzzy version of the analytic form of Hahn-Banach theorem. As application, the Hahn-Banach theorem for r-fuzzy bounded linear functionals on r-fuzzy normed linear spaces is obtained.

## 2. Preliminaries

We begin with a number of definitions related to fuzzy orders. We follow the notation and vocabulary of Zadeh [21] closely, and refer the reader to Amroune and Davvaz [1], Beg [3], Bernadette [4], Billot [5], Bodenhofer and et.al. [8], Kundu [10], Li and Yen [11], Ovchinnikov [13, 14], Stouti and Zedam [17], Venugopalan [18], Zadeh [21] and Zimmermann [22] for elementary definitions and facts about fuzzy order relations.

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [20] in 1965.

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Let X be a nonempty set, a fuzzy subset A of X is characterized by its membership function  $A: X \to [0, 1]$  and A(x) is interpreted as the degree of membership of the element x in the fuzzy subset A for each  $x \in X$ .

In [21], Zadeh gave the following definition of fuzzy order.

**Definition 2.1.** [21] Let X be a nonempty set. A Zadeh's binary fuzzy partial order (briefly, fuzzy order) on X is a fuzzy subset r on  $X \times X$  in which the following conditions are satisfied:

(i) for all  $x \in X$ , r(x, x) = 1, (fuzzy reflexivity);

(ii) for all  $x, y \in X$ , (r(x, y) > 0 and  $x \neq y$ ) implies (r(y, x) = 0), (fuzzy antisymmetry);

(iii) for all  $x, y, z \in X$ ,  $r(x, z) \ge \max_{y \in X} [\min\{r(x, y), r(y, z)\}]$ , (fuzzy transitivity).

Note that each crisp order  $\leq$  on X can be considered a fuzzy order defined by r(x, y) = 1 if  $x \leq y$  and r(x, y) = 0 if x and y are incomparable elements.

A nonempty set X with a fuzzy order r defined on it is called fuzzy ordered set (for short, foset) and we denote it by (X, r).

If Y is a subset of a foset (X, r), then the restriction of r to Y is a fuzzy order in Y and is called induced fuzzy order.

A fuzzy order r is linear (or total) on X if for every  $x, y \in X$ , we have r(x, y) > 0or r(y, x) > 0. If  $x \neq y$ , by the fuzzy antisymmetry of r, clearly only one of these conditions can be satisfied. A fuzzy ordered set (X, r) in which r is total is called a r-fuzzy chain. Conversely, if for any  $x, y \in X$ , r(x, y) > 0 if and only if x = y, then (X, r) is called r-fuzzy antichain.

Next, we give some examples of fuzzy order.

**Example 2.2.** Let  $X = \{a, b, c, d, e, f, g\}$ . Then the fuzzy subset r defined on  $X \times X$  by the following table:

	a	b	с	d	е	f	g
a	1	0	0	0.55	0.40	0.45	0.60
b	0	1	0	0.60	0.50	0.35	0.75
c	0.15	-0	1	0.30	0.70	0.80	0.90
d	0	0	0	1	0	0.15	0
е	0	0	0	0	1	0.30	0.25
f	0	0	0	0	0	1	0
g	0	0	0	0	0	0.20	1

is a fuzzy order on X.

**Example 2.3.** Let  $x, y \in \mathbb{R}$ . Then the fuzzy subset  $r_{\lambda}$  defined for all  $x, y \in \mathbb{R}$  by:

$$r_{\lambda}(x,y) = \begin{cases} 1, & \text{if } x = y; \\ \min(1, \frac{y-x}{\lambda}), & \text{if } x < y \\ 0, & \text{if } x > y; \end{cases}$$

is a total fuzzy order on  ${\rm I\!R}.$ 

Clearly,  $0 \leq r_{\lambda}(x, y) \leq 1$  for all  $x, y \in \mathbb{R}$ . Thus  $r_{\lambda}$  is well defined. Now let us show that  $r_{\lambda}$  is a fuzzy order on  $\mathbb{R}$ .

1) For all  $x \in \mathbb{R}$ ,  $r_{\lambda}(x, x) = 1$ . Thus  $r_{\lambda}$  is fuzzy reflexive.

2) Let  $x, y \in \mathbb{R}$  with  $x \neq y$ . Then,  $r_{\lambda}(x, y) > 0$  is true only in the case x < y. So,  $r_{\lambda}$  is fuzzy antisymmetric.

3) Let  $x, y, z \in \mathbb{R}$ . Then, we have three cases to study.

3.i) If  $r_{\lambda}(x,z) = 1$ , then  $r_{\lambda}(x,z) \ge \min\{r_{\lambda}(x,y), r_{\lambda}(y,z)\}$ , for all  $y \in \mathbb{R}$ .

3.ii) If  $r_{\lambda}(x, z) = \frac{z-x}{\lambda} > 0$ , then x < z. Hence, for  $y \in \mathbb{R}$  we have three cases to consider:

(a) if x < z < y, then  $r_{\lambda}(y, z) = 0$ .

(b) If  $x \le y \le z$ , so  $\frac{z-x}{\lambda} \ge \frac{z-y}{\lambda}$ . Hence, we get  $r_{\lambda}(x,z) \ge r_{\lambda}(y,z)$ .

(c) If y < x < z, then  $r_{\lambda}(x, \hat{y}) = 0$ . Thus  $r_{\lambda}(x, z) \ge \min\{r_{\lambda}(x, y), r_{\lambda}(y, z)\}$ , for all  $y \in \mathbb{R}$ .

3.iii) If  $r_{\lambda}(x,z) = 0$ , then x > z. So, for every  $y \in \mathbb{R}$  we have three cases:

(a) if  $x > z \ge y$ , then  $r_{\lambda}(x, y) = 0$ .

(b) If  $x \ge y > z$ , so  $r_{\lambda}(y, z) = 0$ .

(c) If y > x > z, hence  $r_{\lambda}(y, z) = 0$ .

Hence,  $r(x, z) \ge \min\{r_{\lambda}(x, y), r_{\lambda}(y, z)\}$ , for all  $y \in \mathbb{R}$ . Thus,  $r_{\lambda}$  is fuzzy transitive. Therefore,  $r_{\lambda}$  is a fuzzy order on  $\mathbb{R}$ .

Since for all  $x, y \in \mathbb{R}$ , such that  $x \neq y$  we have either x < y or y < x. Then, we get either  $\min(1, \frac{y-x}{\lambda}) > 0$  or  $\min(1, \frac{x-y}{\lambda}) > 0$  Thus,  $r_{\lambda}$  is a total fuzzy order.

**Example 2.4.** Let  $X = \mathbb{R}$ . Then, the fuzzy relation r defined for all  $x, y \in \mathbb{R}$  by:

$$r(x,y) = \begin{cases} 1 \ , \ if \ x = y; \\ 0 \ , \ if \ x > y; \\ 1 - \frac{x}{y} \ , \ if \ 0 \le x < y; \\ 1 - \frac{y}{x} \ , \ if \ x < y \le 0; \\ 1 \ , \ if \ x < 0 \ and \ y > 0; \end{cases}$$

is a total fuzzy order on  ${\rm I\!R}.$  ]

Clearly,  $0 \le r(x, y) \le 1$  for all  $x, y \in \mathbb{R}$ . Thus r is well defined. Now let us show that r is a fuzzy order on  $\mathbb{R}$ .

1) For all  $x \in \mathbb{R}$ , r(x, x) = 1. Thus r is fuzzy reflexive.

2) Let  $x, y \in \mathbb{R}$  such that  $x \neq y$ . Then, we have r(x, y)r(y, x) = 0. So, r is fuzzy antisymmetric.

3) Let  $x, y, z \in \mathbb{R}$ . Then, we have four cases to study.

3.i) If r(x, z) = 1, then  $r(x, z) \ge \min\{r(x, y), r(y, z)\}$ , for all  $y \in \mathbb{R}$ .

3.ii) If r(x, z) = 0, then x > z. Hence, for every  $y \in \mathbb{R}$  we distinguish the following subcases.

(a) If  $x > z \ge y$ , then it holds that r(x, y) = 0.

(b) If  $x \ge y > z$ , then it holds that r(y, z) = 0.

(c) If y > x > z, then it holds that r(y, z) = 0.

Thus,  $r(x, z) \ge \min\{r(x, y), r(y, z)\}$ , for all  $y \in \mathbb{R}$ .

3.iii) If  $r(x, z) = 1 - \frac{x}{z}$ , then  $0 \le x < z$ . Hence, for  $y \in \mathbb{R}$  we have four cases to consider:

(a) If  $0 \le x < z < y$ , then r(y, z) = 0.

(b) If  $0 \le x < y < z$ , so  $1 - \frac{x}{z} \ge 1 - \frac{y}{z}$ . Hence, we get  $r(x, z) \ge r(y, z)$ . (c) If  $0 \le y < x < z$ , then r(x, y) = 0.

(d) If  $y < 0 \le x < z$ , so r(x, y) = 0.

Thus  $r(x, z) \ge \min\{r(x, y), r(y, z)\}$ , for all  $y \in \mathbb{R}$ .

3.iv) If  $r(x,z) = 1 - \frac{z}{r}$ , then by using a similar argument as in the case (3.iii) we can see that  $r(x, z) \ge \min\{r(x, y), r(y, z)\}$ , for all  $y \in \mathbb{R}$ .

Hence, r is fuzzy transitive. Thus, r is a fuzzy order on  $\mathbb{R}$ .

As for all  $x, y \in \mathbb{R}$ , such that  $x \neq y$  we have either x < y or y < x, then we get either  $r(x,y) = 1 - \frac{x}{y} > 0$  or  $r(y,x) = 1 - \frac{y}{x} > 0$ . Thus, r is a total fuzzy order.

**Definition 2.5.** Let (X, r) be a fuzzy ordered set and A be a subset of X.

(a) An element  $u \in X$  is an r-upper bound of A if r(x, u) > 0 for all  $x \in A$ . The set of all r-upper bounds of A is denoted by  $A^u$ . If u is the r-upper bound of A and  $u \in A$ , then u is called a greatest element of A. The r-lower bound and least element are defined analogously and the set of all r-lower bounds of A is denoted by  $A^{\ell}$ .

(b) An element  $m \in A$  is called a maximal element of A if there is no  $x \neq m$  in A for which r(m, x) > 0. x = m. Minimal elements are defined similarly.

(c) As usual, the r-supremum of A is defined by  $\sup_r(A) =$  the least element of r-upper bounds of A (if it exists). Similarly, the r-infimum of A defined by  $\inf_r(A) =$  the greatest element of *r*-lower bounds of A (if it exists).

We write  $x \vee_r y$  the r-supremum and  $x \wedge_r y$  the r-infimum of the set  $\{x, y\}$ . For linear fuzzy order,  $x \lor_r y = \max_r \{x, y\}$  and  $x \land_r y = \min_r \{x, y\}$ .

**Definition 2.6.** Let r be a fuzzy order on  $\mathbb{R}$  and  $x \in \mathbb{R}$ . If r(0, x) > 0, then x is called an r-positive real number. The set of them all is denoted by  $\mathbb{R}_r^+$ . Similarly, if r(x,0) > 0 then x is called an r-negative real number, and the set of them all is denoted by  $\mathbb{R}_r^-$ .

**Definition 2.7.** 1) Let r be a fuzzy order on  $\mathbb{R}$ . We say that r is compatible with the addition if for all  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ , we have

 $(r(x_1, y_1) > 0 \text{ and } r(x_2, y_2) > 0) \Longrightarrow (r(x_1 + x_2, y_1 + y_2) > 0).$ 

2) The fuzzy order r is said to be compatible with the multiplication by scalars if for all  $(x, y) \in \mathbb{R}^2$  and  $\lambda > 0$ , we have

$$r(x, y) > 0) \Longrightarrow (r(\lambda x, \lambda y) > 0).$$

**Example 2.8.** The fuzzy order relation given in Example 2.4 is compatible with the addition and multiplication by scalars on  ${\rm I\!R}.$ 

(i) r is compatible with the addition. Indeed, let  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$  such that  $r(x_1, y_1) > 0$  and  $r(x_2, y_2) > 0$ . By the definition of r we get that  $x_1 \leq y_1$  and

 $x_2 \leq y_2$ . Then,  $x_1 + x_2 \leq y_1 + y_2$ . Hence,  $r(x_1 + x_2, y_1 + y_2) > 0$ . Thus, r is compatible with the addition.

(ii) r is compatible with the multiplication by scalars. Indeed, let  $(x, y) \in \mathbb{R}^2$  such that r(x, y) > 0 and  $\lambda > 0$ . By the definition of r we get that  $x \leq y$ . Then,  $\lambda x \leq \lambda y$ . Hence,  $r(\lambda x, \lambda y) > 0$ . Thus, r is compatible with the multiplication by scalars.

Therefore, r is compatible with the addition and multiplication by scalar on  $\mathbb{R}$ . Next, we show the following two propositions which we shall need for proving a

fuzzy version of Hahn-Banach theorem.

**Proposition 2.9.** Let  $\mathbb{R}_r = (\mathbb{R}, r)$  be the set of all real numbers endowed with a fuzzy order r compatible with the addition and the multiplication by scalar, and  $x, y \in \mathbb{R}$ . Then we have the following:

- i) If r(0, x) > 0 then r(-x, 0) > 0.
- *ii)* If r(0, x) > 0 then r(-x, x) > 0.

*Proof.* Let  $x, y \in \mathbb{R}_r$ . i) Since r(0, x) > 0 and by the fuzzy reflexivity r(-x, -x) = 0

1 > 0, then from the compatibility of r with the addition we have that r(0 + (-x), x + (-x)) > 0. Hence, r(-x, 0) > 0.

ii) We assume that r(0, x) > 0. It is clear from (i) that r(-x, 0) > 0. Then, from the compatibility of r with the addition we have that r(-x, x) > 0.

**Proposition 2.10.** Let  $\mathbb{R}_r = (\mathbb{R}, r)$  be the set of all real numbers endowed with a fuzzy order r compatible with the addition and multiplication by scalar, and let  $x, y \in \mathbb{R}$  such that  $x \neq y$ . Then the following are equivalent.

(*i*) r(x, y) > 0;

(ii) There exists  $\tau \in \mathbb{R}$  such that  $r(x,\tau) > 0$  and  $r(\tau,y) > 0$ , (r-fuzzy density).

*Proof.* Let  $x, y \in \mathbb{R}_r$  such that  $x \neq y$  and r(x, y) > 0. For the one direction, let  $\tau = \frac{x+y}{2}$ . Since r(x, x) = 1 > 0 and r(x, y) > 0, from the compatibility of r with the addition we get that

$$r(x+x, x+y) > 0.$$

Now, by the compatibility of r with the multiplication we obtain that

$$r(x, \frac{x+y}{2}) > 0.$$

Thus,  $r(x, \tau) > 0$ .

In the same way we get that  $r(\tau, y) > 0$ .

The other direction follows directly from the fuzzy transitivity.

## 3. Results

In this section we assume that  $\mathbb{R}_r$  is the set of real numbers  $\mathbb{R}$  endowed with a fuzzy order r compatible with the addition and multiplication by scalar instead of the natural order  $\leq$  and we shall prove a fuzzy version of Hahn-Banach extension theorem. The prove of this fuzzy version will follow the same steps as the crisp case. As application, we define the notion of r-fuzzy normed space with the help of r-fuzzy norm as a generalization of crisp normed space, we introduce the notion

of r-fuzzy bounded linear functional and we prove the Hahn-Banach theorem for r-fuzzy bounded linear functionals on r-fuzzy normed linear spaces.

**Definition 3.1.** Let X be an real linear space, and T a mapping of X into  $\mathbb{R}_r$ . We say that T is a r-fuzzy sublinear functional on X if

i) r(T(x+y), T(x) + T(y)) > 0 for all  $x, y \in X$ , (*r*-subadditivity);

ii)  $T(\lambda x) = \lambda T(x)$  for all  $x \in X$  and  $\lambda \in \mathbb{R}^+_r$ , (*r*-positively homogeneous).

**Example 3.2.** The mapping  $T : \mathbb{R}_r \longrightarrow \mathbb{R}_r$  defined by  $T(x) = |x|_r = \max_r \{x, -x\}$  is an *r*-fuzzy sublinear functional on  $\mathbb{R}_r$ .

The following is a useful fact for r-fuzzy sublinear functionals.

**Proposition 3.3.** If T is an r-fuzzy sublinear functional on a real linear space X then  $r(\lambda T(x), T(\lambda x)) > 0$ , for all  $x \in X$  and  $\lambda \in \mathbb{R}_r$ .

Proof. Let  $x \in X$  and  $\lambda \in \mathbb{R}_r$ . If  $\lambda \in \mathbb{R}_r^+$  we have  $T(\lambda x) = \lambda T(x)$ . Hence,  $r(\lambda T(x), T(\lambda x)) = 1 > 0.$  (1)

If  $\lambda \in \mathbb{R}_r^-$ , then from Proposition 2.9(i) we get that  $-\lambda \in \mathbb{R}_r^+$ . As  $\lambda T(x) = -(-\lambda T(x))$  so by the *r*-positively homogeneous of *T* we have  $\lambda T(x) = -(-\lambda T(x)) = -T(-\lambda x)$ . On the other hand, since  $T(\lambda x - \lambda x) = T(0) = 0$ , by the *r*-subadditivity of *T* we have  $r(T(\lambda x + (-\lambda x)), T(\lambda x) + T(-\lambda x)) > 0$ . Hence,  $r(0, T(\lambda x) + T(-\lambda x)) > 0$ . Now, from the compatibility of *r* with the addition we have  $r(-T(-\lambda x), T(\lambda x)) > 0$ . Thus,

$$r(\lambda T(x), T(\lambda x)) > 0.$$
<sup>(2)</sup>

Therefore, (1) and (2) implies that  $r(\lambda T(x), T(\lambda x)) > 0$ , for all  $x \in X$  and  $\lambda \in \mathbb{R}_r$ .

**Theorem 3.4** (Fuzzy version of Hahn-Banach theorem). Let  $X_0$  be a subspace of a real linear space X, T a r-fuzzy sublinear functional on X, and  $u_0$  be an linear functional on  $X_0$  such that  $r(u_0(x), T(x)) > 0$  for all  $x \in X_0$ . Then there exists a linear functional u on X extends  $u_0$  to X and satisfies r(u(x), T(x)) > 0, for all  $x \in X$ .

*Proof.* Let  $y \in X$  such that  $y \notin X_0$  and denote by Y the vector subspace generated by  $X_0 \cup \{y\}$ , so

$$Y = \{x_0 + \lambda y \mid x_0 \in X_0 \text{ and } \lambda \in \mathbb{R}_r - \{0\}\}$$

Let  $\tau \in \mathbb{R}_r$ , and provisionally define

$$u(x_0 + \lambda y) = u_0(x_0) + \lambda \tau.$$

It is easy to show that u is a linear extension of  $u_0$  to Y; hence it remains to choose  $\tau \in \mathbb{R}_r$  such that for all  $x_0 \in X_0$ , and  $\lambda \in \mathbb{R}_r - \{0\}$ ,

$$r(u_0(x_0) + \lambda \tau, T(x_0 + \lambda y)) > 0.$$
 (3)

For all  $\lambda \in \mathbb{R}_r^+ - \{0\}$ , replacing  $x_0$  by  $\lambda x_0$ , using the *r*-positive homogeneity of T, and from the compatibility of r with the multiplication, it suffices to see that

$$r(u_0(x_0) + \tau, T(x_0 + y)) > 0.$$
(4)

$$r(u_0(x_0) + \tau, T(x_0 + y)) > 0.$$

Therefore, from the r-fuzzy compatibility of r with the addition, it suffices to  $r(\tau T(r_0 \pm u) = u_1(m)) > 0$ see that

$$r(\tau, T(x_0 + y) - u_0(x_0)) > 0.$$
 (5)

For all  $\lambda \in \mathbb{R}_r^- - \{0\}$ , replacing  $x_0$  by  $\lambda x_0$ , using the *r*-positive homogeneity of T, and from the compatibility of r with the multiplication, we observe that it suffices to see that

$$r(-u_0(x_0) - \tau, T(-x_0 - y)) > 0 \tag{6}$$

Therefore, from the r-fuzzy compatibility of r with the addition, it suffices to see that

$$r(-u_0(x_0) - T(-x_0 - y), \tau) > 0.$$
<sup>(7)</sup>

To see the existence of  $\tau \in \mathbb{R}_r$  satisfying (5) and (6), start by observing that

$$r(-u_0(x_0) - T(-x_0 - y), T(x_0 + y) - u_0(x_0)) \ge \min\{r(-u_0(x_0), -u_0(x_0)), r(-T(-x_0 - y), T(x_0 + y))\} \ge \min\{1, r(-T(-x_0 - y), T(x_0 + y))\},$$

In addition, from Proposition 3.3 we have  $r(-T(-x_0 - y), T(x_0 + y)) > 0$ . Then  $r(-u_0(x_0) - T(-x_0 - y), T(x_0 + y) - u_0(x_0)) > 0$ ,

and therefore by Proposition 2.10 there exists  $\tau \in \mathbb{R}_r$  satisfies (5) and (6). Hence, there exists  $\tau \in \mathbb{R}_r$  satisfies (3).

Next, we shall give an application of r-fuzzy Hahn-Banach theorem, but in this subsection, we assume that r is linear order on  $\mathbb{R}$  compatible with the addition and multiplication.

**Definition 3.5.** Let X be a real linear space. An r-fuzzy norm on X is a mapping  $x \mapsto \|x\|_r$  from X into  $\mathbb{R}^+_r$  such that for all  $x, y \in X$  and  $\lambda \in \mathbb{R}^+_r$ , the following properties hold:

- i)  $||x||_r = 0$  if and only if x = 0.
- ii)  $\|\lambda x\|_r = |\lambda|_r \|x\|_r$ . iii)  $r(\|x+y\|_r, \|x\|_r + \|y\|_r) > 0$ .

A linear space X equipped with an r-fuzzy norm  $\|.\|_r$  is called an r-fuzzy normed linear space. We denote it by  $(X, \|.\|_r)$ .

**Example 3.6.** The *r*-fuzzy absolute value  $|x|_r = x \vee_r (-x)$  is an *r*-fuzzy norm on  $\mathbb{R}_r$ .

i) Let  $x \in \mathbb{R}_r$ , since r is a total order we have either r(0, x) > 0 or r(0, -x) > 0. Then by Proposition 2.9(ii) we have either r(-x, x) > 0 or r(x, -x) > 0.

Hence,  $r(0, |x|_r) > 0$ .

ii) Obvious.

iii)  $\|\lambda x\|_r = \lambda x \lor_r (-\lambda x) = |\lambda|_r x \lor_r (-|\lambda|_r x) = |\lambda|_r (x \lor_r (-x)) = |\lambda|_r \|x\|_r.$ 

iv) Let  $x, y \in \mathbb{R}_r$ . To prove that  $r(||x + y||_r, ||x||_r + ||y||_r) > 0$  six cases are considered.

a) If r(0, x) > 0 and r(0, y) > 0 then

$$r(|x+y|_r, |x|_r + |y|_r) = r(x+y, x+y) = r > 0.$$

b) If r(x,0) > 0 and r(y,0) > 0 then  $r(|x+y|_r, |x|_r + |y|_r) = r(-x-y, -x-y) = r > 0.$ c) If r(0,x) > 0, r(y,0) > 0 and r(x, -y) > 0 then  $r(|x+y|_r, |x|_r + |y|_r) = r(-x-y, x-y)$   $\ge \min\{r(-x,x), r(-y, -y)\} > 0.$ d) If r(0,x) > 0, r(y,0) > 0 and r(-y,x) > 0 then  $r(|x+y|_r, |x|_r + |y|_r) = r(x+y, x-y)$   $\ge \min\{r(x,x), r(y, -y)\} > 0.$ e) If r(x,0) > 0, r(0,y) > 0 and r(y, -x) > 0 then  $r(|x+y|_r, |x|_r + |y|_r) = r(-x-y, -x+y)$   $\ge \min\{r(-x, -x), r(-y, y)\} > 0.$ f) If r(x,0) > 0, r(0,y) > 0 and r(-x,y) > 0 then  $r(|x+y|_r, |x|_r + |y|_r) = r(x+y, -x+y)$  $\ge \min\{r(x, -x), r(y, y)\} > 0.$ 

**Definition 3.7.** Let  $(X, \|.\|_r)$  and  $(Y, \|.\|_r)$  be *r*-fuzzy normed linear spaces. A linear operator *u* from *X* into *Y* is called an *r*-fuzzy bounded operator if there exists  $K \in \mathbb{R}_r^+$  such that

$$r(||u(x)||_r, K||x||_r) > 0,$$
 for all  $x \in X.$ 

**Remark 3.8.** The *r*-fuzzy norms in X and Y are different. But we use same notation  $\|.\|_r$ , because there is no confusion.

**Example 3.9.** Let  $(X, \|.\|_r)$  be an *r*-fuzzy normed linear space, we define an operator  $u: (X, \|.\|_r) \longrightarrow (X, \|.\|_r)$  by  $u(x) = \lambda x$  where  $\lambda \neq 0 \in \mathbb{R}$  is fixed. Clearly u is an *r*-fuzzy bounded linear operator.

In the following Lemma we describe the r-fuzzy boundedness of a linear operator between r-fuzzy normed linear spaces by means of an r-fuzzy norm of it.

**Lemma 3.10.** Let u be an r-fuzzy bounded linear operator from  $(X, \|.\|_r)$  into  $((Y, \|.\|_r)$ . Then there exists an r-fuzzy norm of u, denoted by  $\|u\|_r$  such that:

 $r(||u(x)||_r, ||u||_r ||x||_r) > 0,$  for all  $x \in X.$ 

*Proof.* Since u is an r-fuzzy bounded linear operator, there exists  $K \in \mathbb{R}_r^+$  such that

 $r(||u(x)||_r, K||x||_r) > 0,$  for all  $x \in X.$ 

From the compatibility of r with the multiplication we obtain

$$r(\frac{\|u(x)\|_r}{\|x\|_r}, K) > 0, \qquad \text{for all } x \in X.$$

Hence,

$$r(\sup_{r} \{ \frac{\|u(x)\|_{r}}{\|x\|_{r}} : x \in X \}, K) > 0, \quad \text{for all } x \in X.$$

This means that  $\sup_r(\frac{\|u(x)\|_r}{\|x\|_r})$  is finite.

Now we put  $||u||_r = \sup_r \{ \frac{||u(x)||_r}{||x||_r} : x \in X \}$ . It is clear that  $||u||_r = 0$  if and only if u = 0, and that  $||\lambda u||_r = |\lambda|_r ||u||_r$ . Since

$$r(||u+v(x)||_r, ||u(x)||_r + ||v(x)||_r) > 0,$$
 for all  $x \in X$ ,

it follows from the compatibility of r with the multiplication that

$$r(\frac{\|u+v(x)\|_{r}}{\|x\|_{r}}, \frac{\|u(x)\|_{r}}{\|x\|_{r}} + \frac{\|v(x)\|_{r}}{\|x\|_{r}}) > 0, \quad \text{for all } x \in X.$$

Then we obtain

$$r(||u+v||_r, ||u||_r + ||v||_r) > 0.$$

Hence,  $||u||_r$  is a *r*-fuzzy norm of *u*.

In addition, as  $||u||_r = \sup_r \{ \frac{||u(x)||_r}{||x||_r} : x \in X \}$ , we get

$$r(\frac{\|u(x)\|_r}{\|x\|_r}, \|u\|_r) > 0,$$
 for all  $x \in$ 

which implies

$$r(||u(x)||_r, ||u||_r ||x||_r) > 0,$$
 for all  $x \in X$ 

**Theorem 3.11.** Let  $X_0$  be a subspace of an r-fuzzy normed linear space X, and  $u_0$  be an r-fuzzy bounded linear functional on  $X_0$ . Then there exists an r-fuzzy bounded linear functional u on X such that  $u(x) = u_0(x)$  for all  $x \in X_0$  and  $||u||_r = ||u_0||_r$ .

*Proof.*  $T(x) = ||u_0||_r ||x||_r$ . It is easy to see that T(x) is an r-fuzzy sublinear functional on X. Since  $u_0$  is an r-fuzzy bounded linear functional on  $X_0$ , we obtain for all  $x \in X_0$ , that

$$r(u_0(x), T(x)) = r(u_0(x), ||u_0||_r ||x||_r) > 0.$$

Then from the r-fuzzy Hahn-Banach theorem there exists a linear functional u on X extends  $u_0$  to X and satisfies  $r(u(x), ||u_0||_r ||x||_r) > 0$ , for all  $x \in X$ . Moreover, for all  $x \in X$  we have

$$r(u(-x), ||u_0||_r|| - x||_r) > 0.$$

This shows that

$$r(-u(x), ||u_0||_r ||x||_r) > 0.$$

Hence,

$$r(|u(x)|_r, ||u_0||_r ||x||_r) > 0.$$

Therefore, u is an r-fuzzy bounded linear functional on X and satisfies

$$r(||u||_r, ||u_0||_r) > 0.$$

But u extends  $u_0$ , so  $r(||u_0||_r, ||u||_r) > 0$  and therefore  $||u||_r = ||u_0||_r$ .

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LEMNAOUAR ZEDAM, DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS AND INFOR-MATICS, M'SILA UNIVERSITY, P.O.BOX 166 ICHBILIA, M'SILA 28105, ALGERIA

E-mail address: L.zedam@yahoo.fr