FIXED POINT THEORY FOR CYCLIC φ-CONTRACTIONS IN FUZZY METRIC SPACES

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ABSTRACT. In this paper, the notion of cyclic φ-contraction in fuzzy metric spaces is introduced and a fixed point theorem for this type of mapping is established. Meantime, an example is provided to illustrate this theorem. The main result shows that a self-mapping on a G-complete fuzzy metric space has a unique fixed point if it satisfies the cyclic φ-contraction. Afterwards, some results in connection with the fixed point are given.

1. Introduction

The contraction type mappings in fuzzy metric spaces play a crucial role in fixed point theory. In 1988, Grabiec [5] first defined the Banach contraction in a fuzzy metric space and extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces. Following Grabiec's approach, Mishra et al. [11] obtained some common fixed point theorems for asymptotically commuting mappings on fuzzy metric spaces in 1994. In 1998, Vasuki [21] offered a generalization of Grabiec's fuzzy Banach contraction theorem and proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Afterwards, Cho [3] presented the concept of compatible mappings of type (α) in fuzzy metric spaces and studied the fixed point theory. Several years later, Singh and Chauhan [20] introduced the concept of compatible mapping and proved two common fixed point theorems in the fuzzy metric space with the continuous triangular norm min. In 2002, Sharma [18] further extended some known results of fixed point theory for compatible mappings in fuzzy metric spaces. In the same year, Gregori and Sapena [6] introduced the notion of fuzzy contractive mapping and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. Soon after, Mihet [8] proposed a fuzzy Banach theorem for (weak) B-contraction in M-complete fuzzy metric spaces. Recently, further studies have been done by different authors [1, 2, 9, 10, 15, 17, 19]. Meanwile, some new notions of contraction mappings have been introduced, such as Edelstein fuzzy contractive mappings, fuzzy

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ψ-contraction of (ε, λ) type, etc. Besides, Qiu et al. [13, 14] also obtained some common fixed point theorems for fuzzy mappings under certain conditions.

In 2010, the concept of cyclic φ-contraction is introduced by Păcurar and Rus in Ref. [12]. Meantime, they constructed a fixed point theorem for the cyclic φ-contraction in a classical complete metric space. In addition, several problems in connection with the fixed point are investigated. Based on these ideas the notion of cyclic φ-contraction in a fuzzy metric space is presented in this paper. Furthermore, a fixed point theorem is established for a G-complete fuzzy metric space in the sense of George and Veeramani. Then, some problems related to the fixed point are discussed.

2. **Preliminaries**

For completeness and clarity, in this section, some related concepts and conclusions are summarized and introduced below. Let $\mathbb{N}$ denote the set of all positive integers.

Definition 2.1. [16] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is called a *continuous triangular norm* (in short, continuous t-norm) if it satisfies the following conditions:

(TN-1) T is commutative and associative;
(TN-2) T is continuous;
(TN-3) $T(a,1) = a$ for every $a \in [0,1]$;
(TN-4) $T(a,b) \leq T(c,d)$ whenever $a \leq c$, $b \leq d$ and $a,b,c,d \in [0,1]$.

An arbitrary t-norm T can be extended (by associativity) in a unique way to an n-ary operator taking for $(x_1, x_2, \cdots, x_n) \in [0,1]^n$, $n \in \mathbb{N}$, the value $T(x_1, x_2, \cdots, x_n)$ is defined, in Ref. [7], by

$$T_{i=1}^0 x_i = 1, \ T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, \cdots, x_n).$$

Definition 2.2. [12] Let X be a nonempty set, m a positive integer and $f : X \to X$ an operator. $X = \bigcup_{i=1}^m X_i$ is a *cyclic representation* of X with respect to f if

(i) X_i, $i = 1, 2, \cdots, m$ are nonempty sets;
(ii) $f(X_1) \subset X_2, \cdots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

Definition 2.3. [4] A *fuzzy metric space* is an ordered triple (X, M, T) such that X is a nonempty set, T is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(FM-1) $M(x,y,t) > 0$;
(FM-2) $M(x,y,t) = 1$ if and only if $x = y$;
(FM-3) $M(x,y,t) = M(y,x,t)$;
(FM-4) $T(M(x,y,t), M(y,z,s)) \leq M(x,z,t+s)$;
(FM-5) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Definition 2.4. [5] Let (X, M, T) be a fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to *converge to* x in X, denoted by $x_n \to x$, if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in (0,1)$ and $t > 0$, there

exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.

(ii) A sequence $\{x_n\}$ is a *G-Cauchy sequence* if and only if $\lim\limits_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for any $p > 0$ and $t > 0$.

(iii) The fuzzy metric space (X, M, T) is called *G-complete* if every G-Cauchy sequence is convergent.

Definition 2.5. A function $\varphi : [0,1] \to [0,1]$ is called a *comparison function* if it satisfies

(i) φ is nondecreasing and left continuous;

(ii) $\varphi(t) > t$ for all $t \in (0,1)$.

Lemma 2.6. *If φ be a comparison function, then*

(i) $\varphi(1) = 1$;

(ii) $\lim\limits_{n \to +\infty} \varphi^n(t) = 1$ for all $t \in (0,1)$, where $\varphi^n(t)$ denotes the composition of $\varphi(t)$ with itself n times.

Inspired by the cyclic φ-contraction in Ref. [12] we present the same notion in fuzzy metric space, where $P_{cl}(X)$ denotes the collection of nonempty closed subsets of X.

Definition 2.7. Let (X, M, T) be a fuzzy metric space, m a positive integer, $A_1, A_2, \cdots, A_m \in P_{cl}(X)$, $Y = \bigcup\limits_{i=1}^{m} A_i$ and $f : Y \to Y$ an operator. If

(i) $\bigcup\limits_{i=1}^{m} A_i$ is cyclic representation of Y with respect to f;

(ii) there exists a comparison function $\varphi : [0,1] \to [0,1]$ such that

$$M(f(x), f(y), t) \geq \varphi(M(x, y, t))$$

for any $x \in A_i$, $y \in A_{i+1}$ and $t > 0$, where $A_{m+1} = A_1$, then f is called *cyclic φ-contraction* in the fuzzy metric space (X, M, T).

Definition 2.8. Let (X, M, T) be a fuzzy metric space and let $\{f_n\}$ be a sequence of self-mappings on X. $f_0 : X \to X$ is a given mapping. The sequence $\{f_n\}$ is said to *converge uniformly to f_0* if for each $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(f_n(x), f_0(x), t) > 1 - \epsilon$$

for all $n \geq n_0$ and $x \in X$.

3. Main Results

Theorem 3.1. *Let (X, M, T) be a G-complete fuzzy metric space, m a positive integer, $A_1, A_2, \cdots, A_m \in P_{cl}(X)$, $Y = \bigcup\limits_{i=1}^{m} A_i$, $\varphi : [0,1] \to [0,1]$ a comparison function and $f : Y \to Y$ an operator. Assume that*

(C1) $\bigcup\limits_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(C2) f is a cyclic φ-contraction.

Then f has a unique fixed point $x^* \in \bigcap\limits_{i=1}^{m} A_i$ and the iterative sequence $\{x_n\}_{n \geq 0}$ ($x_n = f(x_{n-1})$, $n \in \mathbb{N}$) converges to x^* for any starting point $x_0 \in Y$.

Proof. For any starting point $x_0 \in Y = \bigcup_{i=1}^{m} A_i$, since $x_n = f(x_{n-1})$ $(n \geq 1)$, we have

$$M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t) \text{ for any } t > 0.$$

Besides, for any $n \geq 0$, there exists $i_n \in \{1, 2, \cdots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Therefore, we can obtain

$$M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t) \geq \varphi(M(x_{n-1}, f(x_n), t)).$$

Consider the definition of φ, we get by induction that

$$M(x_n, x_{n+1}, t) \geq \varphi^n(M(x_0, x_1, t)).$$

Thus, for any $p > 0$, we have

$$M(x_n, x_{n+p}, t) \geq T(M(x_n, x_{n+1}, t/p), M(x_{n+1}, x_{n+2}, t/p), \cdots, M(x_{n+p-1}, x_{n+p}, t/p))$$
$$\geq T(\varphi^n(M(x_0, x_1, t/p), \varphi^{n+1}(M(x_0, x_1, t/p), \cdots, \varphi^{n+p-1}(M(x_0, x_1, t/p)))$$
$$= T_{i=0}^{p-1} \varphi^{n+i}(M(x_0, x_1, t/p)).$$

By Lemma 2.6, for every $i \in \{0, 1, \cdots, p-1\}$, we obtain that

$$\lim_{n \to \infty} \varphi^{n+i}(M(x_0, x_1, t/p)) = 1.$$

According to the continuity of t-norm T, it can easily be verified that $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$. It shows that $\{x_n\}_{n \geq 0}$ is a G-Cauchy sequence in the G-complete subspace Y. So there exists $x^* \in Y$ such that $\lim_{n \to \infty} x_n = x^*$.

On the other hand, by the condition (C1), it follows that the iterative sequence $\{x_n\}_{n \geq 0}$ has an infinite number of terms in each A_i, $i = 1, 2, \cdots, m$. Since Y is G-complete, from each A_i, $i = 1, 2, \cdots, m$, one can extract a subsequence of $\{x_n\}_{n \geq 0}$ which converges to x^* as well. Because each A_i, $i = 1, 2, \cdots, m$ is closed, we conclude that $x^* \in \bigcap_{i=1}^{m} A_i$ and thus $\bigcap_{i=1}^{m} A_i \neq \emptyset$.

Set $Z = \bigcap_{i=1}^{m} A_i$. Obviously, Z is also closed and G-complete. Consider the restriction of f to Z, that is, $f|_Z : Z \to Z$. Next, we will prove that $f|_Z$ has a unique fixed point in $Z \subset Y$.

For the foregoing $x^* \in Z$, since $f|_Z(x^*) \in Z$ and $x_n \in A_{i_n}$, we can choose A_{i_n+1} such that $f|_Z(x^*) \in A_{i_n+1}$. Hence, for any $t > 0$, we have

$$M(f|_Z(x^*), x^*, t) = M(f(x^*), x^*, t)$$
$$\geq T(M(f(x^*), f(x_n), t/2), M(x_{n+1}, x^*, t/2))$$
$$\geq T(\varphi(x^*, x_n, t/2), M(x_{n+1}, x^*, t/2)) \to T(1, 1) = 1 \text{ as } n \to \infty.$$

Clearly, we get $f|_Z(x^*) = x^*$, namely, x^* a fixed point, which is obtained by iteration from starting point x_0. To show uniqueness, we assume that $z \in \bigcap_{i=1}^{m} A_i$ is another fixed point of $f|_Z$. Since $x^*, z \in A_i$ for all $i \in \mathbb{N}$, we can obtain

$$M(x^*, z, t) = M(f|_Z(x^*), f|_Z(z), t) = M(f(x^*), f(z), t) \geq \varphi(M(x^*, z, t)) > M(x^*, z, t).$$

This leads to a contradiction. Thus, x^* is the unique fixed point of $f|_Z$ for any starting point $x_0 \in Z \subset Y$.

Now, we still have to prove that the iterative sequence $\{x_n\}_{n \geq 0}$ converges to x^* for any initial point $x_0 \in Y$. Let $x \in Y = \bigcup_{i=1}^{m} A_i$, there exists $i_0 \in \{1, 2, \cdots, m\}$ such that $x \in A_{i_0}$. As $x^* \in \bigcap_{i=1}^{m} A_i$, it follows that $x^* \in A_{i_0+1}$ as well. Then, for any $t > 0$, we have

$$M(f(x), f(x^*), t) \geq \varphi(M(x, x^*, t)).$$

By induction and Definition 2.4, we can obtain

$$\begin{aligned}
M(x_n, x^*, t) = M(f^n(x_0), x^*, t) &= M(f^n(x_0), f(x^*), t) \\
&= M(f(f^{n-1}(x_0)), f(x^*), t) \\
&\geq \varphi(M(f^{n-1}(x_0), x^*, t)) \\
&\geq \cdots \geq \varphi^n(M(x_0, x^*, t)).
\end{aligned}$$

Supposing $x_0 \neq x^*$, it follows immediately that $x_n \to x^*$ as $n \to \infty$. So the iterative sequence $\{x_n\}_{n \geq 0}$ converges to the unique fixed point x^* of f for any starting point $x_0 \in Y$. $\qquad \square$

Example 3.2. Let X be the subset of $\mathbb{R}^2$ defined by

$$X = \{A, B, C, D, E\},$$

where $A = (0,0)$, $B = (0,1)$, $C = (1,1)$, $D = (0,2)$, $E = (3,3)$. $\varphi(\tau) = \sqrt{\tau}$ for all $\tau \in [0,1]$. Define $M(x, y, t) = e^{-\frac{2d(x,y)}{t}}$, where $d(x,y)$ denotes the Euclidean distance of $\mathbb{R}^2$. Clearly, (X, M, T) is a G-complete fuzzy metric space with respect to t-norm $T(a, b) = ab$.

Let $f : X \to X$ be given by

$$f(A) = f(B) = f(C) = f(D) = B, \quad f(E) = A.$$

Set $A_1 = \{A, B, C, D\}$, $A_2 = \{B, D, E\}$. $f(A_1) = \{B\} \subseteq A_2$, $f(A_2) = \{A, B\} \subseteq A_1$. According to Definition 2.2, $X = A_1 \cup A_2$ is a cyclic representation of X with respect to f. In addition, it can easily be verified that $M(f(x), f(y), t) \geq \varphi(M(x, y, t))$ for every $x \in A_1$, $y \in A_2$, and $t > 0$. This shows that f is a cyclic φ-contraction. Hence, all the conditions of Theorem 3.1 are satisfied and then f has a unique fixed point, that is, $x = B$.

Theorem 3.3. *Let $f : Y \to Y$ be a self-mapping as in Theorem 3.1. If there exists an iterative sequence $\{y_n\}_{n \in \mathbb{N}}$ in Y such that $M(y_n, f(y_n), t) \to 1$ as $n \to \infty$ for any $t > 0$, then $y_n \to x^*$ as $n \to \infty$.*

Proof. According to the proof of Theorem 3.1, we know that x^* is the unique fixed point of f for any starting point $x_0 \in Y$. Therefore, for any $t > 0$, we have

$$\begin{aligned}
1 \geq M(y_n, x^*, t) &\geq T(M(y_n, f(y_n), t/2), M(f(y_n), f(x^*), t/2)) \\
&\geq T(M(y_n, f(y_n), t/2), \varphi(M(y_n, x^*, t/2))) \\
&\geq T(M(y_n, f(y_n), t/2), \varphi^n(M(x_0, x^*, t/2))).
\end{aligned}$$

Since $M(y_n, f(y_n), t/2) \to 1$ and $\varphi^n(M(x_0, x^*, t/2)) \to 1$ as $n \to \infty$, it means that $M(y_n, x^*, t) \to 1$ which is equivalent to saying that $y_n \to x^*$ as $n \to \infty$. □

Theorem 3.4. *Let $f : Y \to Y$ be a self-mapping as in Theorem 3.1. If there exists a convergent sequence $\{y_n\}_{n \in \mathbb{N}}$ in Y such that $M(y_{n+1}, f(y_n), t) \to 1$ as $n \to \infty$ for any $t > 0$, then there exists $x_0 \in Y$ such that $M(y_n, f^n(x_0), t) \to 1$ as $n \to \infty$.*

Proof. For any $t > 0$, let $y_n \in Y$, $n \in \mathbb{N}$ such that $M(y_{n+1}, f(y_n), t) \to 1$, $n \to \infty$. Set y as a limit of $\{y_n\}_{n \in \mathbb{N}}$. By the proof of Theorem 3.3, we note that $x^* \in \bigcap_{i=1}^{m} A_i$ is the unique fixed point of f for any starting point $x_0 \in Y$ and $t > 0$. Therefore, for any $t = t_1 + t_2$ with $t_1, t_2 > 0$ and $n \geq 0$, we have

$$M(y_{n+1}, x^*, t) \geq T(M(y_{n+1}, f(y_n), t_1), M(f(y_n), f(x^*), t_2)).$$

Now, we show that $M(y_{n+1}, x^*, t) \to 1$ as $n \to \infty$ for any $t > 0$. Suppose that $M(y_{n+1}, x^*, t) \nrightarrow 1$, $n \to \infty$, there exists $0 < \epsilon < 1$ and $t > 0$ such that $\lim_{n \to \infty} M(y_{n+1}, x^*, t) = M(y, x^*, t) = 1 - \epsilon$. Then there exists $0 < t_0 < t$ such that $M(y, x^*, t_0) \leq 1 - \epsilon$ and $\limsup_{n \to \infty} M(y_n, x^*, t_0) = 1 - \epsilon$. Since $y_n \in Y = \bigcup_{i=1}^{m} A_i$ for each $n \geq 0$, there is $i_n \in \{1, 2, \cdots, m\}$ such that $y_n \in A_{i_n}$. But $x^* \in \bigcap_{i=1}^{m} A_i$, so we can choose $A_{i_{n+1}}$ such that $x^* \in A_{i_{n+1}}$. Therefore, we can obtain

$$M(y_{n+1}, x^*, t) \geq T(M(y_{n+1}, f(y_n), t - t_0), \varphi(M(y_n, x^*, t_0))), \quad n \geq 0.$$

Thus, according to the continuity of t-norm T, we have

$$1 - \epsilon = \lim_{n \to \infty} M(y_{n+1}, x^*, t) = M(y, x^*, t)$$

$$\geq \limsup_{n \to \infty} T(M(y_{n+1}, f(y_n), t - t_0), \varphi(M(y_n, x^*, t_0)))$$

$$= T(\limsup_{n \to \infty} M(y_{n+1}, f(y_n), t - t_0), \limsup_{n \to \infty} \varphi(M(y_n, x^*, t_0)))$$

$$= T(1, \limsup_{n \to \infty} \varphi(M(y_n, x^*, t_0)))$$

$$= \limsup_{n \to \infty} \varphi(M(y_n, x^*, t_0))$$

$$= \varphi(1 - \epsilon) > 1 - \epsilon,$$

which is a contradiction. Hence, $M(y, x^*, t) = 1$, namely, $y = x^*$. Thus, for any $t > 0$, we have $M(y_n, f^n(x_0), t) \to M(y, x^*, t)$ as $n \to \infty$. □

Theorem 3.5. *Let $f : Y \to Y$ be a self-mapping as in Theorem 3.1 and $f_n : Y \to Y$, $n \in \mathbb{N}$. If the following conditions are satisfied:*
(i) there exists a fixed point x_n^* for each f_n;
(ii) $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to f;
(iii) the sequence $\{x_n^*\}_{n \in \mathbb{N}}$ is convergent.
Then, $x_n^* \to x^*$ as $n \to \infty$.

Proof. Suppose that $\{x_n^*\}_{n\in\mathbb{N}}$ converges to $\widetilde{x}^*$. Since $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f, by Definition 2.6, for any $\epsilon \in (0,1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M(f_n(x), f(x), t) > 1 - \epsilon$ for all $n \geq n_0$ and $x \in Y$. That is, for every $x \in Y$, $M(f_n(x), f(x), t) \to 1$ as $n \to \infty$. By induction, for any $t = t_1 + t_2$ with $t_1, t_2 > 0$, we can obtain

$$M(x_n^*, x^*, t) = M(f_n(x_n^*), f(x^*), t_1 + t_2)$$
$$\geq T(M(f_n(x_n^*), f(x_n^*), t_1), M(f(x_n^*), f(x^*), t_2))$$
$$\geq T(M(f_n(x_n^*), f(x_n^*), t_1), \varphi(M(x_n^*, x^*, t_2))).$$

Similarly, we assume that $x_n^* \nrightarrow x^*$ as $n \to \infty$, i.e., there exist $\eta \in (0,1)$ and $t > 0$ such that $\lim_{n\to\infty} M(x_n^*, x^*, t) = M(\widetilde{x}^*, x^*, t) = 1 - \eta$. Then there exists $0 < t_0 < t$ such that $M(\widetilde{x}^*, x^*, t_0) \leq 1 - \eta$ and $\limsup_{n\to\infty} M(x_n^*, x^*, t_0) = 1 - \eta$. Therefore, we have

$$1 - \eta = \lim_{n\to\infty} M(x_n^*, x^*, t) = M(\widetilde{x}^*, x^*, t)$$
$$\geq \limsup_{n\to\infty} T(M(f_n(x_n^*), f(x_n^*), t - t_0), \varphi(M(x_n^*, x^*, t_0)))$$
$$= T(1, \limsup_{n\to\infty} \varphi(M(x_n^*, x^*, t_0)))$$
$$= \limsup_{n\to\infty} \varphi(M(x_n^*, x^*, t_0))$$
$$= \varphi(1 - \eta) > 1 - \eta,$$

which is a contradiction. Hence, $M(x_n^*, x^*, t) \to 1$ as $n \to \infty$, i.e., $x_n^* \to x^*$ as $n \to \infty$. $\qquad\square$

Theorem 3.6. *Let $f : (X, M, T_1) \to (X, M, T_1)$ be a self-mapping as in Theorem 3.1, if (X, M, T_2) is a G-complete metric space with $T_1(a,b) \leq T_2(a,b)$ for all $a, b \in [0,1]$, then $f : (X, M, T_2) \to (X, M, T_2)$ has the same unique fixed point.*

Proof. The result can be easily obtained from the proof of Theorem 3.1. $\qquad\square$

Theorem 3.7. *Let (X, M_1, T) and (X, M_2, T) be two fuzzy metric spaces. m a positive integer, $A_1, A_2, \cdots A_m \in P_{cl}(X)$, $Y = \bigcup_{i=1}^m A_i$ and $f : Y \to Y$ an operator. Assume that*

(i) $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;
(ii) $M_1(x, y, t) \geq M_2(x, y, t)$ for any $x, y \in Y$ and $t > 0$;
(iii) (X, M_1, T) is a G-complete metric space;
(iv) $f : (Y, M_1, T) \to (Y, M_1, T)$ is continuous;
(v) $f : (Y, M_2, T) \to (Y, M_2, T)$ is a cyclic φ-contraction with $\varphi : [0,1] \to [0,1]$ a comparison function.

Then the sequence $\{f^n(x_0)\}_{n\geq 0}$ converges to x^* in (X, M_1, T) for any starting point $x_0 \in Y$ and x^* is the unique fixed point of f.

Proof. According to Theorem 3.1, let $x_0 \in Y$, the condition (v) ensures that the iterative sequence $\{f^n(x_0)\}_{n\geq 0}$ is a G-Cauchy sequence in (Y, M_2, T). By the condition (ii), it is easy to verify that $\{f^n(x_0)\}_{n\geq 0}$ is also a G-Cauchy sequence in (Y, M_1, T). Due to the condition (iii), $\{f^n(x_0)\}_{n\geq 0}$ converges to x^* in (Y, M_1, T)

for any starting point $x_0 \in Y$. Furthermore, the condition (iv) implies the uniqueness of x^*. □

4. **Conclusion**

In this paper, we presented the notion of cyclic φ-contraction in a fuzzy metric space and constructed a fixed point theorem for this type of mapping in a G-complete fuzzy metric space. Noted that the iterative sequence induced by the cyclic φ-contraction could converge to the fixed point for any starting point. Moreover, some problems related to the fixed point were discussed. In our future research, we intend to establish a fixed point theorem for cyclic φ-contractive mappings in an M-complete fuzzy metric space.

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