A NOTE ON THE RELATIONSHIP BETWEEN HUTTON'S QUASI-UNIFORMITIES AND SHI'S QUASI-UNIFORMITIES

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Abstract. This note studies the relationship between Hutton's quasi-uniformities and Shi's quasi-uniformities. It is shown that when L satisfies"multiple choice principle" for co-prime elements, the category of Hutton's quasi-uniform spaces is a bireflective full subcategory of the category of Shi's quasi-uniform spaces. Especially, if the remote-neighborhood mapping defined by Shi preserves arbitrary joins, then the two categories are isomorphic to each other.

1. **Introduction and Preliminaries**

Uniformity is a very important concept close to topology and a convenient tool for investigating topology (See [2]). Many researchers have tried to establish the theories of uniformities in the fuzzy setting and obtained a series of interesting results (See [4, 6, 7, 8, 9, 10, 11, 12, 14, 17, 18, 19]). There are also many papers studying the relationships between all kinds of uniformities (See [4, 13, 20]). The aim of this note is to deal with the relationship between Hutton quasi-uniformities and Shi quasi-uniformities. It is shown that when L satisfies "multiple choice principle" for co-prime elements, the category of Hutton quasi-uniform spaces is a bireflective full subcategory of the category of Shi quasi-uniform spaces. Especially, if remote-neighborhood mapping defined by Shi preserves joins, then the two categories are isomorphic to each other.

An element a in a complete lattice L is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of all coprimes of L is denoted by $c(L)$. In this paper, L is a completely distributive lattice with an order reversing involution $'$, and we always assume that L satisfies "multiple choice principle" for coprime elements, i.e.,

$$e \leq \bigvee_{t\in T} l_t \Rightarrow \exists t_0 \in T \text{ s.t., } e \leq l_{t_0}.$$

From [5], we know that a completely distributive lattice satisfies " multiple choice principle" above is exactly a completely distributive algebraic lattice or equivalently, the set of all lower sets of a poset. In domain theory, it is well known that the category of completely distributive algebraic lattice is dually equivalent to the category of posets. For more lattice theories, please refer to [3, 5]

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L^X denotes the set of all L-fuzzy sets on X. $A' \in L^X$ is defined by $A'(x) = (A(x))'$. The set of all coprimes of L^X is denoted by $c(L^X)$. Let $F : X \to Y$ be an ordinary mapping, define $F_L^{\to} : L^X \to L^Y$ and $F_L^{\leftarrow} : L^Y \to L^X$ by $F_L^{\to}(A)(y) = \bigvee\{A(x)|\ x \in X, F(x) = y\}$ for $A \in L^X$ and $y \in Y$, and $F_L^{\leftarrow}(B)(x) = B(F(x))$ for $B \in L^Y$ and $x \in X$, respectively.

$H(L^X)$ denotes the family of all mappings $f : L^X \to L^X$ such that:

(1) $A \le f(A)$ for all $A \in L^X$;

(2) $f(\bigvee_{j\in J} A_j) = \bigvee_{j\in J} f(A_j)$ for $\{A_j\}_{j\in J} \subseteq L^X$.

f_1 is the biggest element in $H(L^X)$, i.e., $f_1(A) = 0_X$ when $A = 0_X$ and $f_1(A) = 1_X$ others. Suppose $F : X \to Y$ is a mapping, $g \in H(L^Y)$, define $F^{\Leftarrow}(g) : L^X \to L^X$ by $F^{\Leftarrow}(g)(A) = F_L^{\leftarrow} \circ g \circ F_L^{\to}(A)$ for all $A \in L^X$, then $F^{\Leftarrow}(d) \in H(L^X)$.

Definition 1.1. A Hutton quasi-uniformity on a set X is a subset $HU \subseteq H(L^X)$ satisfying the following conditions:

(HU1) $f_1 \in HU$;

(HU2) $f \in HU, f \le g \in H(L^X) \Rightarrow g \in HU$;

(HU3) $f, g \in HU \Rightarrow f \wedge g \in HU$;

(HU4) $f \in HU \Rightarrow \exists g \in HU,\ g \circ g \le f$.

The pair (L^X, HU) is called a Hutton quasi-uniform space. A mapping $F : (L^X, HU) \to (L^Y, HU_1)$ is called uniformly continuous if $F^{\Leftarrow}(g) \in HU$ holds for all $g \in HU_1$. The categories of Hutton quasi-uniform spaces is denoted by L-**HuQUnif**.

$\phi : c(L^X) \to L^X$ is called a remote-neighborhood mapping if $x_\lambda \not\le \phi(x_\lambda)$ for all $x_\lambda \in c(L^X)$. $D(L^X)$ denotes the set of all remote-neighborhood mappings on $c(L^X)$. ϕ_0 is the smallest element of $D(L^X)$, i.e., $\phi_0(x_\lambda) = 0$ for all $x_\lambda \in c(L^X)$. For $\phi, \psi \in D(L^X)$, we define

(1) $\phi \le \psi$ if and only if $\phi(x_\lambda) \le \psi(x_\lambda)$ for all $x_\lambda \in c(L^X)$,

(2) $(\phi \vee \psi)(x_\lambda) = \phi(x_\lambda) \vee \psi(x_\lambda)$ for all $x_\lambda \in c(L^X)$

(3) $(\phi \diamond \psi)(x_\lambda) = \bigwedge\{\phi(y_\mu)|\ y_\mu \in c(L^X),\ y_\mu \not\le \psi(x_\lambda)\}$ for all $x_\lambda \in c(L^X)$.

Then $\phi \vee \psi \in D(L^X)$, $\phi \diamond \psi \in D(L^X)$, $\phi \diamond \psi \le \phi$, $\phi \diamond \psi \le \psi$ and the operations $\vee$ and $\diamond$ satisfy associative law.

Definition 1.2. A Shi quasi-uniformity is a set $SH \subseteq D(L^X)$ satisfying the following conditions:

(SHU1) $\phi_0 \in SH$;

(SHU2) $\phi \in SH, \phi \ge \psi \in D(L^X) \Rightarrow \psi \in SH$;

(SHU3) $\phi, \psi \in SH \Rightarrow \phi \vee \psi \in SH$;

(SHU4) $\phi \in SH \Rightarrow \exists \psi \in SH,\ \psi \diamond \psi \ge \phi$.

The pair (L^X, SH) is called a Shi quasi-uniform space. A mapping $F : (L^X, SH) \to (L^Y, SH_1)$ is called uniformly continuous if $F^{\Leftarrow}(\psi) \in SH$ for all $\psi \in SH_1$, where $F^{\Leftarrow}(\psi) : c(L^X) \to L^X$ is defined by $F^{\Leftarrow}(\psi)(x_\lambda) = F_L^{\leftarrow} \circ \psi \circ F_L^{\to}(x_\lambda)$. The category of Shi's quasi-uniform spaces is denoted by L-**ShQUnif**.

2. The Relationship Between Hutton's Quasi-uniformities and Shi's Quasi-uniformities

Due to "multiple choice principle" , we have the following lemma.

Lemma 2.1. *Let $f \in H(L^X)$ and define $s^f : c(L^X) \to L^X$ as follows:*

$$s^f(e) = \bigvee_{A \not\leq e'} f(A)'.$$

Then $s^f \in D(L^X)$.

Lemma 2.2. *The following statements are valid:*

(1) If $f, g \in H(L^X)$ and $f \leq g$, then $s^f \geq s^g$;

(2) If $f \in H(L^X)$, then $s^{f\circ f} \leq s^f \diamond s^f$.

Proof. (1) is obvious. We prove (2). From the definition of s^f, we have the following formulas

$$s^f \diamond s^f(e) = \bigwedge_{\lambda \not\leq s^f(e)} s^f(\lambda) = \bigwedge_{\lambda \not\leq \bigvee_{e \not\leq B'} f(B)'} \bigvee_{\lambda \not\leq A'} f(A)'$$

and

$$s^{f\circ f}(e) = \bigvee_{e \not\leq A'} [f \circ f(A)]' = \bigvee_{e \not\leq A'} f(f(A))'.$$

Now we show that $f(f(A))' \leq s^f \diamond s^f(e)$ for all $e \not\leq A'$. Let $e \not\leq A'$. Then $e \not\leq f(A)'$. $\forall \lambda \not\leq \bigvee_{e \not\leq B'} f(B)'$, then $\lambda \not\leq f(A)'$. Hence $f(f(A))' \leq \bigvee_{\lambda \not\leq C'} f(C)'$. Thus $f(f(A))' \leq \bigwedge_{\lambda \not\leq \bigvee_{e \not\leq B'} f(B)'} \bigvee_{\lambda \not\leq C'} f(C)' = s^f \diamond s^f(e)$, as desired. $\square$

Lemma 2.3. *Let HU be a Hutton quasi-uniformity and set $SH^{HU} = \{\phi \in D(L^X) | \exists f \in HU, s.t., \phi \leq s^f\}$. Then SH^{HU} is a Shi quasi-uniformity.*

Proof. We only prove (SHU4). Let $\phi \in SH^{HU}$. Then there exists $f \in HU$ such that $\phi \leq s^f$. Since $f \in HU$, there exists $g \in HU$ such that $g \circ g \leq f$. From Lemma 2.2 , we have $s^g \diamond s^g \geq s^{g\circ g} \geq s^f \geq \phi$, as desired. $\square$

In [15], Shi gave the following way to generate Hutton quasi-uniformity from Shi quasi-uniformity:

$$HU^{SH} = \{f \in H(L^X) | \exists \phi \in SH, \text{s.t.}, f \geq h^\phi\},$$

where $h^\phi : L^X \to L^X$ is defined by $h^\phi(A) = \bigvee_{e \not\leq A'} \phi(e)'$ and $h^\phi \in H(L^X)$.

Theorem 2.4. *Let HU be a Hutton quasi-uniformity. Then $HU = HU^{SH^{HU}}$.*

Proof. Let $f \in HU$. Then $s^f \in SH^{HU}$. Now we show $f \geq h^{s^f}$. In fact,

$$h^{s^f}(A) = \bigvee_{e \not\leq A'} s^f(e)' = \bigvee_{e \not\leq A'} \bigwedge_{e \not\leq B'} f(B) \leq \bigvee_{e \not\leq A'} f(A) = f(A)$$

Hence $f \in HU^{SH^{HU}}$. This is to say $HU \subseteq HU^{SH^{HU}}$.

On the other hand, let $f \in HU^{SH^{HU}}$. Then there exists $\phi \in SH^{HU}$ such that $f \geq h^\phi$ and there exists $g \in HU$ such that $\phi \leq s^g$. Hence $f \geq h^\phi \geq h^{s^g} \geq g$. The last inequality is due to

$$\begin{aligned} h^{s^g}(A) &= \bigvee_{e \not\leq A'} \bigwedge_{e \not\leq B'} g(B) \\ &= \bigwedge_{w \in \prod_{e \not\leq A'} J_e} \bigvee_{e \not\leq A'} g(w(e)) \\ &= \bigwedge_{w \in \prod_{e \not\leq A'} J_e} g(\bigvee_{e \not\leq A'} w(e)) \\ &\geq g(A) \end{aligned}$$

since $\bigvee_{e\not\leq A'} w(e) \geq A$. Thus $f \in HU$ which implies $HU^{SH^{HU}} \subseteq HU$. So $HU = HU^{SH^{HU}}$. $\square$

Theorem 2.5. *Let SH be a Shi quasi-uniformity. Then $SH^{HU^{SH}} \subseteq SH$. Furthermore, if each element in SH preserves arbitrary joins, then $SH^{HU^{SH}} = SH$.*

Proof. Let $\phi \in SH^{HU^{SH}}$. Then there exists $f \in HU^{SH}$ such that $\phi \leq s^f$ and there exists $\psi \in SH$ such that $f \geq h^\psi$. Hence $\phi \leq s^f \leq s^{h^\psi}$. We will show $s^{h^\psi} \leq \psi$. In fact,

$$s^{h^\psi}(e) = \bigvee_{A\not\leq e'} h^\psi(A)' = \bigvee_{A\not\leq e'} \bigwedge_{\lambda\not\leq A'} \psi(\lambda) \leq \bigvee_{A\not\leq e'} \psi(e) = \psi(e).$$

Hence $s \in SH$. This is to say $SH^{HU^{SH}} \subseteq SH$.

If each element in SH preserves arbitrary joins , then

$$\begin{aligned} s^{h^\psi}(e) &= \bigvee_{A\not\leq e'} \bigwedge_{\lambda\not\leq A'} \psi(\lambda) \\ &= \bigwedge_{w\in\prod_{A\not\leq e'} J_A} \bigvee_{A\not\leq e'} \psi(w(A)) \\ &= \bigwedge_{w\in\prod_{A\not\leq e'} J_A} \psi(\bigvee_{A\not\leq e'} w(A)) \\ &\geq \psi(e) \end{aligned}$$

Since $\bigvee_{A\not\leq e'} w(A) \geq e$. Hence $\psi \in SH$ implies $s^{h^\psi} \in SH^{HU^{SH}}$, and $\psi \in SH^{HU^{SH}}$ from the above result. So $SH \subseteq SH^{HU^{SH}}$, as desired. $\square$

Theorem 2.6. *(1) If $F : (L^X, HU_1) \to (L^Y, HU_2)$ is uniformly continuous, then $F : (L^X, SH^{HU_1}) \to (L^Y, SH^{HU_2})$ is also uniformly continuous.*

(2) If $F : (L^X, SH_1) \to (L^Y, SH_2)$ is uniformly continuous, then $F : (L^X, HU^{SH_1}) \to (L^Y, HU^{SH_2})$ is also uniformly continuous.

Proof. We only prove (1). Let $\phi \in SH^{HU_2}$. It suffices to show $F^{\Leftarrow}(\phi) \in SH^{HU_1}$. Since $\phi \in SH^{HU_2}$, there exists $g \in HU_2$ such that $\phi \leq s^g$. From the continuity of F, we have $F^{\Leftarrow}(g) \in HU_1$. Then $s^{F^{\Leftarrow}(g)} \in SH^{HU_1}$. Now we show that $F^{\Leftarrow}(\phi) \leq s^{F^{\Leftarrow}(g)}$. In fact, from

$$\begin{aligned} F^{\Leftarrow}(\phi)(e) &= F^{\leftarrow} \circ \phi \circ F^{\rightarrow}(e) \leq F^{\leftarrow} \circ s^g \circ F^{\rightarrow}(e) \\ &= F^{\leftarrow}(s^g(F^{\rightarrow}(e))) \\ &= \bigvee_{F(e)\not\leq B'} F^{\leftarrow}(g(B)') \\ &= \bigvee_{F(e)\not\leq B'} F^{\leftarrow}(g(B))' \\ &= \bigvee_{e\not\leq F^{\leftarrow}(B)'} F^{\leftarrow}(g(B))' \end{aligned}$$

and $s^{F^{\Leftarrow}(g)}(e) = \bigvee_{e\not\leq A'} F^{\leftarrow}(g(F^{\rightarrow}(A)))'$, we have $F^{\Leftarrow}(\phi) \leq s^{F^{\Leftarrow}(g)}$. Then $F^{\Leftarrow}(\phi) \in SH^{HU_1}$. Hence $F : (L^X, SH^{HU_1}) \to (L^Y, SH^{HU_2})$ is also uniformly continuous. $\square$

From Theorem 2.4, 2.5 and Theorem 2.6, we have the main result in this note.

Theorem 2.7. L-**HuQUnif** *is a bireflective full subcategory of* L-**ShQUnif**. *Especially, if remote-neighborhood mappings defined by Shi preserve arbitrary joins, then the two categories are isomorphic to each other. It is easy to verify that if* $c(L)$ *is an antichain (for example* L *is a Boolean algebra), then every remote- neighborhood mapping preserves arbitrary joins.*

Remark 2.8. The results in this note are valid under the condition that L satisfies "multiple choice principle" for coprime elements. Especially, when L is a finite distributive lattice, the results are right. From Theorem 2.7, we know that if the remote-neighborhood mappings preserve arbitrary joins, then L-**HuQUnif** and L-**ShQUnif** are isomorphic. Maybe the remote-neighborhood mappings which do not preserve arbitrary joins have no influence on the generated topologies.

At the end of this note, we have the following example :

Example 2.9. Let X be a single-point set and $L = \{0, a = \frac{1}{4}, b = \frac{3}{4}, 1\}$, where $a' = b$,$b' = a$. Hence, we know that $c(L) = \{a, b, 1\}$ and we do not distinguish L and L^X in the following discussion. It is easy to verify that $H(L^X)$ is the set $\{f^{ab}_{ab}, f^{a1}_{ab}, f^{bb}_{ab}, f^{b1}_{ab}, f^{11}_{ab}\}$ and $D(L^X)$ is the set $\{s^{000}_{ab1}, s^{00a}_{ab1}, s^{00b}_{ab1}, s^{0aa}_{ab1}, s^{0ab}_{ab1}, s^{0a0}_{ab1}\}$, where f^{ab}_{ab} and s^{000}_{ab1} are defined as follows:

$$f^{ab}_{ab}(\lambda) = \begin{cases} 0, & \lambda = 0, \\ 1, & \lambda = 1, \\ a, & \lambda = a, \\ b, & \lambda = b. \end{cases}$$

and

$$s^{000}_{ab1}(\lambda) = \begin{cases} 0, & \lambda = a, \\ 0, & \lambda = b. \\ 0, & \lambda = 1, \end{cases}$$

Others can be similarly defined.

There are four Hutton's quasi-uniformities as follows:

$HU_1 = \{f^{11}_{ab}\}$, $HU_2 = \{f^{11}_{ab}, f^{a1}_{ab}, f^{b1}_{ab}\}$, $HU_3 = \{f^{11}_{ab}, f^{b1}_{ab}, f^{bb}_{ab}\}$, $HU_4 = \{f^{ab}_{ab}, f^{a1}_{ab}, f^{bb}_{ab}, f^{b1}_{ab}, f^{11}_{ab}\}$.

There are four Shi's quasi-uniformities;

$SH_1 = \{s^{000}_{ab1}\}$, $SH_2 = \{s^{000}_{ab1}, s^{00a}_{ab1}, s^{00b}_{ab1}\}$,
$SH_3 = \{s^{000}_{ab1}, s^{00a}_{ab1}, s^{0aa}_{ab1}, s^{0a0}_{ab1}\}$, $SH_4 = \{s^{000}_{ab1}, s^{00a}_{ab1}, s^{00b}_{ab1}, s^{0aa}_{ab1}, s^{0ab}_{ab1}, s^{0a0}_{ab1}\}$.

The readers can easily check that s^{0a0}_{ab1} does not preserve joins, and also easily construct the one to one correspondence between two kind of mappings and quasi-uniformities.

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