A NOTE ON STRATIFIED LM-FILTERS

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ABSTRACT. We develop a theory of stratified LM-filters which generalizes the theory of stratified L-filters. Our stratification condition explicitly depends on a suitable mapping between the lattices L and M. If L and M are identical and the mapping is the identity mapping, then we obtain the theory of stratified L-filters. Based on the stratified LM-filters, a general theory of lattice-valued convergence spaces can be developed.

1. Introduction

In [18] Yao stated at the end that it is unclear how to define stratified LM-filters. In this short note we point to a possible answer. We consider two frames L, M, i.e. two complete lattices where finite meets distribute over arbitrary joins. We then have an implication in L defined by $\alpha \to \beta = \bigvee\{\gamma \in L \ : \ \alpha \wedge \gamma \leq \beta\}$ (and similarly for M). We denote the bottom and top elements in L and M by $\bot_L, \top_L$ and $\bot_M, \top_M$, respectively.

We consider mappings $\varphi : L \longrightarrow M$ with the following properties: (M1) $\varphi(\bot_L) = \bot_M$, (M2) $\varphi(\top_L) = \top_M$, and (M3) $\varphi(\alpha \wedge \beta) = \varphi(\alpha) \wedge \varphi(\beta)$. Here and in the sequel we use for both lattices the same symbols for the meets, joins and implications. This should not lead to confusion. In particular, any frame morphism between L and M can be used as an example of such a mapping. Further examples are the following.

- $\varphi_0(\alpha) = \begin{cases} \top_M & \text{if } \alpha = \top_L \\ \bot_M & \text{otherwise.} \end{cases}$ satisfies (M1), (M2) and (M3). It is the pointwise smallest such mapping.

- If $\bot_L$ is prime, i.e. of $\alpha \wedge \beta = \bot_L$ implies that $\alpha = \bot_L$ or $\beta = \bot_L$, then the mapping $\varphi_1(\alpha) = \begin{cases} \bot_M & \text{if } \alpha = \bot_L \\ \top_M & \text{otherwise.} \end{cases}$ is satisfying the conditions (M1), (M2) and (M3). (For (M3) we need the primeness of $\bot_L$.) It is the pointwise largest such mapping.

For a set X we denote the L-set by $a, b, c, ... : X \longrightarrow L$. The set of all L-sets on X is denoted by L^X. A constant L-set with value $\alpha \in L$ is denoted by α_X or by α_X^L in order to emphasize the lattice L. The lattice operations are extended pointwise from L to L^X.

2. **Stratified LM-filters: Examples and Characterizations**

We hold one mapping $\varphi : L \longrightarrow M$ with the properties (M1), (M2) and (M3) fixed. A mapping $\mathcal{F} : L^X \longrightarrow M$ is called an LM-*filter on X* if it satisfies the following properties: For all $a, b \in L^X, \alpha \in L$

(F1) $\mathcal{F}(\perp_X) = \perp_M$, $\mathcal{F}(\top_X) = \top_M$;

(F2) $a \le b$ implies $\mathcal{F}(a) \le \mathcal{F}(b)$;

(F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \le \mathcal{F}(a \wedge b)$.

If $\mathcal{F}$ additionally satisfies the condition

(Fs) $\varphi(\alpha) \wedge \mathcal{F}(a) \le \mathcal{F}(\alpha_X \wedge a)$ for all $a \in L^X, \alpha \in L$

then we call $\mathcal{F}$ a φ-*stratified LM-filter on X*.

It is not difficult to show that (Fs) is equivalent to

(Fs') $\varphi(\alpha) \le \mathcal{F}(\alpha_X)$ for all $\alpha \in L$.

One of the reviewers pointed out that a mapping $\varphi : L \longrightarrow M$ with (M1), (M2) and (M3) can be considered as a fuzzy filter with value set M. The condition (Fs') can then be interpreted as a compatibility condition between this fuzzy filter and the LM-filter $\mathcal{F}$. Note that (F2) and (F3) are equivalent to $\mathcal{F}(a) \wedge \mathcal{F}(b) = \mathcal{F}(a \wedge b)$ for all $a, b \in L^X$. We make this compatibility condition precise as follows. For an LM-filter $\mathcal{F} : L^X \longrightarrow M$ we define the mapping $\varphi_\mathcal{F} : L \longrightarrow M$ by $\varphi_\mathcal{F}(\alpha) = \mathcal{F}(\alpha_X)$. Then $\varphi_\mathcal{F}$ satisfies the conditions (M1), (M2) and (M3) and $\mathcal{F}$ is φ-stratified if and only if $\varphi \le \varphi_\mathcal{F}$ pointwise. This also shows that $\varphi_\mathcal{F}$ is the (pointwise) largest possible mapping $\varphi : L \longrightarrow M$ such that (M1), (M2) and (M3) is satisfied and for which $\mathcal{F}$ is φ-stratified.

We obtain the following Corollary.

Lemma 2.1. *Let φ and $\overline{\varphi}$ be mappings from L to M which satisfy (M1), (M2) and (M3) and let for all $\alpha \in L$, $\overline{\varphi}(\alpha) \le \varphi(\alpha)$. If $\mathcal{F}$ is a φ-stratified LM-filter, then $\mathcal{F}$ is $\overline{\varphi}$-stratified.*

For a family of mappings $(\varphi_i)_{i \in J}$, we define $\bigvee_{i \in J} \varphi_i$ as the minimal upper bound of the φ_i in the set of mappings that satisfy (M1), (M2) and (M3), whenever there is an upper bound $\varphi : L \longrightarrow M$ that satisfies (M1), (M2) and (M3). Clearly, for the existence of such an upper bound we can reformulate a corresponding criterion for L-filters (see e.g. [7]), cf. Section 3 below.) If an LM-filter $\mathcal{F}$ is φ_i-stratified for all φ_i $(i \in J)$, then $\mathcal{F}$ is also $(\bigvee_{i \in J} \varphi_i)$-stratified.

Example 2.2.   (1) If $L = M$ and $\varphi = id_L$ is the identity mapping, then id_L-stratified LL-filters are just stratified L-filters ([7]). So if $L \subseteq M$ and L is a subframe of M, then it seems most natural to demand $\iota_L \le \varphi_\mathcal{F}$, where $\iota_L(\alpha) = \alpha$ for all $\alpha \in L$, as stratification condition.

(2) If $L = \{0, 1\}$ then a stratified LM-filter is an M-filter of ordinary subsets [6]. Note that in this case the stratification condition (Fs) is true for any φ that satisfies (M1) and (M2).

(3) If $L = [0, 1]$ is the unit interval and $M = \{0, 1\}$, then an LM-filter, $\mathcal{F}$, can be identified with a prefilter, i.e. a filter in L^X [13, 12]. If we define the *characteristic value* of such a prefilter by $c(\mathcal{F}) = \bigwedge_{\mathcal{F}(\alpha_X) = 1} \alpha$ (see [13, 12])

and the mapping

$$\varphi_{c(\mathcal{F})}(\alpha) = \left\{ \begin{array}{ll} 1 & \text{if } \alpha > c(\mathcal{F}) \\ 0 & \text{otherwise} \end{array} \right.$$

then $\mathcal{F}$ is $\varphi_{c(\mathcal{F})}$-stratified.

(4) Every *LM*-filter is φ_0-stratified.

(5) If $L = M = \{0, 1\}$ then a φ-stratified *LM*-filter can be identified with a filter. Note that there is only one mapping $\varphi : \{0, 1\} \longrightarrow \{0, 1\}$ with (M1) and (M2), namely the identity mapping. Hence filters are always φ-stratified.

Further examples are given below.

Example 2.3. We define the *φ-stratified LM-point filter of $x \in X$* by

$$[x]_\varphi(a) = \varphi(a(x)) \quad \text{for } a \in L^X.$$

Example 2.4. An *enriched LM-fuzzy topological space* (X, Δ) [7] is a set X together with a mapping $\Delta : L^X \longrightarrow M$ which satisfies the axioms: For all $a, b, a_i \in L^X$ $(i \in J), \alpha \in L$

(O1) $\Delta(\top_X) = \top_M$;
(O2) $\Delta(a) \wedge \Delta(b) \leq \Delta(a \wedge b)$;
(O3) $\bigwedge_{i \in J} \Delta(a_i) \leq \Delta(\bigvee_{i \in J} a_i)$;
(Oe) $\Delta(\alpha_X) = \top_M$ for all $\alpha \in L$.

We define the following mapping $\mathcal{U}_\Delta^x : L^X \longrightarrow M$,

$$\mathcal{U}_\Delta^x(a) = \bigvee_{b \leq a} (\varphi(b(x)) \wedge \Delta(b)).$$

Then $\mathcal{U}_\Delta^x$ is a φ-stratified *LM*-filter. If $L = M$ and $\varphi = id_L$ we get Wei Yao's approach [16]. We note further that if $M = \{0, 1\}$, then we can identify enriched *LM*-fuzzy topological spaces with stratified *L*-topological spaces [7]. However, in this case $\mathcal{U}_\Delta^x$ is not a stratified *L*-filter unless $L = \{0, 1\}$.

Example 2.5. An *enriched L-fuzzy filter on X* [7] is a mapping $\mathcal{F} : L^X \times L \longrightarrow L$ with the following properties. For all $a, b \in L^X, \alpha, \beta \in L$

(FF1) $\mathcal{F}(\top_X, \alpha) = \top_L, \quad \mathcal{F}(\bot_X, \alpha) = \bot_L$;
(FF2) $a \leq b$ and $\beta \leq \alpha$ implies $\mathcal{F}(a, \alpha) \leq \mathcal{F}(b, \beta)$;
(FF3) $\mathcal{F}(a, \alpha) \wedge \mathcal{F}(b, \alpha) \leq \mathcal{F}(a \wedge b, \alpha)$;
(FFe) $\beta \wedge \mathcal{F}(a, \alpha) \leq \mathcal{F}(\beta_X \wedge a, \alpha)$.

We can identify such an enriched *L*-fuzzy filter with a φ-stratified *LM*-filter, $\widetilde{\mathcal{F}}$, if we put $M = L^L$, $\varphi : L \longrightarrow L^L$, $\alpha \longmapsto \alpha_L$ by defining $\widetilde{\mathcal{F}} : L^X \longrightarrow L^L$, $\widetilde{\mathcal{F}}(a)(\alpha) = \mathcal{F}(a, \alpha)$. We note that in this way we obtain a special class of φ-stratified *LM*-filters which satisfy additionally the condition

$$\alpha \leq \beta \text{ implies } \widetilde{\mathcal{F}}(a)(\beta) \leq \widetilde{\mathcal{F}}(a)(\alpha) \quad \forall a \in L^X,$$

i.e. the mappings $\widetilde{\mathcal{F}}(a) : L \longrightarrow L$ are order-reversing.

Example 2.6. We consider $L = [0, 1]$. A mapping $\varphi : L \longrightarrow L$ then satisfies (M3) if and only if it is non-decreasing. If $X = \{x\}$ is a one-point set, then we can

identify L^X with L and a φ-stratified LL-filter is then a mapping $\mathcal{F} : L \longrightarrow L$ that satisfies (M1), (M2) and (M3) and $\varphi(\alpha) \leq \mathcal{F}(\alpha)$ for all $\alpha \in L$. In particular, then any LL-filter $\mathcal{F}$ is $\mathcal{F}$-stratified and $\varphi = \mathcal{F}$ is the largest of possible mappings φ.

The following result generalizes a characterization of stratified L-filters given by Fang [2].

Lemma 2.7. *A mapping $\mathcal{F} : L^X \longrightarrow M$ is a φ-stratified LM-filter if and only if it satisfies the following conditions.*

(F1) $\mathcal{F}(\top_X) = \top_M$ *and* $\mathcal{F}(\bot_X) = \bot_M$;

(F2') $\varphi(\bigwedge_{x \in X}(a(x) \rightarrow b(x))) \leq \mathcal{F}(a) \rightarrow \mathcal{F}(b)$;

(F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b)$.

Proof. We need to show that (F2') is equivalent to (F2) and (Fs). Let first condition (F2') be true. Then $\mathcal{F}(\alpha) = \top_M \rightarrow \mathcal{F}(\alpha) = \mathcal{F}(\top_X) \rightarrow \mathcal{F}(\alpha) \geq \varphi(\bigwedge_{x \in X}((\top_X)(x) \rightarrow \alpha_X(x))) = \varphi(\alpha)$, i.e. (Fs) is true. If $a \leq b$, then $\bigwedge_{x \in X}(a(x) \rightarrow b(x)) = \top_L$ and hence $\top_M = \varphi(\top_L) \leq \mathcal{F}(a) \rightarrow \mathcal{F}(b)$ which is the same as $\mathcal{F}(a) \leq \mathcal{F}(b)$.

Let now (F2) and (Fs) be satisfied. We then conclude, with (F2) and (Fs), $\varphi(\bigwedge_{x \in X}(a(x) \rightarrow b(x))) \wedge \mathcal{F}(a) \leq \mathcal{F}((\bigwedge_{x \in x}(a(x) \rightarrow b(x)) \wedge a) \leq \mathcal{F}(b)$, because $(\bigwedge_{x \in x}(a(x) \rightarrow b(x)) \wedge a \leq b$. Hence $\varphi(\bigwedge_{x \in X}(a(x) \rightarrow b(x))) \leq \mathcal{F}(a) \rightarrow \mathcal{F}(b)$.   □

3. Properties of Stratified LM-filters

In this section we briefly discuss generalizations of results about stratified L-filters. Most of these generalizations are easy and require no new methods. Therefore we can be brief and mainly refer to the papers, where the L-filter results are originally stated and proved. We denote the set of all φ-stratified LM-filters on X by $\mathcal{F}^{\varphi s}_{LM}(X)$. This set is endowed with the pointwise order, i.e. we define $\mathcal{F} \leq \mathcal{G}$ if for all $a \in L^X$ we have $\mathcal{F}(a) \leq \mathcal{G}(a)$. It is easy to see that for a family of φ-stratified LM-filters, $(\mathcal{F}_i)_{i \in J}$ the meet, $\bigwedge_{i \in J} \mathcal{F}_i$, can be calculated by $(\bigwedge_{i \in J} \mathcal{F}_i)(a) = \bigwedge_{i \in J} \mathcal{F}_i(a)$. It is also easy to see that the join, $\bigvee_{i \in J} \mathcal{F}_i$ exists if and only if for each finite subfamily we have $\mathcal{F}_{i_1}(a_1) \wedge \mathcal{F}_{i_2}(a_2) \wedge \cdots \wedge \mathcal{F}_{i_n}(a_n) = \bot_M$ whenver $a_1 \wedge a_2 \wedge \cdots \wedge a_n = \bot_X$. The join is then given by $\mathcal{F}(a) = \bigvee\{\mathcal{F}_{i_1}(a_1) \wedge \mathcal{F}_{i_2}(a_2) \wedge \cdots \wedge \mathcal{F}_{i_n}(a_n) : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq a\}$, which is clearly φ-stratified. (see [7] for the case $L = M$).

Lemma 3.1. [7] *Let $\mathcal{F} \in \mathcal{F}^{\varphi s}_{LM}(X)$ and let $a \in L^X$. Then*

$$\varphi(\bigwedge_{x \in x} a(x)) \leq \mathcal{F}(a) \leq \varphi((\bigvee_{x \in X} a(x)) \rightarrow \bot_L) \rightarrow \bot_M.$$

Proof. With (Fs) and (F2) we obtain $\varphi(\bigwedge_{x \in X} a(x)) \leq \mathcal{F}(\bigwedge_{x \in X} a(x)) \leq \mathcal{F}(a)$. Moreover, again by (Fs), we have

$$\varphi((\bigvee_{x \in X} a(x)) \rightarrow \bot_L) \wedge \mathcal{F}(a) \leq \mathcal{F}(((\bigvee_{x \in X} a(x)) \rightarrow \bot_L) \wedge a)$$

$$\leq \mathcal{F}((a \rightarrow \bot_X) \wedge a) = \mathcal{F}(\bot_X) = \bot_M.$$

Hence $\mathcal{F}(a) \leq \varphi((\bigvee_{x \in X} a(x)) \rightarrow \bot_L) \rightarrow \bot_M$.   □

It is not difficult to show that $\mathcal{F}_0(a) = \varphi(\bigwedge_{x \in X} a(x))$ defines the smallest φ-stratified *LM*-filter on X. If φ respects arbitrary meets, then $\mathcal{F}_0 = \bigwedge_{x \in X}[x]_\varphi$.

As in the case $L = M$ we can see that the set $\mathcal{F}_{LM}^{\varphi s}(X)$ has maximal elements, which are called *φ-stratified LM-ultrafilters*. The following characterization generalizes a result of Höhle [5].

Lemma 3.2. *$\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X)$ is a φ-stratified LM-ultrafilter if and only if $\mathcal{F}(a) = \mathcal{F}(a \to \perp_X^L) \to \perp_M$ for all $a \in L^X$.*

Proof. We can repeat the proof in [7]. We only have to make sure that for $g \in L^X$ fixed

$$\overline{\mathcal{F}}(a) = \mathcal{F}(a) \vee (\mathcal{F}(g \to a) \wedge (\mathcal{F}(g \to \perp_X^L) \to \perp_M))$$

is φ-stratified. This however is trivial. □

Lemma 3.3. *Let L be a complete Boolean algebra, $\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X)$ and let $a \in L^X$. Then there is $\mathcal{G} \in \mathcal{F}_{LM}^{\varphi s}(X)$ such that $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F}(a) = \mathcal{G}(a)$ and $\mathcal{G}(a) \to \perp_M = \mathcal{G}(a \to \perp_X^L)$.*

Proof. Again we can copy the proof in [7]. We only have to check that $\mathcal{G}(b) = \mathcal{F}(b) \vee (\mathcal{F}(a \vee b) \wedge (\mathcal{F}(a) \to \perp_M))$ defines a φ-stratified *LM*-filter. But also this is easy to see. For the proof of (F3) we need that L is a complete Boolean algebra. □

Corollary 3.4. [7] *Let L and M be complete Boolean algebras and let $\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X)$. Then*

$$\mathcal{F} = \bigwedge_{\mathcal{U} \geq \mathcal{F}, \mathcal{U} \in \mathcal{F}_{LM}^{\varphi s}(X) \ ultra} \mathcal{U}.$$

If we have two sets, X and Y, and a mapping $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X)$ and $\mathcal{G} \in \mathcal{F}_{LM}^{\varphi s}(Y)$, then we define the *image of $\mathcal{F}$ under f, $f(\mathcal{F})$*, by $f(\mathcal{F})(b) = \mathcal{F}(f^{\leftarrow}(b))$ for $b \in L^Y$ [7]. It is easy to see that $f(\mathcal{F})$ is a φ-stratified *LM*-filter on Y. Further, the *inverse image of $\mathcal{G}$ under f, $f^{\leftarrow}(\mathcal{G})$*, is defined by $f^{\leftarrow}(\mathcal{F})(a) = \bigvee\{\mathcal{F}(b) : f^{\leftarrow}(b) \leq a\}$ for $a \in L^X$. $f^{\leftarrow}(\mathcal{G})$ is a φ-stratified *LM*-filter on X if and only if $f^{\leftarrow}(b) = \perp_X$ implies $\mathcal{F}(b) = \perp_M$ [8]. We further define the *product of two φ-stratified LM-filters $\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X)$, $\mathcal{G} \in \mathcal{F}_{LM}^{\varphi s}(Y)$* by

$$\mathcal{F} \times \mathcal{G}(a) = \bigvee\{\mathcal{F}(f) \wedge \mathcal{G}(g) : f \times g \leq a\}$$

for $a \in L^{X \times Y}$. Here, it is defined $f \times g(x, y) = f(x) \wedge g(y)$ for $f \in L^X$ and $g \in L^Y$. The following result is needed in the proof that the category of lattice-valued convergence spaces is Cartesian closed (see the next section).

Lemma 3.5. [8] *Let $f : X \times Y \longrightarrow Z$ and $x \in X$. We define* $f_x : \begin{cases} Y & \longrightarrow & Z \\ y & \longmapsto & f(x, y) \end{cases}$. *If $\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(Y)$ then $f_x(\mathcal{F}) \geq f([x]_\varphi \times \mathcal{F})$.*

Proof. We have for $a \in L^Y$ that $f([x]_\varphi \times \mathcal{F})(a) = [x]_\varphi \times \mathcal{F}(f^{\leftarrow}(a)) = \bigvee\{\varphi(a_1(x)) \wedge \mathcal{F}(a_2) : a_1 \times a_2 \leq f^{\leftarrow}(a)\}$. If $a_1 \times a_2 \leq f^{\leftarrow}(a)$, then for $(x, y) \in X \times Y$ we have $a_1(x) \wedge a_2(y) \leq f^{\leftarrow}(a)(x, y) = a(f(x, y)) = a(f_x(y)) = f_x^{\leftarrow}(a)(y)$. Hence, with the condition (Fs),

$$f([x]_\varphi \times \mathcal{F})(a) \leq \bigvee\{\mathcal{F}(a_1(x) \wedge a_2) : a_1(x) \wedge a_2 \leq f_x^{\leftarrow}(a)\} \leq \mathcal{F}(f_x^{\leftarrow}(a)) = f_x(\mathcal{F})(a).$$

□

4. **Remarks on Stratified** LMN**-convergence Spaces**

Let L, M and N be frames and $\varphi : L \longrightarrow M$ be a mapping that satisfies (M1), (M2) and (M3). A φ-stratified LMN-generalized convergence space is a set X together with a limit map $\lim : \mathcal{F}_{LM}^{\varphi s} \longrightarrow N^X$ which satisfies the axioms (for all $x \in X, \mathcal{F}, \mathcal{G} \in \mathcal{F}_{LM}^{\varphi s}(X)$)

(L1) $\lim[x]_\varphi(x) = \top_N$;

(L2) $\mathcal{F} \leq \mathcal{G}$ implies $\lim \mathcal{F} \leq \lim \mathcal{G}$.

A mapping $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ between two φ-stratified LMN-generalized convergence spaces is called *continuous* if $\lim_X \mathcal{F}(x) \leq \lim_Y f(\mathcal{F})(f(x))$ for all $\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X)$ and all $x \in X$. The category with objects the φ-stratified LMN-generalized convergence spaces and continuous mappings as morphisms is denoted by $\varphi s LMN$-GCS.

Example 4.1.   (1) If $L = M = N$ and $\varphi = id_L$, then a φ-stratified LMN-generalized convergence space is a stratified L-generalized convergence space [8, 9, 11]. Note that these spaces are the same as left-continuous L-convergence spaces introduced in [3]. Also note that if $L = M = N = \{0, 1\}$ we obtain Preuss' generalized convergence spaces [14].

(2) If $L = M = \{0, 1\}$ and $N = [0, 1]$, then a φ-stratified LMN-generalized convergence space can be identified with a left continuous probabilistic convergence space [15].

(3) If $L = \{0, 1\}$ and $M = N$ then we obtain the fuzzifying M-convergence spaces introduced in [17].

(4) If $L = [0, 1]$, $M = \{0, 1\}$ and $N = [0, 1]$, then we obtain convergence spaces where each prefilter gets assigned a grade of convergence to the points of X. Note that the axioms are different from the ones in Lowen and Lowen [12]. Also note that prefilters are in general only φ_0-stratified. This has the consequence that the point prefilter, $[x]_{\varphi_0}$, must be defined by $[x]_{\varphi_0} = \{a \in [0, 1]^X : a(x) = 1\}$.

(5) $L = M$, $N = \{0, 1\}$ then we obtain Gähler's fuzzy convergence structures [4].

If $M = N$, then we can define a φ-stratified LM-neighbourhood filter by

$$\mathcal{U}^x(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)).$$

Then $\mathcal{U}^x \in \mathcal{F}_{LM}^{\varphi s}(X)$ and $\mathcal{U}^x \leq [x]_\varphi$. In this case we can also define *pretopological φ-stratified LM-convergence spaces* by requiring the axiom

(Lp) $\lim \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a))$.

For the case $L = M$, see in this regard the papers [8, 9, 10]. Note that we can formulate the *L-topological axiom*,

$$\mathcal{U}^x(a) = \bigvee \{\mathcal{U}(b) \ : \ b(y) \leq \mathcal{U}^y(a) \, \forall y \in X\},$$

(see [7, 8]) only in case $L = M = N$.

We do not want to develp the theory of φ-stratified LMN-convergence spaces here. This theory is very similar to the theory of stratified L-generalized convergence spaces and almost all proofs parallel the corresponding proofs in [8]. We

leave the details to the reader. We only remark that the category $\varphi sLMN\text{-}GCS$ is topological over SET, where the initial structure on X with respect to the source $(f_i : X \longrightarrow (X_i, \lim_i))_{i \in J}$ is given by

$$\lim \mathcal{F}(x) = \bigwedge_{i \in J} \lim_i f_i(\mathcal{F})(f_i(x)) \qquad (\mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(X), x \in X).$$

Further, on the set $C(X, Y) = \{f : (X, \lim_X) \longrightarrow (Y, \lim_Y) \text{ continuous}\}$, we can define the limit map

$$c - \lim \mathcal{F}(g) = \bigwedge_{\mathcal{G} \in \mathcal{F}_{LM}^{\varphi s}(X)} \bigwedge_{x \in X} \left(\lim^X \mathcal{G}(x) \to \lim^Y ev(\mathcal{F} \times \mathcal{G})(g(x)) \right).$$

$(g \in C(X, Y), \mathcal{F} \in \mathcal{F}_{LM}^{\varphi s}(C(X, Y)))$. This limit map is called the *structure of continuous convergence* and makes the category $\varphi sLMN\text{-}GCS$ Cartesian closed. We need Lemma 3.5 to show this (see [8]) and remark that the φ-stratification is essentially used in the proof of Lemma 3.5.

5. **Conclusions**

We outlined in this note a theory of stratified *LM*-filters, where the stratification condition depends on the choice of a mapping $\varphi : L \longrightarrow M$ with suitable properties. We thus were able to answer a question raised by Wei Yao [18]. The theory of these φ-stratified *LM*-filters can be used to define very general lattice-valued convergence spaces, which, by suitable choices of L, M and N, encompass many examples of convergence spaces found in the literature. We need the φ-stratification of the *LM*-filters in order to show that this general category of lattice-valued convergence spaces is Cartesian closed.

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